Non-linear evolution of large waves in deep water – the influence of directional spreading and spectral bandwidth

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ABSTRACT

As large waves form in the open ocean their dynamics are modified from classic linear dispersion by non-linear physics. In this study we use a numerical model to study the evolution of isolated wave-groups in deep water. We find that the non-linear changes are rather sensitive to the initial conditions, with small changes in parameters such as directional spreading giving significantly different results. In all cases, although as aforementioned in differing degrees, we find changes to the shape of the extreme wave-group – the wave-groups contracting in the mean wave direction, expanding laterally to increase the width of the extreme crest, and with the largest wave moving to the front of the groups. We find that significant extra elevation of the wave-group only occurs for cases which are close to uni-directional.

INTRODUCTION

The formation and dynamics of the largest waves in the sea are of interest to scientists and engineers. In the open ocean, the dominant dynamics is described by dispersive focussing, where at certain points of time and space linear components come randomly into phase. There is much interest as to whether physics beyond this is needed to satisfactorily model the dynamics of large waves (Kharif and Pelinovsky; 2003; Dysthe et al.; 2008; Adcock and Taylor; 2014).

The most likely extra physics which may be significant is to account for the non-linearity of ocean wave propagation. The influence of this on large waves has been studied by numerous Authors over the past few decades (Socquet-Juglard et al.; 2005; Mori et al.; 2007; Onorato et al.; 2009).

There have been two main approaches to studying how non-linearity changes the dynamics of steep waves. One is to consider random wave-fields and study the statistics and properties of their largest waves. An alternative approach is to study the focussing of isolated wave-groups. This deterministic approach is useful to examining some of the details of the non-linear physics. A recent study by Adcock et al. (2015) found excellent general agreement between the non-linear changes to isolated wave-groups which under linear evolution would form the average shape of a large wave in the open ocean (a NewWave), and the average non-linear change that extreme groups undergo in random seas. This gives us confidence that conclusions drawn from studying isolated wave-groups can be applied to random waves in the real ocean. Thus, the isolated wave-group approach is the one taken in the present paper.

The results of the majority of studies of large waves suggest that there is a fundamental difference between waves in uni-directional, or nearly uni-directional, seas and those which are directionally spread. In uni-directional seas non-linearity causes large waves to be amplified relative to linear evolution. This is accompanied by a contraction of the spatial extent of the group along the direction of motion (Baldock et al.; 1996). The key parameter for the strength of this mechanism is the ratio of the wave steepness to the spectral bandwidth – a parameter often described as the Benjamin-Feir index (Janssen; 2003). However, in the open ocean, waves are not uni-directional but always have a directional spread of energy. This fundamentally changes the non-linear physics. For waves with a realistic directional spreading the non-linear physics appears to only give very small increases in the amplitudes of the largest waves over that expected by linear theory.

In this paper we seek to explore in more detail the sensitivity of the nonlinear physics of extreme waves to the underlying spectrum.

METHODS

The basic approach used in this paper is well established. We start with a wave-group which has the expected shape of a large wave-group in a linear random sea-state (Lindgren; 1970; Boccotti; 1983) – in offshore engineering this shape has become known as NewWave (Tromans et al.; 1991). The expected shape is given by the scaled autocorrelation function

$$\eta(x,y) = A_{max}\frac{\sum_n \sum_m S(k_n,\theta_m)\cos(\Psi_{n,m})}{\sum_n \sum_m S(k_n,\theta_m)}, \quad (1)$$

where A_{max} is the amplitude of the wave and $S(k,\theta)$ is the directional wavenumber spectrum of the underlying spectrum with θ being the angle of a given component relative to the mean wave direction. The phase Ψ is

$$\Psi_{n,m} = k_n \cos(\theta_m)x + k_n \sin(\theta_m)y. \quad (2)$$

This is then run backwards in time under linear evolution. In this study we run back twenty wave periods (as done by Gibbs and Taylor (2005)). The wave-group is then propagated forward in time until the peak height of the wave-group is a maximum. For this we use the broadbaneded non-linear Schrödinger equation derived by Dysthe (1979); Trulsen and Dysthe (1996) and given by

