The Second Order Contribution to Wave Crest Amplitude — Random Simulations and NewWave

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ABSTRACT
Estimating the probability of a wave crest exceeding a given threshold is of fundamental importance in offshore engineering. At present second-order theory is used, with the Forristall distribution typically being used for this calculation. In this paper we investigate several points in connection with this approach. Firstly we examine some of the sensitivities involved in deriving second order statistics, including the high frequency cut-off, the directional spreading and spectral bandwidth. The paper then examines how the second order contribution varies for waves of different shapes. We investigate whether the NewWave can be used to predict the second order contribution to a wave-crest but find that the variation in wave shape is too great for this to be practical.

KEY WORDS: Second order wave crest; Forristall distribution; Random ocean waves; NewWave

INTRODUCTION

One of the key problems in offshore engineering is calculating the short-term probability of a wave-crest of given magnitude occurring at a point. A common use for this is in “air-gap” calculations for offshore platforms. The air-gap problem is complicated and this paper only considers one part of the calculation: the second-order contribution to the wave-crest height.
There is an ongoing dispute as to whether physics other than that given by second-order theory is needed for describing the evolution of non-breaking wave-crests over a timescale in which a large wave-group forms and then disperses. Waves for which additional physics are important are termed rogue waves (see reviews by Kharif & Pelinovsky (2003); Dysthe *et al.* (2009); Adcock & Taylor (2014)). Current design practice assumes that physics beyond second order is not significant for practical offshore calculations, an assumption which appears to be generally consistent with field measurements.
The standard statistical model that is used to describe the short-term distribution of wave-crests is the Forristall distribution (Forristall, 2000). This distribution was derived by fitting a probability distribution to the results of numerical simulations of random waves using second-order theory. Forristall derived two distributions: a 2D and a 3D. The 2D is only applicable to laboratory studies. The 3D distribution was based on simulations of JONSWAP spectra with a number of different peak enhancement factors and with a constant, frequency dependent, directional spreading function.
The Forristall distribution is straightforward to apply to practical engineering calculations and seems to agree satisfactorily with most measurements and laboratory tests. The distribution is known to produce inaccurate results in some situations. For instance, if there is significant wave-breaking then the distribution will over-predict the magnitude of wave-crests. Conversely, in crossing seas, the set-down under a large wave, predicted by second-order theory, can become a set-up (Christou *et al.,* 2009) although the changes to the magnitude of the second order sum term mean that this does not always increase the overall crest height. These variations are not captured in the Forristall distribution as this only considers one form of directional spreading.
This paper examines some details and sensitivities of the calculation of second order wave crest statistics. We also examine whether it is possible to avoid the computational demands of simulating random waves by using the second order structure from a NewWave wave-group to calculate the second order correction.
Second order random simulations are also used in other applications such as calculating velocities and loads (Alberello *et al*. 2014).

THEORY

Second order correction

There is no change to the dispersion equation that governs the propagation of wave components in second order theory. Thus, the free surface can be described by a number of “free” or “linear” waves which propagate as described by the linear dispersion equation, and “bound” waves which are a correction to the free surface profile and which are a function only of the linear wave components.
This paper uses standard second order theory. For random directional linear waves this theory was first given in Dean & Sharma (1981) although unfortunately this contains a number of typos. The results

presented in Forristall (2000) and Dalzell (1999) are consistent (although formulated differently).

As highlighted in Dalzell (1999) there is an inconsistency when a linear component interacts with itself in finite water depth and a further assumption has to be made. For random simulations this is generally not an issue as the self-interaction term is small compared to the interactions of different waves. In this paper we have set the value of the interaction kernel to zero for such interactions.

Following Dalzell we present the second order contribution for the interaction of two waves (which is easy to generalise to N interacting waves). Let us assume that the linear wave components have amplitude, a_j, and phase, φ_j, which is given by the components natural frequency multiplied by time plus some arbitrary phase shift. The linear waves are then given by

$$\eta_{linear} = \sum_{j=1}^{2} a_j \cos\varphi_j \tag{1}$$

It is convenient to split the second order contribution into sum and difference terms. These have frequencies that are the sum and difference of the interacting linear components.

