

The automorphisms of class two groups of prime exponent

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Abstract

In 2012, Marcus du Sautoy and Michael Vaughan-Lee gave an example of a class two group G_p of prime exponent p and order p^9 , and they showed that the number of descendants of G_p of order p^{10} is not a PORC function of p . The fact that the number of descendants of G_p is not PORC is directly related to the fact that the order of the automorphism group of G_p is not PORC. The number of conjugacy classes of G_p is also not a PORC function of p . In this note we give a complete list of all class two groups of prime exponent with order p^k for $k \leq 8$. For every group in this list we are able to show that the number of conjugacy classes of the group is polynomial in p , and that the order of the automorphism group is also polynomial in p . Thus, in some sense, the group G_p is minimal subject to having a non-PORC number of conjugacy classes and a non-PORC number of automorphisms.

1 Introduction

Graham Higman wrote two immensely important and influential papers on enumerating p -groups in the late 1950s. The papers were entitled *Enumerating p -groups I* and *II*, and were published in the Proceedings of the London Mathematical Society in 1960 (see [2] and [3]). In the first of these papers Higman proves that if we let $f(p^n)$ be the number of p -groups of order p^n , then $f(p^n)$ is bounded by a polynomial in p . In the second of his two papers Higman formulated his famous PORC conjecture concerning the form of the function $f(p^n)$. He conjectured that for each n there is an integer N (depending on n) such that for p in a fixed residue class modulo N the function $f(p^n)$ is a polynomial in p . For example, for $p \geq 5$ the number of groups of order p^6 is

$$3p^2 + 39p + 344 + 24 \gcd(p-1, 3) + 11 \gcd(p-1, 4) + 2 \gcd(p-1, 5).$$

(See [4].) So for $p \geq 5$, $f(p^6)$ is one of 8 polynomials in p , with the choice of polynomial depending on the residue class of p modulo 60. Thus $f(p^6)$ is **P**olynomial **O**n **R**esidue **C**lasses. The various nineteenth century classifications of groups of order p^n for $n \leq 5$

show that $f(p^n)$ is PORC for $n \leq 5$, and the classification of groups of order p^7 [6] shows that $f(p^7)$ is PORC. It is still an open question whether $f(p^8)$ is PORC, but in a recent article [8] I showed that the function enumerating the number of groups of order p^8 with exponent p is PORC. However Marcus du Sautoy and I have found a class two group G_p of order p^9 and exponent p with the property that the number of class 3 groups H of order p^{10} such that $H/\gamma_3(H) \cong G_p$ is not PORC. It may still be the case that $f(p^{10})$ is PORC, but this example does raise a strong possibility that Higman's conjecture fails for $n = 10$. The details of this example, and a history of the PORC conjecture can be found in [1]. Here we give some of the main properties of G_p .

The group G_p is a six generator class two group of exponent p ($p > 2$) with presentation

$$G_p = \left\langle \begin{array}{l} x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3 \mid [x_1, x_4] = y_3, [x_1, x_5] = y_1, [x_1, x_6] = y_2 \\ [x_2, x_4] = y_1, [x_2, x_5] = y_3, [x_3, x_4] = y_2, [x_3, x_6] = y_1 \end{array} \right\rangle$$

where all other commutators are defined to be 1, and where all generators have order p . The main results in [1] are

Theorem 1 *Let D_p be the number of descendants of G_p of order p^{10} and exponent p . Let V_p be the number of solutions (x, y) in $GF(p)$ that satisfy $x^4 + 6x^2 - 3 = 0$ and $y^2 = x^3 - x$.*

1. *If $p \equiv 5 \pmod{12}$ then $D_p = (p+1)^2/4 + 3$.*
2. *If $p \equiv 7 \pmod{12}$ then $D_p = (p+1)^2/2 + 2$.*
3. *If $p \equiv 11 \pmod{12}$ then $D_p = (p+1)^2/6 + (p+1)/3 + 2$.*
4. *If $p \equiv 1 \pmod{12}$ and $V_p = 0$ then $D_p = (p+1)^2/4 + 3$.*
5. *If $p \equiv 1 \pmod{12}$ and $V_p \neq 0$ then $D_p = (p-1)^2/36 + (p-1)/3 + 4$.*

Theorem 2 *There are infinitely many primes $p \equiv 1 \pmod{12}$ for which $V_p > 0$. However there is no sub-congruence of $p \equiv 1 \pmod{12}$ for which $V_p > 0$ for all p in that sub-congruence class.*

So the number of descendants of G_p of order p^{10} and exponent p is not PORC, and it easily follows that the number of descendants of order p^{10} with no restriction on the exponent is also not PORC.

We also prove in [1] that the order of the automorphism group of G_p is as follows:

1. *If $p \equiv 5 \pmod{12}$ there are $|\mathrm{GL}(2, p)| \cdot 4p^{18}$ automorphisms.*
2. *If $p \equiv 7 \pmod{12}$ there are $|\mathrm{GL}(2, p)| \cdot 2p^{18}$ automorphisms.*
3. *If $p \equiv 11 \pmod{12}$ there are $|\mathrm{GL}(2, p)| \cdot 6p^{18}$ automorphisms.*

4. If $p \equiv 1 \pmod{12}$ and $V_p = 0$ there are $|\mathrm{GL}(2, p)| \cdot 4p^{18}$ automorphisms.
5. If $p \equiv 1 \pmod{12}$ and $V_p \neq 0$ there are $|\mathrm{GL}(2, p)| \cdot 36p^{18}$ automorphisms.