$$\begin{aligned}&\frac{\partial A}{\partial t} + \frac{\omega}{2k}\frac{\partial A}{\partial x} + i\frac{\omega}{8k^2}\frac{\partial^2 A}{\partial x^2} - i\frac{\omega}{4k^2}\frac{\partial^2 A}{\partial y^2} - \frac{\omega}{16k^3}\frac{\partial^3 A}{\partial x^3}\\ &+\frac{3\omega}{8k^3}\frac{\partial^3 A}{\partial y^2 \partial x} - i\frac{5\omega}{128k^4}\frac{\partial^4 A}{\partial x^4} - i\frac{3\omega}{32k^4}\frac{\partial^4 A}{\partial y^4} + i\frac{15\omega}{32k^4}\frac{\partial^4 A}{\partial y^2 \partial x^2}\\ &+\frac{7\omega}{256k^5}\frac{\partial^5 A}{\partial x^5} - \frac{35\omega}{64k^5}\frac{\partial^5 A}{\partial y^2 \partial x^3} + \frac{21\omega}{64k^5}\frac{\partial^5 A}{\partial y^4 \partial x}\\ &= -\frac{i\omega k^2}{2}A\left|A\right|^2 - \frac{3}{2}\omega k A^2\frac{\partial A}{\partial x} - \frac{1}{4}\omega k A^2 \frac{\partial A^*}{\partial x} - ikA\left.\frac{\partial \phi}{\partial x}\right|_{z=0},\end{aligned} \quad (3)$$

where ω and k are the frequency and wavenumber of the carrier wave and A is the complex wave amplitude, and A^* is the complex conjugate of A. The left hand side of this equation is linear and is an approximation to the linear dispersion relationship (Trulsen et al.; 2000). For 'linear' evolution we set the right hand side of this equation to zero.

The final term in equation 3 accounts for the interaction between the wave-group and the induced current. The potential ϕ satisfies

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=0} = \frac{\omega}{2}\frac{\partial \left|A\right|^2}{\partial x}, \quad (4)$$

(with a higher order approximation being given in Dysthe (1979)) and within the fluid

$$\nabla^2 \phi = 0. \quad (5)$$

Equation 3 is time-marched using a pseudo-spectral scheme to solve the linear dispersive part with the non-linear terms evaluated in the spatial domain. We use a spatial discretisation of 10 m and time-march using a timestep of 0.5 s and a Runge-Kutta scheme. For the cases presented in this paper the scheme conserves energy to within 0.2%.

The modified broadbanded non-linear Schrödginer equation has some key limitations. These are discussed in detail by Adcock and Taylor (2016a,b). In this study the critical limitation is the bandwidth. This may not be obvious as we use initial conditions with bandwidths that are narrow enough that the Dysthe equation would be expected to accurately predict the subsequent evolution. However, as wave-groups focus the non-linearity drives a local broadening of the bandwidth and, beyond a certain point, this leads to the broadbanded non-linear Schrodinger equation giving results which differ from potential flow simulations and can be assumed to be inaccurate. For this study we choose to exclude any run in which the local bandwidth in the mean wave direction exceeds 0.135m^{-1}. We derive the local bandwidth by fitting a Gaussian to the points around the spectral peak. This is an important limitation to this study.

In this study we use as initial conditions a Gaussian spectrum in both x and y directions with a central wavenumber of $k = 0.0279\text{m}^{-1}$ corresponding to a wavelength of 225 m and peak period of 12 s.

We vary the bandwidths of this spectrum. The bandwidth in the mean wave direction, s_x, we vary between 0.0023m^{-1} and 0.0092m^{-1}. Note that Gibbs (2004) found that a bandwidth of 0.0046m^{-1} fitted the peak of a JONSWAP spectrum with $\gamma = 3.3$. We vary the directional spreading, s, between $2.5°$ and $30°$. The spectral bandwidth in the lateral direction is then found using

$$s_y = k\frac{\pi s}{180}. \quad (6)$$

We define the amplitudes of the different cases based on the amplitude of the group at focus under linear evolution. Three amplitudes, A_{lin}, are considered: 5 m, 8 m and 11 m ($A_{lin}k$ equal to 0.14, 0.223 and 0.307).

In this study we just consider the complex envelope A and, in terms of results, only analyse the magnitude of this. Thus in this investigation we do not consider the phase. The phase of the underlying waves in the wave-group can be considered arbitrary – although the relative phase between components is not.

RESULTS

We start by presenting some examples picked from the different cases that we have simulated. Figure 1 shows wave-groups at focus for four different initial conditions and shows both the wave-group under linear and non-linear evolution.