The sum terms are given by

$$\eta_{2+} = \sum_{j=1}^{2}\sum_{j}^{2} a_i a_j B_p \cos\left(\varphi_i + \varphi_j\right) \tag{2}$$

where B_p is a given by a complicated expression and is a function of the interacting frequencies and the water depth (Dalzell, 1999) given in the appendix. Similarly the difference terms are given by

$$\eta_{2-} = a_i a_j B_m \cos(\varphi_i - \varphi_j) \tag{3}$$

where again the interaction kernel B_m is given in the appendix with the reader also referred to Dalzell (1999) and Forristall (2000).

We demonstrate the form of the second order correction to the free surface by plotting the results of fully non-linear simulations carried out by Gibbs & Taylor (2005). The simulations are of a focused NewWave wave-group with an amplitude ak = 0.1 and a wrapped normal directional spreading of 15° starting 20 periods before linear focus. The wave-group is split into linear and second order constituents using the method described in Taylor *et al.* (2005) and by standard spectral filtering. Figure 1 shows the linear and second order waves at linear focus, demonstrating the second order sum terms being in phase with a large crest, whilst the difference terms are out of phase. The sum and difference terms are in excellent agreement with those predicted by second order theory (Adcock, 2009).

Random waves

The fundamental assumption underlying much offshore wave theory is that the linear waves at different frequencies are uncorrelated with each other and that the free surface is a random Gaussian process.

For a given spectrum, simulating random waves can be done numerically. In this paper we follow the general approach described in Tucker *et al.* (1984).

Once random linear waves have been generated these can be written as given in equation 1 and the second order correction to the free surface may be calculated from equations (2) and (3) for a given directional distribution of energy.

SENSITIVITY TO HIGH FREQUENCY CUT-OFF

In calculating a second order random time series it is necessary to curtail the calculation of the interactions at some high frequency cut-off. For instance, in his seminal paper Forristall (2000) states: "The calculations described in this paper were typically truncated… [at] four or five times the peak frequency".

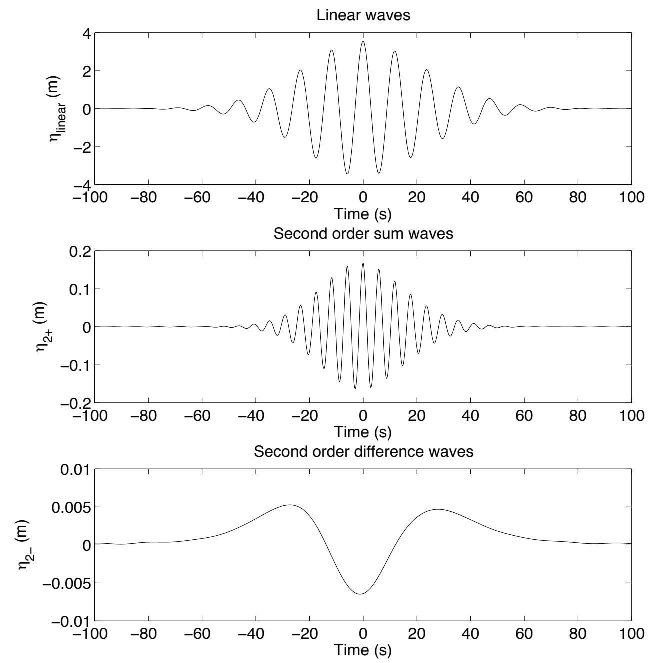

Figure 1 Example of second order contribution to the free surface from fully non-linear potential flow simulations based on a NewWave wavegroup.

One reason for this truncation is computational. Calculating a directionally spread second order sea is computer intensive and for practical purposes many simulations may be required in order to derive robust statistics.