So the order of the automorphism group of G_p is not PORC. To see why the order of the automorphism group of G_p impacts on the number of descendants of G_p we need to briefly recall the p -group generation algorithm [5]. Let P be a p -group. The p -group generation algorithm uses the lower p -central series, defined recursively by $\mathcal{P}_1(P) = P$ and $\mathcal{P}_{i+1}(P) = [\mathcal{P}_i(P), P]\mathcal{P}_i(P)^p$ for $i \geq 1$. The p -class of P is the length of this series. Each p -group P , apart from the elementary abelian ones, is an *immediate descendant* of the quotient P/R where R is the last non-trivial term of the lower p -central series of P . Thus all the groups with order p^n , except the elementary abelian one, are immediate descendants of groups with order p^k for $k < n$. All of the immediate descendants of P are quotients of a certain extension of P (the p -covering group); the isomorphism problem for these descendants is equivalent to the problem of determining orbits of certain subgroups of this extension under an action of the automorphism group of P .

The group G_p also has the property that the function enumerating the number of its conjugacy classes is not PORC. I showed in [7] that the number of conjugacy classes of G_p is

$$p^6 + p^3 - 1 + (p^3 - p^2 - p + 1) \times E,$$

where E is the number of points on the elliptic curve $y^2 = x^3 - x$ over $\mathrm{GF}(p)$ (including the point at infinity). I give a proof in [7] of the well known fact that E is not PORC.

It would be interesting to find more examples of finite p -groups with a non-PORC number of conjugacy classes, or a non-PORC number of automorphisms, and so I undertook a systematic search of all the class two groups of exponent p and order p^n for $n \leq 8$. My search found nothing of interest!

Theorem 3 *For every prime $p > 2$ there are 70 class two groups of exponent p and order p^n with $n \leq 8$. The number of conjugacy classes of each of these groups is polynomial in p , and the number of automorphisms of each of these groups is polynomial in p .*

2 The groups and their automorphisms

For every prime $p > 2$ there is one class two group of exponent p of order p^3 , one of order p^4 , three of order p^5 , seven of order p^6 , fifteen of order p^7 , and forty-three of order p^8 . We give presentations for all these groups below. For example the single class two group of exponent p and order p^3 has presentation

$$\langle a, b \mid \text{class 2, exponent } p \rangle.$$

(So it is the free group of rank 2 in the variety of class two groups of exponent p .) The single class two group of exponent p and order p^4 has presentation

$$\langle a, b, c \mid [c, a], [c, b], \text{ class 2, exponent } p \rangle.$$

Note that the prime p in these presentations is a parameter, so that the two presentations define families of groups of order p^3 and p^4 — one group in each family for each prime value of $p > 2$. However it is easier to think of the presentations as defining a single group with the prime p undetermined. The first of these groups has $p^2 + p - 1$ conjugacy classes, and the second of the two groups has $p^3 + p^2 - p$ conjugacy classes.

Many of the presentations involve a second parameter ω , which is assumed throughout to be an integer which is primitive modulo p . For example one of the class two groups of exponent p and order p^6 has presentation

$$\langle a, b, c, d \mid [c, b], [d, a], [d, b] = [c, a], [d, c] = [b, a]^\omega \rangle.$$

(For the remainder of this note we will omit the words “class 2, exponent p ” from the presentations, taking them as understood.) In this presentation we can take ω to be any integer which is *not* a quadratic residue modulo p . The isomorphism type of the group does not depend on the particular choice of non-quadratic residue. However, for consistency in all the different presentations it is convenient to assume throughout that ω is always a primitive element modulo p .

Finally, two of the presentations involve a parameter m which takes a value such that $x^3 + mx - 1$ is irreducible over $\text{GF}(p)$ in one of the presentations and takes a value such that $x^3 - mx + 1$ is irreducible in the other presentation. The isomorphism types of the two groups do not depend on the choice of m .

We give presentations for each of the 70 class two groups of exponent p and order p^n with $n \leq 8$, and for each of these groups we give the polynomials giving the number of conjugacy classes and the order of the automorphism group. We also give a description of the automorphism group which gives the reader enough information to “write down” a set of generators for the automorphism group. If G is a class two group of exponent p , and if A is the automorphism group of G then A induces a group of automorphisms on G/G' . If G/G' has rank k then the full automorphism group of G/G' is $\text{GL}(k, p)$, and so the automorphisms induced by A on G/G' form a subgroup, B say, of $\text{GL}(k, p)$. Furthermore this subgroup B of $\text{GL}(k, p)$ completely determines A . To see this, suppose that the defining generators of G are a_1, a_2, \dots, a_k and let $\alpha \in A$. If we pick arbitrary $g_1, g_2, \dots, g_k \in G'$ then there is an automorphism mapping a_i to $(a_i \alpha) g_i$ for $i = 1, 2, \dots, k$. So B completely determines A , and $|A| = |B| \cdot |G'|^k$.