For all cases presented in Figure 1 there is minimal change in amplitude between linear and non-linear cases. The changes in the group shape are very dependent on the initial conditions. However, all cases show the same fundamental changes in shape albeit to a greater or lesser extent. The groups expand in the lateral (y) direction; contract around the crest in the mean wave direction; and shift in the peak of the wave-group towards the front of the wave-group. These results are generally consistent with the potential flow results of Gibbs and Taylor (2005) and the random wave simulations of Adcock et al. (2015).

We now consider the difference between the maximum elevation reached under linear evolution to that recorded during the non-linear simulations. Figure 2 presents the maximum elevation during a non-linear run, relative to the elevation of the linearly focussing wave-group.

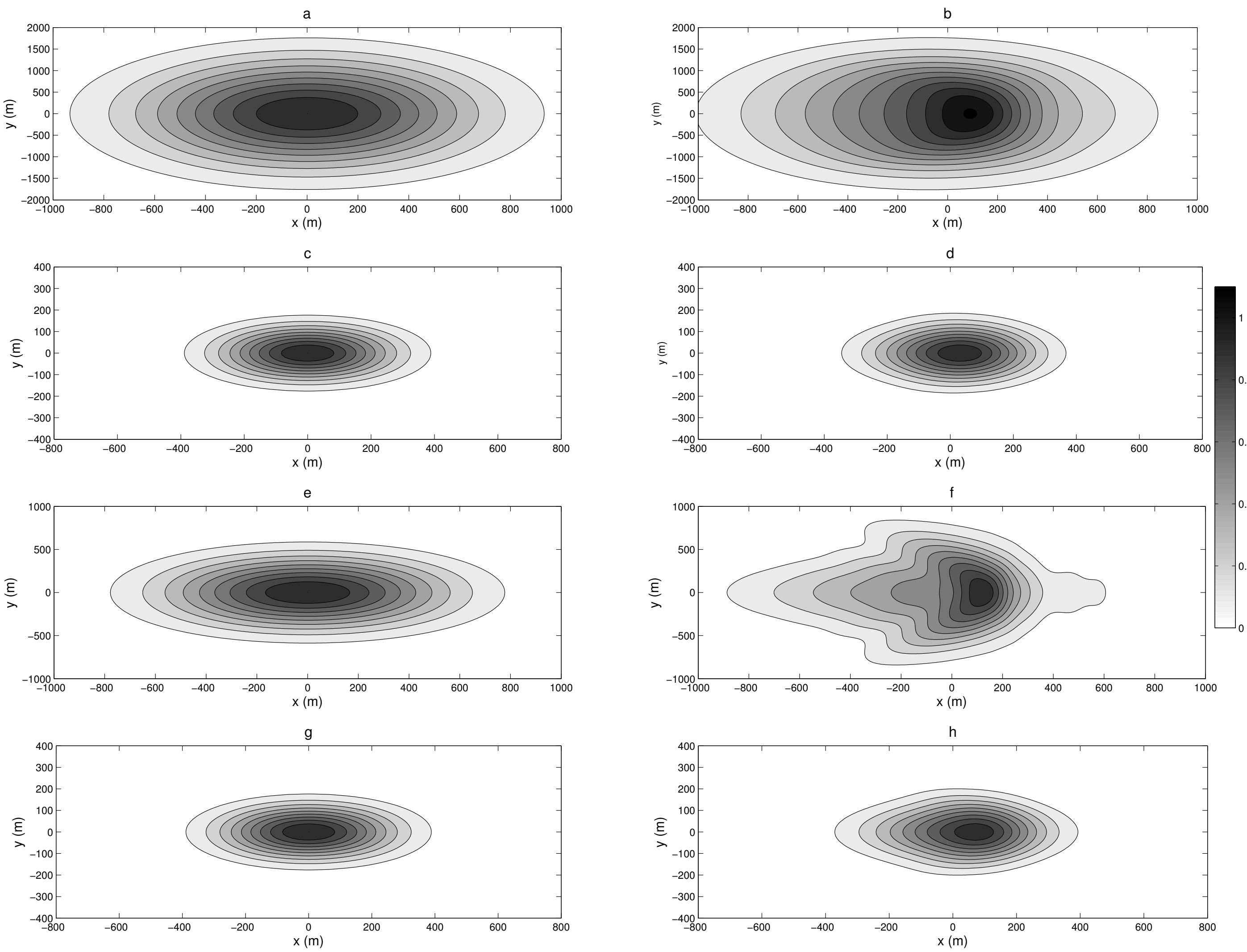

FIGURE 1: Wave-groups at maximum point elevation in simulation. Plot shows wave envelopes – carrier waves moving left to right. Amplitudes normalised by maximum amplitude of linear simulation. Note that different axes are used for different cases. Left – linear evolution; right – non-linear evolution. a and b – $A_{lin} = 5$m; $s = 2.5^{\circ}$. $s_x = 0.0023\text{m}^{-1}$. c and d – $A_{lin} = 5$m; $s = 25^{\circ}$. $s_x = 0.0055\text{m}^{-1}$. e and f – $A_{lin} = 8$m; $s = 10^{\circ}$. $s_x = 0.0028\text{m}^{-1}$. g and h – $A_{lin} = 8$m; $s = 25^{\circ}$. $s_x = 0.0055\text{m}^{-1}$.

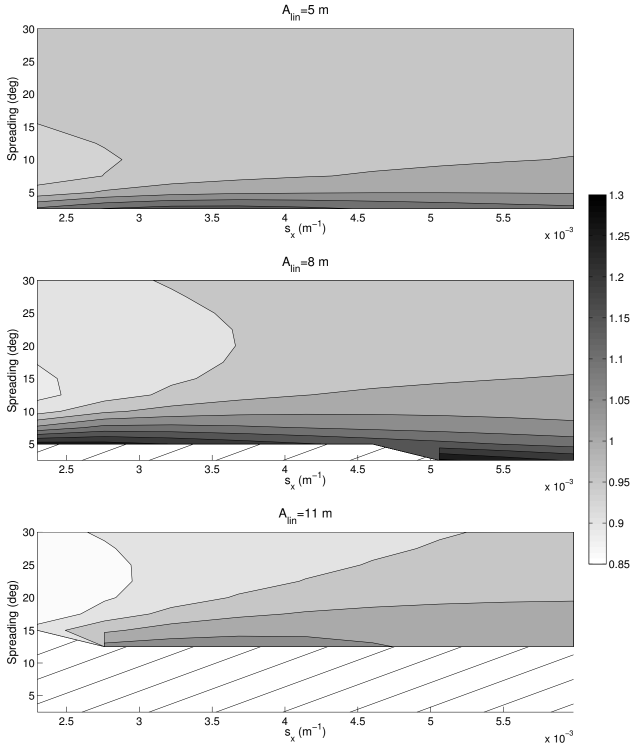

FIGURE 2: Maximum amplitude reached during non-linear run normalised by maximum amplitude of linear simulation. Hatching shows region where bandwidth limit was exceeded and no results are available.

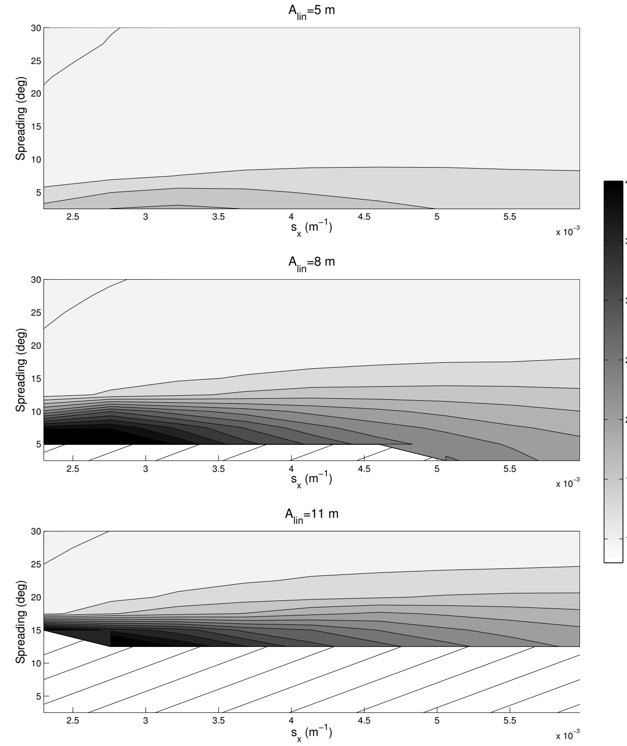

FIGURE 3: Mean-wave direction bandwidth of wave-group at maximum amplitude normalised by bandwidth under linear evolution. Hatching shows region where bandwidth limit was exceeded and no results are available.

The pattern is the same for all three steepnesses considered. For large directional spreads there is little or no increase in amplitude relative to the linear case. As directional spreading is decreased (i.e. the sea-state becomes closer to uni-directional) there is generally a significant increase in amplitude, although this is masked somewhat by the limitations of the present modelling not giving useful results for the most non-linear cases. For all three cases there is a region, for the narrowest bandwidth considered here, where there is a decrease in the maximum amplitude reached in non-linear evolution.