However, there is a more significant reason why this is an issue. To demonstrate, let us consider the interaction between two components with angular frequency ω_1 and ω_2. In this example we consider waves travelling in the same direction but the general result is the same for waves travelling in different directions. Taking $\omega_1 = 0.5$ rad s^{-1} the magnitude of the interactions B_p and B_m for different values of ω_2 are plotted in Figure 2 for deep water (actually 3000 m) and intermediate to shallow water (30 m). Both interaction terms increase in magnitude as the difference in frequency increases. The value of B_p increases at the square of the difference in frequencies, whilst the increase in B_m is more complicated but similar in leading order term. For a JONSWAP spectrum, the high frequency tail of the amplitude decays with ω^{-3}. Thus, the decay in the magnitude of the second order terms is very slow.

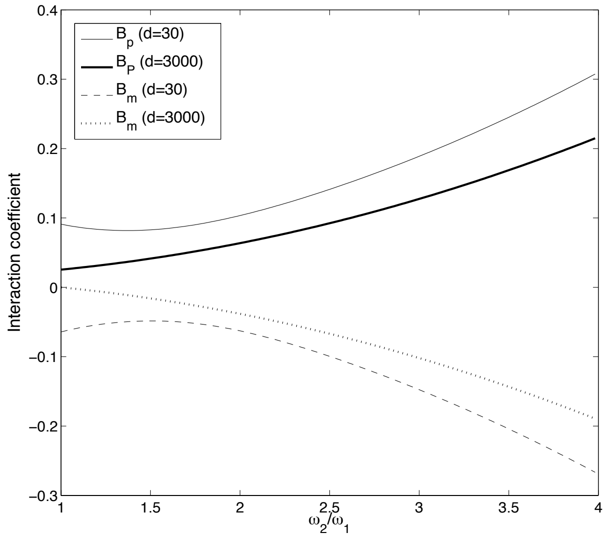

Figure 3 Magnitude of second order interaction kernel for co-linear waves

The issue can be seen by examining the spectra of a random wave simulation. A JONSWAP spectrum is used with γ = 3.3 with a low frequency cut-off at $0.5\omega_p$ and a high frequency cut-off at $4\omega_p$. The directional spreading of the sea-state is given by the Ewans (1998) distribution. Figure 3 shows the omnidirectional spectra of the different constituents of the second-order sea state. The second order difference spectra shows a slow decrease in amplitude between zero frequency and the artificial cut-off caused by curtailing the linear spectrum at $|\omega_p$ - $4\omega_p$ | and decays to zero at $|\omega_p$ - $4\omega_p|$. Similarly, the second order sum term increases between $2\times0.5\omega_p$ and $2\omega_p$. It then decays slowly until the point at which the linear peak is no longer taking part in the interactions. The key result is that, prior to the change-point caused by curtailing the linear spectrum, the decay of both second order sum, and second order difference terms is slow.

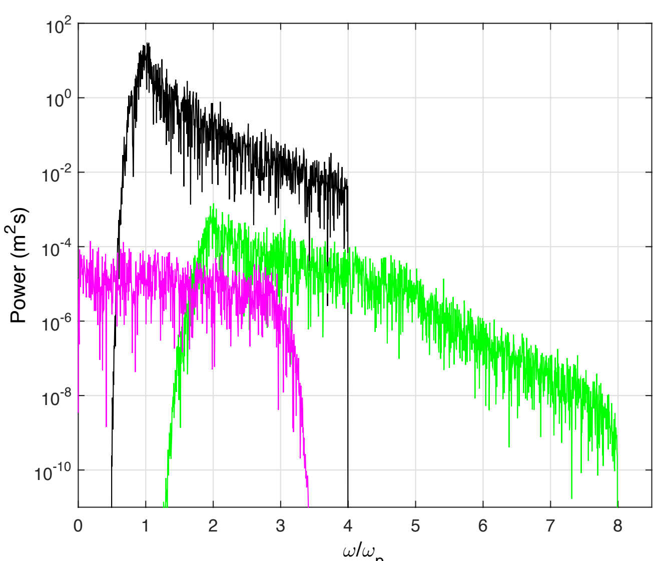

Figure 3 Different components of a simulated second order spectrum. Linear — black; second order difference — magenta; second order sum — green.

