Non-linear evolution of large waves in deep water – the influence of directional spreading and spectral bandwidth

Thomas A. A. Adcock\textsuperscript{1} and Paul H. Taylor\textsuperscript{1}

\textsuperscript{1}Department of Engineering Science, University of Oxford, Parks Road, Oxford. United Kingdom

ABSTRACT

As large waves form in the open ocean their dynamics are modified from classic linear dispersion by non-linear physics. In this study we use a numerical model to study the evolution of isolated wave-groups in deep water. We find that the non-linear changes are rather sensitive to the initial conditions, with small changes in parameters such as directional spreading giving significantly different results. In all cases, although as aforementioned in differing degrees, we find changes to the shape of the extreme wave-group – the wave-groups contracting in the mean wave direction, expanding laterally to increase the width of the extreme crest, and with the largest wave moving to the front of the groups. We find that significant extra elevation of the wave-group only occurs for cases which are close to uni-directional.

INTRODUCTION

The formation and dynamics of the largest waves in the sea are of interest to scientists and engineers. In the open ocean, the dominant dynamics is described by dispersive focussing, where at certain points of time and space linear components come randomly into phase. There is much interest as to whether physics beyond this is needed to satisfactorily model the dynamics of large waves (Kharif and Pelinovsky; 2003; Dysthe et al.; 2008; Adcock and Taylor; 2014).

The most likely extra physics which may be significant is to account for the non-linearity of ocean wave propagation. The influence of this on large waves has been studied by numerous Authors over the past few decades (Socquet-Juglard et al.; 2005; Mori et al.; 2008; Adcock and Taylor; 2014).

The methods used in this paper are well established. We start with a wave-group which has the expected shape of a large wave-group in a linear random sea-state (Lindgren; 1970; Boccotti; 1983) – in offshore engineering this shape has become known...
as NewWave (Tromans et al.; 1991). The expected shape is given by the scaled autocorrelation function

\[ \eta(x, y) = A_{max} \frac{\sum_n \sum_m S(k_n, \theta_m) \cos(\Psi_{n,m})}{\sum_n \sum_m S(k_n, \theta_m)}, \]  

where \( A_{max} \) is the amplitude of the wave and \( S(k, \theta) \) is the directional wavenumber spectrum of the underlying spectrum with \( \theta \) being the angle of a given component relative to the mean wave direction. The phase \( \Psi \) is

\[ \Psi_{n,m} = k_n \cos(\theta_m)x + k_n \sin(\theta_m)y. \]

This is then run backwards in time under linear evolution. In this study we run back twenty wave periods (as done by Gibbs and Taylor (2005)). The wave-group is then propagated forward in time until the peak height of the wave-group is a maximum. For this we use the broadband non-linear Schrödinger equation derived by Dysthe (1979); Trulsen and Dysthe (1996) and given by

\[
\frac{\partial A}{\partial t} + \frac{\omega}{2k} \frac{\partial A}{\partial x} + i \frac{\omega}{8k^2} \frac{\partial^2 A}{\partial x^2} - i \frac{\omega}{4k^2} \frac{\partial^2 A}{\partial y^2} - \frac{\omega}{16k^4} \frac{\partial^4 A}{\partial x^4} + i \frac{3\omega}{8k^3} \frac{\partial A}{\partial y} - i \frac{3\omega}{128k^4} \frac{\partial^3 A}{\partial x^3} + i \frac{15\omega}{32k^4} \frac{\partial A}{\partial y^2} + i \frac{3\omega}{32k^3} \frac{\partial^2 A}{\partial y^2} + i \frac{3\omega}{64k^5} \frac{\partial^2 A}{\partial x \partial y} + i \frac{3\omega}{256k^5} \frac{\partial A}{\partial x^2} = - \frac{\omega k^2}{2} |A|^2 \frac{\partial A}{\partial x} - \frac{1}{4} \omega k^2 A^2 \frac{\partial^2 A}{\partial x^2} - i k A \frac{\partial \phi}{\partial x} \bigg|_{z=0},
\]

where \( \omega \) and \( k \) are the frequency and wavenumber of the carrier wave and \( A \) is the complex wave amplitude, and \( A^* \) is the complex conjugate of \( A \). The left hand side of this equation is linear and is an approximation to the linear dispersion relationship (Trulsen et al.; 2000). For ‘linear’ evolution we set the right hand side of this equation to zero.