We give three examples of the induced automorphism groups B . For our first example we consider the group

$$\langle a, b, c, d \mid [c, a], [c, b], [d, a], [d, b], [d, c] \rangle$$

of order p^5 . The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

This is intended to show that the matrices in B must have zeros in the four positions shown, but that their other entries are arbitrary subject to the restriction that the matrix must have non-zero determinant. If we write the matrix in block form

$$\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$$

where each block is a 2×2 matrix then it is clear that X and Z must lie in $\text{GL}(2, p)$, whereas Y is arbitrary. So the group of all these matrices has order

$$(p^2 - 1)^2(p^2 - p)^2p^4.$$

Perhaps the simplest way to generate the group is to take generators with random entries in all the $*$ positions, checking that the determinant is non-zero, and continue adding generators until they generate a subgroup of $\text{GL}(4, p)$ of the right order. Alternatively, take generators $\begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ for $\text{GL}(2, p)$ (where ω is a primitive element in $\text{GF}(p)$). Then the matrix group B is generated by the following matrices:

$$\begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As a second example, consider the group

$$\langle a, b, c, d \mid [c, b], [d, a], [d, b] = [c, a], [d, c] = [b, a]^\omega \rangle$$

of order p^6 . The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \varepsilon & \zeta & \eta & \theta \\ -\omega\theta & \omega\eta & \zeta & -\varepsilon \\ \omega\delta & -\omega\gamma & -\beta & \alpha \end{pmatrix},$$

and

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \varepsilon & \zeta & \eta & \theta \\ \omega\theta & -\omega\eta & -\zeta & \varepsilon \\ -\omega\delta & \omega\gamma & \beta & -\alpha \end{pmatrix}$$

where $(\alpha, \beta, \gamma, \delta)$ can be any 4-vector other than zero ($p^4 - 1$ possibilities), and where $(\varepsilon, \zeta, \eta, \theta)$ can be any 4-vector which is *not* in the linear span of $(\alpha, \beta, \gamma, \delta)$ and

$(\omega\delta, -\omega\gamma, -\beta, \alpha)$ ($p^4 - p^2$ possibilities). Again, one way of generating this group of matrices is to throw in random non-singular matrices of the form shown as generators until the required order $2(p^4 - 1)(p^4 - p^2)$ is reached. Alternatively, take the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

as the first generator, and then throw in random non-singular matrices of the first form as generators until the required order is reached. A second alternative is to first find generators for the subgroup of matrices of the first kind with first row $(1, 0, 0, 0)$. It is easy to see that this subgroup is generated by matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -\omega & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta & \eta & 0 \\ 0 & \omega\eta & \zeta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with ζ, η not both zero. Furthermore, the matrices $\begin{pmatrix} \zeta & \eta \\ \omega\eta & \zeta \end{pmatrix}$ with ζ, η not both zero form a group of order $p^2 - 1$ which is isomorphic to the multiplicative group of $\text{GF}(p^2)$. So this group is cyclic, and it is easy to find a single element which generates the group. So we can find three matrices which generate the group of matrices of the first type with first row $(1, 0, 0, 0)$, and the full matrix group is then generated by these three matrices together with

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and with matrices of the first type with general first row $(\alpha, \beta, \gamma, \delta)$, and second row $(0, 1, 0, 0)$ if $\alpha \neq 0$ or $\delta \neq 0$, or second row $(1, 0, 0, 0)$ if $\alpha = \delta = 0$. Experimentally it seems that we only need one of these matrices — the one with first row $(0, 1, 0, 0)$ and second row $(0, 1, 0, 0)$.

As a third example we consider the group

$$\langle a, b, c, d \mid [c, b], [d, a], [d, b] = [c, a], [d, c] \rangle$$

of order p^6 . The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} \alpha & \beta & * & * \\ \gamma & \delta & * & * \\ 0 & 0 & \lambda\delta & -\lambda\gamma \\ 0 & 0 & -\lambda\beta & \lambda\alpha \end{pmatrix}$$

with $\lambda(\alpha\delta - \beta\gamma) \neq 0$. Here $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ takes arbitrary values in $\text{GL}(2, p)$, λ takes any non-zero value, and the entries in the four positions denoted $*$ take arbitrary values. This group of matrices has order $(p-1)(p^2-1)(p^2-p)p^4$, and it is easy to see that if we let ω be a primitive element in $\text{GF}(p)$ then it is generated by the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The complete list of 70 class two groups of exponent p and order p^n with $n \leq 8$ follows below. For each group we give the number of conjugacy classes, the order of the automorphism group, and a description of the action of the automorphism group on G/G' . The information shows that in some sense the automorphism groups are independent of p , though of course the entries in the matrices must lie in $\text{GF}(p)$. Also, to find sets of generators for the matrix groups we need to make some choice of primitive elements in $\text{GF}(p)$ and $\text{GF}(p^2)$. There is one case, group 8.5.9, where we need different generators for the matrix group when $p = 3$, but in all other cases the choice of generators is independent of p , except in the sense just described above.

The proofs of the results below are all traditional “hand proofs”, albeit with machine assistance with linear algebra over rational function fields of characteristic zero. But all the results have been checked with a computer for small primes.

3 Order p^3

Group 3.2.1

$$\langle a, b \rangle$$

The number of conjugacy classes is $p^2 + p - 1$, and the automorphism group has order $(p^2 - 1)(p^2 - p)p^2$.

The action of the automorphism group on G/G' is given by $\text{GL}(2, p)$.

4 Order p^4

Group 4.3.1

$$\langle a, b, c \mid [c, a], [c, b] \rangle$$

The number of conjugacy classes is $p^3 + p^2 - p$, and the automorphism group has order $(p-1)(p^2-1)(p^2-p)p^5$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(3, p)$ of the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}.$$

5 Order p^5

5.3 Three generator groups

Group 5.3.1

$$\langle a, b, c \mid [c, b] \rangle$$

The number of conjugacy classes is $2p^3 - p$, and the automorphism group has order $(p-1)(p^2-1)(p^2-p)p^8$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(3, p)$ of the form

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

5.4 Four generator groups

Group 5.4.1

$$\langle a, b, c, d \mid [c, a], [c, b], [d, a], [d, b], [d, c] \rangle$$

The number of conjugacy classes is $p^4 + p^3 - p^2$, and the automorphism group has order $(p^2-1)^2(p^2-p)^2p^8$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

Group 5.4.2

$$\langle a, b \rangle \times_{[b,a]=[d,c]} \langle c, d \rangle$$

The number of conjugacy classes is $p^4 + p - 1$, and the automorphism group has order $(p^5 - p)(p^5 - p^4)(p^3 - p)p^2$.