This behavior is not physically correct. This is a known problem perhaps first described in general terms by Barik & Webber (1977). The physics of the anomaly is resolved in Janssen (2009) where it is shown that linear and third order terms will interact to cancel much of the upper tail of the spectrum.

There remains the problem of where to curtail the spectrum for practical engineering calculations. In practice, the calculation of extreme wave-crests in random simulations is not overly sensitive to the precise cut-off point. The reason for this is that a crest will tend to occur where the highest energy parts around the peak of the spectrum are in phase, with the low energy components at higher frequencies less likely to be in phase at the crest. This means that the second order terms due to interactions between components in the peak (that are correctly given by second order theory) will generally be correlated giving the required second order correction. However, the interactions between the peak and the tail of the spectrum are likely to not have a phase that leads to them being correlated with the crest, and so these introduce relatively small errors into the estimate of the second order correction at the extreme crest. In this paper we use $2.5\omega_p$ as a suitable cut-off. It should be noted however that for high-cycle fatigue calculations on objects near the surface the cut-off point will be more significant.

INFLUENCE OF SPECTRAL SHAPE ON SECOND ORDER CONTRIBUTION

Directional spreading

The magnitude of the second order contribution will be dependent on the spectral shape. Whilst it would be possible to investigate this by running random simulations it is easier to analyse the problem using deterministic wave-groups. In this study we use NewWave wave groups. These are the expected shape of a large wave-group in a linear random sea-state. A NewWave in the time domain focusing at t = 0 is given by equation (4) and an example is shown in Figure 1. The derivation is given in Lindgren (1970) and it was brought into offshore engineering practice by Tromans *et al.* (1991). It has been shown to agree with field measurements in numerous studies (e.g. Jonathan & Taylor, 1997) and to be valid up until the point where waves are dominated by shallow water wave breaking (Haniffah, 2013). It may be noted that a spectrum derived from the time history of a NewWave decays rather quicker than that of the spectrum, S(ω), which was used to derive the NewWave. Thus there is no problem with high frequency cut-off location

$$\eta(t) = \frac{\sum_n S(\omega_n)\cos(\omega_n t)}{\sum_n S(\omega_n)} \tag{4}$$

Except for in very shallow water, all sea states are directionally spread. The degree of directional spreading will vary depending on the type of storm. In realistic sea-states directional spreading is strongly dependent on frequency, but in this investigation we assume a constant directional spreading across all frequencies. We use a wrapped normal directional spreading and calculate the magnitude of the second order contribution under a large crest for different widths of directional spreading. In this we use a constant omni-directional frequency spectrum given by a JONSWAP spectrum with γ = 3.3, water depth 40 m, and a unit NewWave (i.e. a crest amplitude of 1m).

Figure 4 shows the magnitude of the sum and difference contributions under the crest. The magnitude of both terms reduces as directional spreading is increased. It is noticeable the magnitude of the second order difference term, although smaller, decreases by a relatively greater amount which is one reason why Adcock & Taylor (2009) used this to estimate directional spreading from a measured Eulerian timeseries. What is remarkable is that the changes almost cancel out so that the combined extra elevation is insensitive to the degree of directional spreading. This result is consistent across different water depths until the water is sufficiently shallow that the whole approach in this paper breaks down.

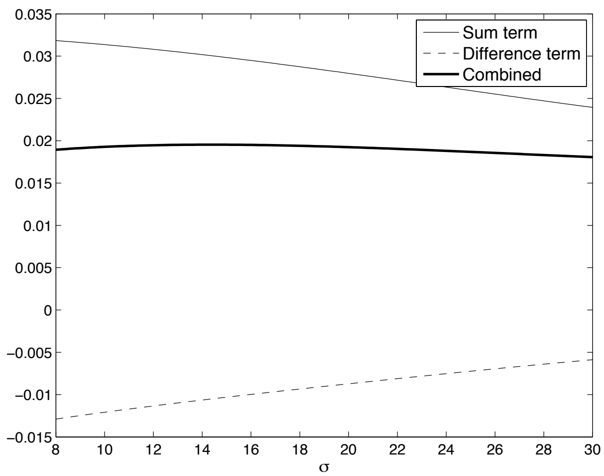

Figure 4 Second order contribution to wave crest height normalized by amplitude under a NewWave group as a function of directional spread σ in degrees.