The final term in equation 3 accounts for the interaction between the wave-group and the induced current. The potential \( \phi \) satisfies

\[ \frac{\partial \phi}{\partial z} \bigg|_{z=0} = \frac{\omega}{2} \frac{\partial |A|^2}{\partial x}, \]

(with a higher order approximation being given in Dysthe (1979)) and within the fluid

\[ \nabla^2 \phi = 0. \]

Equation 3 is time-marched using a pseudo-spectral scheme to solve the linear dispersive part with the non-linear terms evaluated in the spatial domain. We use a spatial discretisation of 10 m and time-march using a timestep of 0.5 s and a Runge-Kutta scheme. For the cases presented in this paper the scheme conserves energy to within 0.2%.

The modified broadband non-linear Schrödinger equation has some key limitations. These are discussed in detail by Adcock and Taylor (2016a,b). In this study the critical limitation is the bandwidth. This may not be obvious as we use initial conditions with bandwidths that are narrow enough that the Dysthe equation would be expected to accurately predict the subsequent evolution. However, as wave-groups focus the non-linearity drives a local broadening of the bandwidth and, beyond a certain point, this leads to the broadbanded non-linear Schrodinger equation giving results which differ from potential flow simulations and can be assumed to be inaccurate. For this study we choose to exclude any run in which the local bandwidth in the mean wave direction exceeds 0.135m⁻¹. We derive the local bandwidth by fitting a Gaussian to the points around the spectral peak. This is an important limitation to this study.

In this study we use as initial conditions a Gaussian spectrum in both \( x \) and \( y \) directions with a central wavenumber of \( k = 0.0279m^{-1} \) corresponding to a wavelength of 225 m and peak period of 12 s.

We vary the bandwidths of this spectrum. The bandwidth in the mean wave direction, \( s_x \), we vary between 0.0023m⁻¹ and 0.0092m⁻¹. Note that Gibbs (2004) found that a bandwidth of 0.0046m⁻¹ fitted the peak of a JONSWAP spectrum with \( \gamma = 3.3 \). We vary the directional spreading, \( s \), between 2.5° and 30°. The spectral bandwidth in the lateral direction is then found using

\[ s_y = k \pi s_{180}. \]

We define the amplitudes of the different cases based on the amplitude of the group at focus under linear evolution. Three amplitudes, \( A_{lin} \), are considered: 5 m, 8 m and 11 m (\( A_{lin}k \) equal to 0.14, 0.223 and 0.307).

In this study we just consider the complex envelope \( A \) and, in terms of results, only analyse the magnitude of this. Thus in this investigation we do not consider the phase. The phase of the underlying waves in the wave-group can be considered arbitrary – although the relative phase between components is not.

**RESULTS**

We start by presenting some examples picked from the different cases that we have simulated. Figure 1 shows wave-groups at focus for four different initial conditions and shows both the wave-group under linear and non-linear evolution. For all cases presented in Figure 1 there is minimal change in amplitude between linear and non-linear cases. The changes in the group shape are very dependent on the initial conditions. However, all cases show the same fundamental changes in shape albeit to a greater or lesser extent. The groups expand in the lateral (\( y \)) direction; contract around the crest in the mean wave direction; and shift in the peak of the wave-group towards the front of the wave-group. These results are generally consistent with the potential flow results of Gibbs and Taylor (2005) and the random wave simulations of Adcock et al. (2015).

We now consider the difference between the maximum elevation reached under linear evolution to that recorded during the nonlinear simulations. Figure 2 presents the maximum elevation during a non-linear run, relative to the elevation of the linearly focussing wave-group.
FIGURE 1: Wave-groups at maximum point elevation in simulation. Plot shows wave envelopes – carrier waves moving left to right. Amplitudes normalised by maximum amplitude of linear simulation. Note that different axes are used for different cases. Left – linear evolution; right – non-linear evolution. a and b – $A_{lin} = 5m$; $s = 2.5^\circ$, $s_x = 0.0023m^{-1}$. c and d – $A_{lin} = 5m$; $s = 25^\circ$, $s_x = 0.0055m^{-1}$. e and f – $A_{lin} = 8m$; $s = 10^\circ$, $s_x = 0.0028m^{-1}$. g and h – $A_{lin} = 8m$; $s = 25^\circ$, $s_x = 0.0055m^{-1}$. 
In all cases we observe a contraction of the wave-group in the mean wave direction. In general the change is greater for steeper (larger amplitude) wave-groups, with smaller $s_x$ and smaller directional spreading. A marked feature is that for certain initial conditions this change is very dramatic whereas for others, with only slightly different initial conditions, the change is quite small. To put this another way, the contours on the plots are either very spread out or very close together.