The centre of the group has order p . The image of a can be anything outside the centre ($p^5 - p$ choices). The image of b can be anything outside the centralizer of the

image of a ($p^5 - p^4$ choices). The image of c must centralize the images of a and b and lie outside the centre ($p^3 - p$ choices). The image of d must centralize then images of a and b , but not the image of c , and must be scaled so that the image of $[d, c]$ equals the image of $[b, a]$ (p^2 choices).

6 Order p^6

6.3 Three generator groups

Group 6.3.1

$$\langle a, b, c \rangle$$

The number of conjugacy classes is $p^4 + p^3 - p$, and the automorphism group has order $(p^3 - 1)(p^3 - p)(p^3 - p^2)p^9$.

The action of the automorphism group on G/G' is given by $\text{GL}(3, p)$.

6.4 Four generator groups

Group 6.4.1

$$\langle a, b, c, d \mid [c, b], [d, a], [d, b], [d, c] \rangle$$

The number of conjugacy classes is $2p^4 - p^2$ and the order of the automorphism group is $(p - 1)^2(p^2 - 1)(p^2 - p)p^{13}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

Group 6.4.2

$$\langle a, b, c, d \mid [c, b], [d, a], [d, b] = [b, a], [d, c] \rangle$$

The number of conjugacy classes is $p^4 + 2p^3 - p^2 - 2p + 1$ and the order of the automorphism group is $2(p^2 - 1)^2(p^2 - p)^2p^8$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} -\alpha & \beta & -\gamma & -\alpha + \delta \\ \varepsilon & 0 & \zeta & \varepsilon \\ 0 & \eta & 0 & \theta \\ \alpha & 0 & \gamma & \alpha \end{pmatrix}$$

with $(\alpha\zeta - \gamma\varepsilon)(\beta\theta - \delta\eta) \neq 0$ and

$$\begin{pmatrix} \alpha & -\beta & \gamma & \alpha - \delta \\ 0 & \varepsilon & 0 & \zeta \\ \eta & 0 & \theta & \eta \\ 0 & \beta & 0 & \delta \end{pmatrix}$$

with $(\alpha\theta - \gamma\eta)(\beta\zeta - \delta\varepsilon) \neq 0$.

Group 6.4.3

$$\langle a, b, c, d \mid [c, b], [d, a], [d, b] = [c, a], [d, c] \rangle$$

The number of conjugacy classes is $p^4 + p^3 - p$ and the order of the automorphism group is $(p - 1)(p^2 - 1)(p^2 - p)p^{12}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} \alpha & \beta & * & * \\ \gamma & \delta & * & * \\ 0 & 0 & \lambda\delta & -\lambda\gamma \\ 0 & 0 & -\lambda\beta & \lambda\alpha \end{pmatrix}$$

with $\lambda(\alpha\delta - \beta\gamma) \neq 0$.

Group 6.4.4

$$\langle a, b, c, d \mid [c, b], [d, a], [d, b] = [c, a], [d, c] = [b, a]^\omega \rangle$$

The number of conjugacy classes is $p^4 + p^2 - 1$ and the order of the automorphism group is $2(p^4 - 1)(p^4 - p^2)p^8$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \varepsilon & \zeta & \eta & \theta \\ \omega\theta & -\omega\eta & -\zeta & \varepsilon \\ -\omega\delta & \omega\gamma & \beta & -\alpha \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \varepsilon & \zeta & \eta & \theta \\ -\omega\theta & \omega\eta & \zeta & -\varepsilon \\ \omega\delta & -\omega\gamma & -\beta & \alpha \end{pmatrix},$$

where $(\alpha, \beta, \gamma, \delta)$ can be any 4-vector other than zero ($p^4 - 1$ possibilities), and where $(\varepsilon, \zeta, \eta, \theta)$ can be any 4-vector which is *not* in the linear span of $(\alpha, \beta, \gamma, \delta)$ and $(\omega\delta, -\omega\gamma, -\beta, \alpha)$ ($p^4 - p^2$ possibilities).

6.5 Five generator groups

Group 6.5.1

$$\langle a, b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$$

The number of conjugacy classes is $p^5 + p^4 - p^3$ and the order of the automorphism group is $(p^2 - 1)(p^2 - p)(p^3 - 1)(p^3 - p)(p^3 - p^2)p^{11}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}.$$

Group 6.5.2

$$\langle a, b \rangle \times_{[b,a]=[d,c]} \langle c, d \rangle \times \langle e \rangle$$

The number of conjugacy classes is $p^5 + p^2 - p$ and the order of the automorphism group is $(p^6 - p^2)(p^6 - p^5)(p^4 - p^2)p^3(p^2 - p)$.

The centre of the group has order p^2 . The image of a can be anything not in the centre ($p^6 - p^2$ choices). The image of b can be anything which does not centralize the image of a ($p^6 - p^5$ choices). The image of c must centralize the images of a and b but lie outside the centre ($p^4 - p^2$ choices). The image of d must centralize the images of a and b , but not the image of c , and must be scaled so that the image of $[d, c]$ equals the image of $[b, a]$ (p^3 choices). The image of e must lie in the centre, but not in the derived group ($p^2 - p$ choices).