A more realistic spreading function is that given by Ewans (1998). If this is used then the second-order sum contribution is 0.0244 and the difference -0.0061, giving a combined second-order contribution of 0.0184. This is broadly consistent with the frequency independent spreading estimates.

We conclude that for "following" sea-states (i.e. ones where the waves are going in a narrowly confined direction) an accurate knowledge of spreading is not crucial in determining second-order amplitudes implying that the Forristall distribution can be applied. The relative independence from directional spreading also suggests that any non-linear reduction in spreading under a large wave (Gibbs & Taylor, 2005; Adcock *et al*., 2012) can be ignored when calculating second order statistics. Of course, as noted in the introduction, in crossing sea states there can be a significant difference in the second order contribution to crest amplitude.

Spectral bandwidth

The above simulations considered a constant spectral width described by the JONSWAP spectral enhancement factor γ. Similar simulations of second-order NewWaves were carried out but showed that second order sum, and second order difference contributions changed negligibly for values of γ between 1 and 5.

VARIATION OF SECOND ORDER CONTRIBUTION IN RANDOM SIMULATIONS

In a real sea-state, large waves that are of practical interest to engineers show a significant variation in shape, although the average profile is given by the theoretical NewWave form (Tucker, 1999). Figure 5 shows plots of the 2D free surface around two large waves with linear crest amplitudes greater than significant wave height H_s, generated randomly from the same underlying spectrum. The difference in the general shape, and particularly the broadness of the crest, is marked.

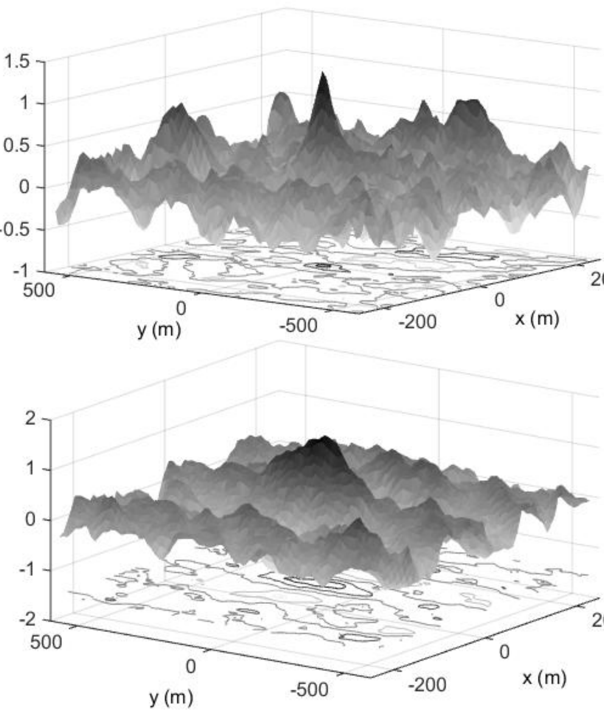

Figure 5 Examples of two large waves drawn from numerical simulations of identical sea-states with different shapes. Vertical axis free surface elevation normalized by H_s. Mean wave direction in x direction

The difference in the shape of the wave will lead to variations in the magnitude of the second order contribution to the crest elevation. It is not immediately obvious to the Authors whether the variation in the shape of an extreme wave event will lead to a small or large change in the second order contribution. If the magnitude of the second order contribution were strongly dependent only on the amplitude of the wave then knowledge of the linear wave amplitude statistics would immediately allow one to determine second order statistics, greatly simplifying design.