Figure 4 presents the change to the lateral width of the group due to non-linear dynamics. In all cases non-linearity leads to an increase in the width of the wave-group. The changes are greater for cases with larger amplitudes and narrower mean wave direction spectral bandwidth. For large directional spreads there is relatively little change. This is also the case for very small directional spreads – this latter may be due to these wave-groups being very close to uni-directional already and so large lateral expansions are suppressed. The biggest relative lateral expansion thus occurs for wave-groups whose initial directional spreading is around 5° to 10°. A notable feature of these results is that unlike the contraction of the group in the mean wave direction, the contour plots are much flatter – i.e. there is no point in the plots where there is a sudden increase in the non-linear changes.

Figure 5 presents the asymmetry of the group at non-linear focus (noting that at linear focus the group is symmetrical so using the measure adopted here the asymmetry is unity). In all cases the asymmetry measure is greater than unity implying a movement.
of the peak towards the front of the wave-group. Like the lateral expansion, the movement towards the front of the wave-group is greater for larger amplitudes and narrower mean-wave direction bandwidths. The trend with directional spreading is more complex. In general for high directional spread (\(\sim 30^\circ\)) there is relatively little movement. The directional spreading which gives the most asymmetry is dependent on the amplitude and mean-wave direction bandwidth.

**DISCUSSION**

This study shows that the evolution of large wave-groups are strongly dependent on the spectral bandwidths of the underlying sea-state. In particular, the non-linear dynamics are very strongly dependent upon the directional spreading.

An important result which these simulations demonstrate the sensitivity of the different changes to the underlying spectrum. The key results are:

- The contraction in the mean wave direction is very strongly dependent on the non-linearity and directional spreading. For realistic sea-states it would only occur for large waves in the steepest and most narrow banded sea-states.

- The expansion in the lateral direction occurs even for moderate steepness and spectral bandwidth. Although we find that this is greater for steeper sea-states with narrower band-

These conclusions appear to be consistent with, and may help explain, past results.

Gibbs (2004) (see also Gibbs and Taylor (2005)) used a potential flow solver to simulate the focussing of isolated wave-groups. They did not vary the mean-wave direction bandwidth but only the steepness. They only considered two directional spreads: either uni-directional or a directional spread of \(15^\circ\). For both cases they found that the contraction in the mean wave direction varied more strongly than a quadratic in steepness. By contrast they found the lateral expansion approximately varied quadratically with wave steepness.

Adcock and Taylor (2009) and Adcock et al. (2012) derived approximate analytical expressions for the change in the contraction in the mean wave direction and lateral expansion. The two papers were for uni-directional and directionally spread results respectively. Unfortunately the directionally spread result does not re-
duce to the uni-directional case and these equations must be used with caution. However, they do again suggest a similar trend to those observed here – that the contraction in the mean-wave direction is much more sensitive that the expansion in the lateral direction.

Adcock et al. (2015) carried out simulations of random waves. Although directional spreading and mean-wave direction bandwidth of the background sea-state were not varied, the dependencies of the different changes in shape were entirely consistent with those found in this paper.

CONCLUSIONS

In this study we examined the non-linear changes that occur to large waves in the open ocean. We have examined the focussing of isolated wave packets using the broadbanded non-linear Schrödinger equation to model the non-linear dynamics. We have explored the sensitivity of the changes to the size (or steepness) of the waves, the bandwidth, and the directional spread.

Consistent with previous studies we find that extra amplitude only occurs for waves that are close to uni-directional. We also observe significant changes to the shape of extreme wave-groups. Relative to linear evolution the extreme wave-group contract in the mean-wave direction, expand in a lateral direction, and the largest wave moves to the front of the group. We find that the first and last of these is a strong function of the initial conditions and will only happen for steep sea-states with a narrow bandwidth; the lateral expansion will occur even for more moderate sea-states.

REFERENCES