We next consider the changes to group shape. We define these simply by finding where the amplitude of the group drops below $2/3$ of its maximum value. To evaluate the change in shape we equate this to the width of an equivalent Gaussian to find the 'bandwith' in the x and y directions. We also use this to evaluate the asymmetry of the wave-group. We define asymmetry as the ratio of distances from maximum point of the wave envelope to where the envelope drops below $2/3$ of the maximum behind and in front of the crest in the mean wave direction. Thus an asymmetry of 1 implies a symmetrical group whereas a value greater than 1 implies the crest has moved towards the front of the group. In all cases we present the ratio of the bandwidth at non-linear focus to the bandwidth at linear focus.

Figure 3 presents the change in the bandwidth in the mean wave direction. In all cases we observe a contraction of the wave-group in the mean wave direction. In general the change is greater for steeper (larger amplitude) wave-groups, with smaller s_x and smaller directional spreading. A marked feature is that for certain initial conditions this change is very dramatic whereas for others, with only slightly different initial conditions, the change is quite small. To put this another way, the contours on the plots are either very spread out or very close together.

Figure 4 presents the change to the lateral width of the group due to non-linear dynamics. In all cases non-linearity leads to an increase in the width of the wave-group. The changes are greater for cases with larger amplitudes and narrower mean wave direction spectral bandwidth. For large directional spreads there is relatively little change. This is also the case for very small directional spreads – this latter may be due to these wave-groups being very close to uni-directional already and so large lateral expansions are suppressed. The biggest relative lateral expansion thus occurs for wave-groups whose initial directional spreading is around $5°$ to $10°$. A notable feature of these results is that unlike the contraction of the group in the mean wave direction, the contour plots are much flatter – i.e. there is no point in the plots where there is a sudden increase in the non-linear changes.

Figure 5 presents the asymmetry of the group at non-linear focus (noting that at linear focus the group is symmetrical so using the measure adopted here the asymmetry is unity). In all cases the asymmetry measure is greater than unity implying a movement

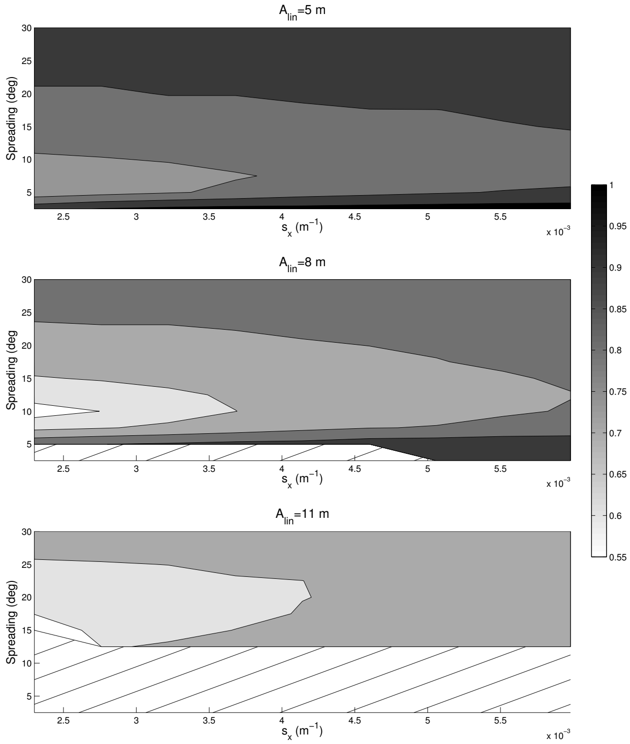

FIGURE 4: Lateral bandwidth of wave-group at maximum amplitude normalised by bandwidth under linear evolution. Hatching shows region where bandwidth limit was exceeded and no results are available.

of the peak towards the front of the wave-group. Like the lateral expansion, the movement towards the front of the wave-group is greater for larger amplitudes and narrower mean-wave direction bandwidths. The trend with directional spreading is more complex. In general for high directional spread ($\sim 30°$) there is relatively little movement. The directional spreading which gives the most asymmetry is dependent on the amplitude and mean-wave direction bandwidth.