We investigate how second order contributions vary by running 10000 simulations of three hours of waves and recording the second order contribution to the largest (linear) wave observed in each simulation. We have used a JONSWAP spectrum with $\gamma = 3.3$ and a Ewans spreading function. The magnitude of the linear component, the second order sum and second order difference components were recorded.

Figure 6 plots the magnitude of the second order components against the linear crest amplitude. If we consider a given linear amplitude of wave we can see that there is a significant variation in both second order terms. Despite the large waves considered in these simulations there is no obvious sign that the absolute magnitude of the variation in the size of the second order terms reduces for larger waves as would be the case if the largest waves were all close to the NewWave in form. It can also be observed that the total second order term (sum + difference) shows slightly less variation than the sum term alone suggesting that, to some extent, a larger sum term than the mean is correlated with a larger (negative) difference term than the mean; hence some of the variation cancels out.

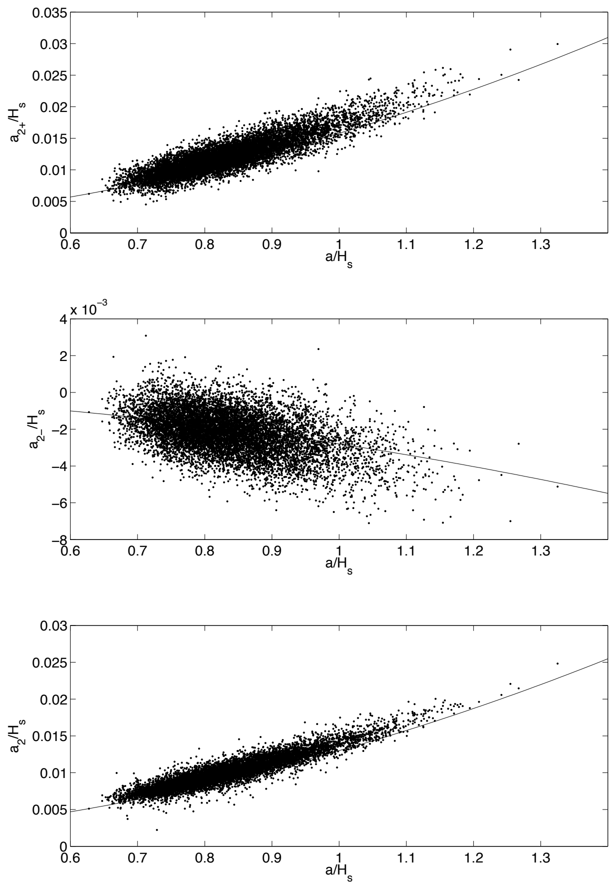

Figure 6 Magnitude of second order contribution to wave crest amplitude from random wave simulations (dots) and NewWave estimation (continuous line) as a function of linear crest amplitude. Top — second order sum; middle — second order difference; bottom — combined (sum + difference)

It is also noticeable than despite this simulation being in a following sea a few large waves have a positive second order difference term (i.e. they have a set-up under the crest). This occurs when much of the highly directionally spread high frequency tail is in phase with the wave crest. There are however, only a handful of set-ups that are of equal magnitude to the expected set-down as was observed at the famous Draupner wave recorded in the North Sea (Walker *et al.,* 2005; Adcock *et al*., 2011).

USING NEWWAVE TO ESTIMATE THE SECOND ORDER CORRECTION

A number of Authors have, at various times, tried to estimate the second order correction from the local properties of the waves without carrying out full second order simulations (e.g. Kriebel & Dawson, 1991). These have typically assumed that the linear waves are a slowly modulated sine wave and have calculated a local second order correction by assuming that locally the linear wave is a Stokes wave. This obviously does not account for the second order difference wave or for finite directional spreading. To account for these we could instead assume that the free surface around an extreme event is given by NewWave and use this to calculate the second order correction. This is straightforward, computationally quick, and as noted above does not suffer from ambiguity in the high frequency cut-off. It would also be straightforward to extend it to third, or higher, order which would be computationally difficult for random wave simulations.