7 Order p^7

7.4 Four generator groups

Group 7.4.1

$$\langle a, b, c, d \mid [c, b], [d, b], [d, c] \rangle$$

The number of conjugacy classes is $p^5 + p^4 - p^2$, and the automorphism group has order $(p - 1)(p^3 - 1)(p^3 - p)(p^3 - p^2)p^{15}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}.$$

Group 7.4.2

$$\langle a, b, c, d \mid [d, a], [d, b], [d, c] \rangle$$

The number of conjugacy classes is $p^5 + p^4 - p^2$, and the automorphism group has order $(p-1)(p^3-1)(p^3-p)(p^3-p^2)p^{15}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

Group 7.4.3

$$\langle a, b, c, d \mid [d, a], [c, b], [d, c] \rangle$$

The number of conjugacy classes is $3p^4 - 3p^2 + p$, and the automorphism group has order $2(p-1)^4 p^{16}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} * & 0 & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \text{ and } \begin{pmatrix} 0 & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{pmatrix}.$$

Group 7.4.4

$$\langle a, b, c, d \mid [d, a] = [c, b], [d, b], [d, c] \rangle$$

The number of conjugacy classes is $2p^4 + p^3 - 2p^2$, and the automorphism group has order $(p-1)(p^2-1)(p^2-p)p^{17}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} \lambda & * & * & * \\ 0 & \alpha & \beta & * \\ 0 & \gamma & \delta & * \\ 0 & 0 & 0 & \lambda^{-1}(\alpha\delta - \beta\gamma) \end{pmatrix},$$

with $\lambda(\alpha\delta - \beta\gamma) \neq 0$.

Group 7.4.5

$$\langle a, b, c, d \mid [c, a], [d, a] = [c, b], [d, b] \rangle$$

The number of conjugacy classes is $2p^4 + p^3 - 2p^2$, and the automorphism group has order $(p^2-1)^2(p^2-p)p^{13}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} \lambda\alpha & \lambda\beta & \mu\alpha & -\mu\beta \\ -\lambda\gamma & -\lambda\delta & -\mu\gamma & \mu\delta \\ \nu\alpha & \nu\beta & \xi\alpha & -\xi\beta \\ \nu\gamma & \nu\delta & \xi\gamma & -\xi\delta \end{pmatrix},$$

with $(\alpha\delta - \beta\gamma)(\lambda\xi - \mu\nu) \neq 0$

Group 7.4.6

$$\langle a, b, c, d \mid [d, b] = [c, a]^\omega, [d, c] = [b, a], [c, b] \rangle$$

The number of conjugacy classes is $p^4 + 2p^3 - p^2 - p$, and the automorphism group has order $2(p^2 - 1)^2 p^{16}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} \alpha & * & * & \beta \\ 0 & \gamma & \omega\delta & 0 \\ 0 & \delta & \gamma & 0 \\ \omega\beta & * & * & \alpha \end{pmatrix} \text{ and } \begin{pmatrix} \alpha & * & * & \beta \\ 0 & \gamma & \omega\delta & 0 \\ 0 & -\delta & -\gamma & 0 \\ -\omega\beta & * & * & -\alpha \end{pmatrix},$$

with α and β not both zero, and with γ and δ not both zero.

7.5 Five generator groups

Group 7.5.1

$$\langle a, b, c, d, e \mid [c, b], [d, a], [d, b], [d, c], [e, a], [e, b], [e, c], [e, d] \rangle$$

The number of conjugacy classes is $2p^5 - p^3$ and the order of the automorphism group is $(p - 1)(p^2 - 1)^2(p^2 - p)^2 p^{18}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix}.$$

Group 7.5.2

$$\langle a, b, c, d, e \mid [c, b], [d, a], [d, b] = [b, a], [d, c], [e, a], [e, b], [e, c], [e, d] \rangle$$

The number of conjugacy classes is $p^5 + 2p^4 - p^3 - 2p^2 + p$ and the order of the automorphism group is $2(p - 1)(p^2 - 1)^2(p^2 - p)^2 p^{14}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} -\alpha & \beta & -\gamma & -\alpha + \delta & * \\ \varepsilon & 0 & \zeta & \varepsilon & * \\ 0 & \eta & 0 & \theta & * \\ \alpha & 0 & \gamma & \alpha & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}$$

with $(\alpha\zeta - \gamma\varepsilon)(\beta\theta - \delta\eta) \neq 0$ and

$$\begin{pmatrix} \alpha & -\beta & \gamma & \alpha - \delta & * \\ 0 & \varepsilon & 0 & \zeta & * \\ \eta & 0 & \theta & \eta & * \\ 0 & \beta & 0 & \delta & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}$$

with $(\alpha\theta - \gamma\eta)(\beta\zeta - \delta\varepsilon) \neq 0$.

Group 7.5.3

$$\langle a, b, c, d, e \mid [c, b], [d, a], [d, b] = [c, a], [d, c], [e, a], [e, b], [e, c], [e, d] \rangle$$

The number of conjugacy classes is $p^5 + p^4 - p^2$ and the order of the automorphism group is $(p-1)^2(p^2-1)(p^2-p)p^{18}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} \alpha & \beta & * & * & * \\ \gamma & \delta & * & * & * \\ 0 & 0 & \lambda\delta & -\lambda\gamma & * \\ 0 & 0 & -\lambda\beta & \lambda\alpha & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

with $\lambda(\alpha\delta - \beta\gamma) \neq 0$.