DISCUSSION

This study shows that the evolution of large wave-groups are strongly dependent on the spectral bandwidths of the underlying sea-state. In particular, the non-linear dynamics are very strongly dependent upon the directional spreading.
An important result which these simulations demonstrate the sensitivity of the different changes to the underlying spectrum. The key results are:

- The contraction in the mean wave direction is very strongly dependent on the non-linearity and directional spreading. For realistic sea-states it would only occur for large waves in the steepest and most narrow banded sea-states.

- The expansion in the lateral direction occurs even for moderate steepness and spectral bandwidth. Although we find that this is greater for steeper sea-states with narrower bandwidths this dependence is much weaker than the contraction in the mean wave direction.

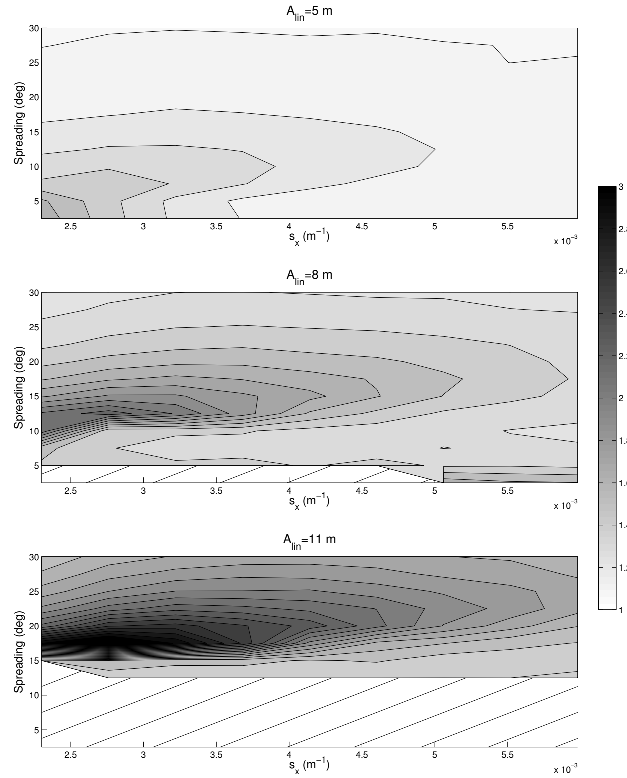

FIGURE 5: Asymmetry of wave-group at maximum amplitude. Hatching shows region where bandwidth limit was exceeded and no results are available.

- The movement of the largest wave to the front of the wave-group is again a strong function of the steepness and mean-wave direction bandwidth. Although the dependence is not quite as strong as for the contraction in the mean-wave direction we would still only expect to observe this in steep and narrow-banded seas.

These conclusions appear to be consistent with, and may help explain, past results.
Gibbs (2004) (see also Gibbs and Taylor (2005)) used a potential flow solver to simulate the focussing of isolated wave-groups. They did not vary the mean-wave direction bandwidth but only the steepness. They only considered two directional spreads: either uni-directional or a directional spread of $15°$. For both cases they found that the contraction in the mean wave direction varied more strongly than a quadratic in steepness. By contrast they found the lateral expansion approximately varied quadratically with wave steepness.
Adcock and Taylor (2009) and Adcock et al. (2012) derived approximate analytical expressions for the change in the contraction in the mean wave direction and lateral expansion. The two papers were for uni-directional and directionally spread results respectively. Unfortunately the directionally spread result does not re-

duce to the uni-directional case and these equations must be used with caution. However, they do again suggest a similar trend to those observed here – that the contraction in the mean-wave direction is much more sensitive that the expansion in the lateral direction.

Adcock et al. (2015) carried out simulations of random waves. Although directional spreading and mean-wave direction bandwidth of the background sea-state were not varied, the dependencies of the different changes in shape were entirely consistent with those found in this paper.

CONCLUSIONS

In this study we examined the non-linear changes that occur to large waves in the open ocean. We have examined the focussing of isolated wave packets using the broadbanded nonlinear Schrödinger equation to model the non-linear dynamics. We have explored the sensitivity of the changes to the size (or steepness) of the waves, the bandwidth, and the directional spread.

Consistent with previous studies we find that extra amplitude only occurs for waves that are close to uni-directional. We also observe significant changes to the shape of extreme wave-groups. Relative to linear evolution the extreme wave-group contract in the mean-wave direction, expand in a lateral direction, and the largest wave moves to the front of the group. We find that the first and last of these is a strong function of the initial conditions and will only happen for steep sea-states with a narrow bandwidth; the lateral expansion will occur even for more moderate sea-states.

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