Figure 6 shows the sum, difference and combined amplitude predicted by the NewWave approach. The approach shows good general agreement with the random wave simulations although it appears to slightly underestimate the mean size of the second order contribution. However, this approach obviously does not capture the variation in wave shapes in a realistic sea state. In estimating the probability of an extreme crest it is vital to capture this variation and so the NewWave approach cannot be used for this purpose.

VARIATION IN SECOND ORDER CONTRIBUTION IN FIELD DATA

Examining the variation in second order contribution in field data is problematic as sea-states do not remain stationary long enough to observe sufficient extreme waves to build up a robust picture of what is happening. In examining field data it is considerably easier to separate the low frequency difference terms compared to the high frequency sum terms as there are generally few other waves present in the low frequency part of the spectrum. Despite the lack of data, analysis of the low frequency waves present in North Sea data (Adcock & Taylor, 2009) and hurricane data (Santo *et al*, 2013) does indicate that large waves of similar magnitude occurring a short time from each other will have very different second order contributions.

CONCLUSIONS

This paper investigates the contribution second order bound waves make to the amplitude of the crests of large ocean waves. We note than when computing these the point chosen for the high-frequency cut-off can influence the results due to the limitations of second order theory. The extra amplitude under a wave crest is remarkably independent of directional spreading as the changes in sum and difference contributions to the amplitude roughly cancel out. The magnitude of the second order contribution is strongly dependent on the local time-history of the wave and not just on linear crest amplitude. This leads to considerable variation in the magnitude of the second order contribution across waves of similar amplitude. This has implication both for deriving crest statistics and for analysis of field measurements. Calculating the magnitude of the second order contribution using NewWave gives a reasonable first estimate but does not capture the important variations in second order contribution due to the random nature of waves.

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APPENDIX

This interaction kernels describe the magnitude of the second order difference interaction between two interacting components in equations 2 and 3. The frequency and wavenumber of the linear components, ω and k, and related by the linear dispersion relationship. The angle between components is θ, and the water depth is d. The "plus" and "minus" terms, B_p and B_m respectively are

$$B_p = \alpha - \frac{\beta_p \gamma_p}{\delta_p} + \frac{\varepsilon_p \zeta_p}{\delta_p},$$

and

$$B_m = \alpha + \frac{\beta_m \gamma_m}{\delta_m} + \frac{\varepsilon_m \zeta_m}{\delta_m},$$

where

$$\alpha = \frac{\omega_1^2 + \omega_2^2}{2g},$$

$$\beta_p = \frac{\omega_1 \omega_2}{2g}\left(1 - \frac{\cos(\theta)}{\tanh(|k_1|d)\tanh(|k_2|d)}\right),$$

$$\gamma_p = (\omega_1 + \omega_2)^2 + g|k_1 + k_2|\tanh(|k_1 + k_2|d),$$

$$\delta_p = (\omega_1 + \omega_2)^2 - g|k_1 + k_2|\tanh(|k_1 + k_2|d),$$

$$\varepsilon_p = \frac{\omega_1 + \omega_2}{2g},$$

$$\zeta_p = \frac{\omega_1^3}{\sinh^2(|k_1|d)} + \frac{\omega_2^3}{\sinh^2(|k_2|d)},$$

$$\beta_m = \frac{\omega_1 \omega_2}{2g}\left(1 + \frac{\cos(\theta)}{\tanh(|k_1|d)\tanh(|k_2|d)}\right),$$

$$\gamma_m = (\omega_1 - \omega_2)^2 + g|k_1 - k_2|\tanh(|k_1 - k_2|d),$$

$$\delta_m = (\omega_1 - \omega_2)^2 - g|k_1 - k_2|\tanh(|k_1 - k_2|d),$$

$$\varepsilon_m = \frac{\omega_1 - \omega_2}{2g},$$

$$\zeta_m = \frac{\omega_1^3}{\sinh^2(|k_1|d)} - \frac{\omega_2^3}{\sinh^2(|k_2|d)}.$$