Group 7.5.4

$$\langle a, b, c, d, e \mid [c, b], [d, a], [d, b] = [c, a], [d, c] = [b, a]^\omega, [e, a], [e, b], [e, c], [e, d] \rangle$$

The number of conjugacy classes is $p^5 + p^3 - p$ and the order of the automorphism group is $2(p-1)(p^4-1)(p^4-p^2)p^{14}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & * \\ \varepsilon & \zeta & \eta & \theta & * \\ \omega\theta & -\omega\eta & -\zeta & \varepsilon & * \\ -\omega\delta & \omega\gamma & \beta & -\alpha & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & * \\ \varepsilon & \zeta & \eta & \theta & * \\ -\omega\theta & \omega\eta & \zeta & -\varepsilon & * \\ \omega\delta & -\omega\gamma & -\beta & \alpha & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix},$$

where $(\alpha, \beta, \gamma, \delta)$ can be any 4-vector other than zero ($p^4 - 1$ possibilities), and where $(\varepsilon, \zeta, \eta, \theta)$ can be any 4-vector which is *not* in the linear span of $(\alpha, \beta, \gamma, \delta)$ and $(\omega\delta, -\omega\gamma, -\beta, \alpha)$ ($p^4 - p^2$ possibilities).

Group 7.5.5

$$\langle a, b, c, d, e \mid [c, b], [d, a], [d, b], [d, c], [e, a], [e, b], [e, c], [e, d] = [b, a] \rangle$$

The number of conjugacy classes is $p^5 + p^4 - p^2$ and the order of the automorphism group is $(p - 1)^2(p^2 - 1)(p^2 - p)p^{15}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon \\ 0 & \alpha^{-1}(\lambda\xi - \mu\nu) & 0 & 0 & 0 \\ 0 & \zeta & \eta & 0 & 0 \\ 0 & \alpha^{-1}(-\delta\mu + \varepsilon\lambda) & 0 & \lambda & \mu \\ 0 & \alpha^{-1}(-\delta\xi + \varepsilon\nu) & 0 & \nu & \xi \end{pmatrix}$$

with $\alpha, \eta, \lambda\xi - \mu\nu \neq 0$.

Group 7.5.6

$$\langle a, b, c, d, e \mid [c, b], [d, a], [d, b] = [c, a], [d, c], [e, a], [e, b], [e, c], [e, d] = [b, a] \rangle$$

The number of conjugacy classes is $p^5 + p^3 - p$ and the order of the automorphism group is $(p - 1)(p^2 - 1)(p^2 - p)p^{14}$.

The action of the automorphism group on G/G' is given by a subgroup H of $\text{GL}(5, p)$, where the matrices in H have the form

$$\begin{pmatrix} \alpha & * & * & \beta & * \\ 0 & * & * & 0 & * \\ 0 & * & * & 0 & * \\ \gamma & * & * & \delta & * \\ 0 & * & * & 0 & * \end{pmatrix}.$$

As we run through the elements of H , $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ takes all values in $\text{GL}(2, p)$. If we take generators $\begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ for $\text{GL}(2, p)$ then we obtain the following

generating matrices for H :

$$\begin{pmatrix} \omega & \alpha & \beta & 0 & \gamma \\ 0 & \omega\lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & \omega^{-1}\gamma & -\omega^{-1}\alpha & 1 & \delta \\ 0 & 0 & 0 & 0 & \omega^2\lambda \end{pmatrix} \quad (\lambda \neq 0)$$

and

$$\begin{pmatrix} -1 & \alpha & \beta & 1 & \gamma \\ 0 & \lambda & 0 & 0 & 2\lambda \\ 0 & 0 & 0 & 0 & \lambda \\ -1 & \delta & \beta - \delta & 0 & \delta - \alpha \\ 0 & -\lambda & \lambda & 0 & -\lambda \end{pmatrix} \quad (\lambda \neq 0).$$

7.6 Six generator groups

Group 7.6.1

$$\langle a, b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle \times \langle f \rangle$$

The number of conjugacy classes is $p^6 + p^5 - p^4$, and the automorphism group has order $(p^7 - p^5)(p^7 - p^6)(p^5 - p)(p^5 - p^2)(p^5 - p^3)(p^5 - p^4)$.

Group 7.6.2

$$\langle a, b \rangle \times_{[b,a]=[d,c]} \langle c, d \rangle \times \langle e \rangle \times \langle f \rangle$$

The number of conjugacy classes is $p^6 + p^3 - p^2$, and the automorphism group has order $(p^7 - p^3)(p^7 - p^6)(p^5 - p^3)p^4(p^3 - p)(p^3 - p^2)$.

Group 7.6.3

$$\langle a, b \rangle \times_{[b,a]=[d,c]=[f,e]} \langle c, d \rangle \times_{[b,a]=[d,c]=[f,e]} \langle e, f \rangle$$

The number of conjugacy classes is $p^6 + p - 1$, and the automorphism group has order $(p^7 - p)(p^7 - p^6)(p^5 - p)p^4(p^3 - p)p^2$.

8 Order p^8

8.4 Four generator groups

Group 8.4.1

$$\langle a, b, c, d \mid [b, a], [c, a] \rangle$$

The number of conjugacy classes is $2p^5 + p^4 - 2p^3$, and the automorphism group has order $(p - 1)^2(p^2 - 1)(p^2 - p)p^{21}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} * & 0 & 0 & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix}.$$

Group 8.4.2

$$\langle a, b, c, d \mid [b, a], [d, c] \rangle$$

The number of conjugacy classes is $p^5 + 3p^4 - 2p^3 - 2p^2 + p$, and the automorphism group has order $2(p^2 - 1)^2(p^2 - p)^2p^{16}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}.$$

Group 8.4.3

$$\langle a, b, c, d \mid [b, a], [d, b][c, a] \rangle$$

The number of conjugacy classes is $p^5 + 2p^4 - p^3 - p^2$, and the automorphism group has order $(p - 1)(p^2 - 1)(p^2 - p)p^{20}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ * & * & -\lambda\delta & \lambda\gamma \\ * & * & \lambda\beta & -\lambda\alpha \end{pmatrix},$$

with $\lambda(\alpha\delta - \beta\gamma) \neq 0$.

Group 8.4.4

$$\langle a, b, c, d \mid [d, b][c, a], [d, c][b, a]^\omega \rangle$$

The number of conjugacy classes is $p^5 + p^4 - p$, and the automorphism group has order $2(p^4 - 1)(p^4 - p^2)p^{16}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(4, p)$ of the form

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \varepsilon & \zeta & \eta & \theta \\ -\omega\theta & \omega\eta & \zeta & -\varepsilon \\ \omega\delta & -\omega\gamma & -\beta & \alpha \end{pmatrix} \text{ and } \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \varepsilon & \zeta & \eta & \theta \\ \omega\theta & -\omega\eta & -\zeta & \varepsilon \\ -\omega\delta & \omega\gamma & \beta & -\alpha \end{pmatrix}.$$

In these two matrices the first row can be anything non-zero. The first row determines the fourth row up to sign, and the second row can be anything not in the span of rows one and four. The subgroup of matrices with first row $(1, 0, 0, 0)$ is generated by matrices with second row $(\varepsilon, \zeta, \eta, \theta)$ with one (or both) of ζ, η non-zero. The full matrix group is then generated by this subgroup and matrices with a general first row $(\alpha, \beta, \gamma, \delta)$, and second row $(0, 1, 0, 0)$ if $\alpha \neq 0$ or $\delta \neq 0$, or second row $(1, 0, 0, 0)$ if $\alpha = \delta = 0$.

8.5 Five generator groups

Group 8.5.1

$$\langle a, b, c, d, e \mid [e, a], [c, b], [d, b], [e, b], [d, c], [e, c], [e, d] \rangle$$

The number of conjugacy classes is $p^6 + p^5 - p^3$, and the automorphism group has order $(p-1)^2(p^3-1)(p^3-p)(p^3-p^2)p^{22}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

Group 8.5.2

$$\langle a, b, c, d, e \mid [d, a], [e, a], [d, b], [e, b], [d, c], [e, c], [e, d] \rangle$$

The number of conjugacy classes is $p^6 + p^5 - p^3$, and the automorphism group has order $(p^2-1)(p^2-p)(p^3-1)(p^3-p)(p^3-p^2)p^{21}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix}.$$

Group 8.5.3

$$\langle a, b, c, d, e \mid [d, a], [e, a], [c, b], [e, b], [d, c], [e, c], [e, d] \rangle$$

The number of conjugacy classes is $3p^5 - 3p^3 + p^2$, and the automorphism group has order $2(p-1)^5p^{23}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} * & 0 & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix} \text{ and } \begin{pmatrix} 0 & * & * & * & * \\ * & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

Group 8.5.4

$$\langle a, b, c, d, e \mid [d, a] = [c, b], [e, a], [d, b], [e, b], [d, c], [e, c], [e, d] \rangle$$

The number of conjugacy classes is $2p^5 + p^4 - 2p^3$, and the automorphism group has order $(p-1)^2(p^2-1)(p^2-p)p^{24}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} \lambda & * & * & * & * \\ 0 & \alpha & \beta & * & * \\ 0 & \gamma & \delta & * & * \\ 0 & 0 & 0 & \lambda^{-1}(\alpha\delta - \beta\gamma) & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix},$$

with $\lambda(\alpha\delta - \beta\gamma) \neq 0$.

Group 8.5.5

$$\langle a, b, c, d, e \mid [c, a], [d, a] = [c, b], [e, a], [d, b], [e, b], [e, c], [e, d] \rangle$$

The number of conjugacy classes is $2p^5 + p^4 - 2p^3$, and the automorphism group has order $(p^2-1)^2(p^2-p)^2p^{19}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} \lambda\alpha & \lambda\beta & \mu\alpha & -\mu\beta & * \\ -\lambda\gamma & -\lambda\delta & -\mu\gamma & \mu\delta & * \\ \nu\alpha & \nu\beta & \xi\alpha & -\xi\beta & * \\ \nu\gamma & \nu\delta & \xi\gamma & -\xi\delta & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix},$$

with $(\alpha\delta - \beta\gamma)(\lambda\xi - \mu\nu) \neq 0$

Group 8.5.6

$$\langle a, b, c, d, e \mid [d, b] = [c, a]^\omega, [d, c] = [b, a], [e, a], [c, b], [e, b], [e, c], [e, d] \rangle$$

The number of conjugacy classes is $p^5 + 2p^4 - p^3 - p^2$, and the automorphism group has order $2(p-1)(p^2-1)^2p^{23}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} \alpha & * & * & \beta & * \\ 0 & \gamma & \omega\delta & 0 & * \\ 0 & \delta & \gamma & 0 & * \\ \omega\beta & * & * & \alpha & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix} \text{ and } \begin{pmatrix} \alpha & * & * & \beta & * \\ 0 & \gamma & \omega\delta & 0 & * \\ 0 & -\delta & -\gamma & 0 & * \\ -\omega\beta & * & * & -\alpha & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix},$$

with α and β not both zero, and with γ and δ not both zero.

Group 8.5.7

$$\langle a, b, c, d, e \mid [e, b] = [c, a][d, b]^m, [d, c] = [b, a], [e, c] = [d, b], [d, a], [e, a], [c, b], [e, d] \rangle,$$

where m is chosen so that $x^3 + mx - 1$ is irreducible over $\text{GF}(p)$. Different choices of m give isomorphic groups. Note that the discriminant of $x^3 + mx - 1$ is $-4m^3 - 27$ and this must be a square in $\text{GF}(p)$ — we let $u^2 = -4m^3 - 27$.

The number of conjugacy classes is $p^5 + p^4 - p$, and the automorphism group has order $3(p-1)(p^3-1)p^{18}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} * & 0 & 0 & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & 0 & 0 & * & * \\ * & 0 & 0 & * & * \end{pmatrix}$$

where the first row can be anything non-zero of the form shown ((p^3-1) possibilities). If we restrict ourselves to matrices with first row $(1, 0, 0, 0, 0)$ then we get a subgroup of $\text{GL}(5, p)$ of order $3(p-1)p^3$ generated by matrices of the following form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \alpha & \beta & 0 & \gamma & \delta \\ -\gamma & 0 & \beta & m\alpha + \delta & -\alpha \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\beta \neq 0$, and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{u-9}{2m^2} & \frac{3}{m} & 0 & 0 \\ 0 & 1 & \frac{u+9}{2m^2} & 0 & 0 \\ -\frac{(u+3)m}{2u} & 0 & 0 & \frac{u^2-2u+9}{4u} & \frac{m^2}{u} \\ -\frac{(u+1)m^2}{2u} & 0 & 0 & \frac{(u^2+3)m}{4u} & -\frac{u^2+2u+9}{4u} \end{pmatrix}.$$

(The cube of the second matrix has the form of the first, with $\alpha = \gamma = \delta = 0$, and $\beta = 4u^3m^{-3}$.) The full subgroup of $\text{GL}(5, p)$ giving the action of the automorphism

group on G/G' is generated by the matrices above together with $p^3 - 1$ matrices of the form

$$\begin{pmatrix} \alpha & 0 & 0 & \beta & \gamma \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \gamma & 0 & 0 & \alpha & -\beta - m\gamma \\ \beta + m\gamma & 0 & 0 & -\gamma & \alpha - m\beta - m^2\gamma \end{pmatrix},$$

where α, β, γ are not all zero.

Group 8.5.8

$$\langle a, b, c, d, e \mid [d, a][c, b]^\omega, [e, a], [e, b] = [c, a], [d, b], [d, c] = [b, a], [e, c], [e, d] \rangle$$

The number of conjugacy classes is $p^5 + p^4 - p$, and the automorphism group has order $2(p-1)(p^2-1)p^{18}$.

The action of the automorphism group on G/G' is given by a group H of matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} * & 0 & * & 0 & * \\ * & * & * & * & * \\ * & 0 & * & 0 & * \\ * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

The group H has a subgroup of order $p^3(p-1)$ consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2\mu\xi \\ \lambda & \xi^{-1} & \mu & 0 & \nu \\ 0 & 0 & 1 & 0 & -2\lambda\xi \\ \mu & 0 & \omega\lambda & \xi^{-1} & -\omega\lambda^2\xi + \mu^2\xi \\ 0 & 0 & 0 & 0 & \xi \end{pmatrix}$$

with $\xi \neq 0$.

If we premultiply a general matrix

$$\begin{pmatrix} * & 0 & * & 0 & * \\ * & * & * & * & * \\ * & 0 & * & 0 & * \\ * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}$$

in H by a suitable matrix from this subgroup of order $p^3(p-1)$ we can obtain a matrix of the form

$$\begin{pmatrix} * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 \\ * & 0 & * & 0 & * \\ * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix},$$

and the most general matrices of this form arising in the action of the automorphism group on G/G' are

$$\xi \begin{pmatrix} \alpha & 0 & \omega\beta & 0 & 0 \\ 0 & \alpha^2 - \omega\beta^2 & 0 & 0 & 0 \\ \beta & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \alpha^2 - \omega\beta^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\xi \begin{pmatrix} \alpha & 0 & \omega\beta & 0 & 0 \\ 0 & -\alpha^2 + \omega\beta^2 & 0 & 0 & 0 \\ -\beta & 0 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & \alpha^2 - \omega\beta^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

There are $2(p-1)(p^2-1)$ of these matrices, and so the order of the subgroup of $\text{GL}(5, p)$ generated by all these matrices is $2p^3(p-1)(p^2-1)$.

Group 8.5.9

$$\langle a, b, c, d, e \mid [d, a], [e, a], [e, b] = [c, a], [d, b], [d, c] = [b, a], [e, c], [e, d] = [c, b] \rangle$$

The number of conjugacy classes is $p^5 + p^4 - p$, and the automorphism group has order $(p+1)(p-1)^2 p^{16}$.

The action of the automorphism group on G/G' is given by a subgroup H of $\text{GL}(5, p)$ of order $(p+1)(p-1)^2 p$. There is a subgroup K of H of order $p(p-1)^2$ consisting of matrices of the form

$$\lambda \begin{pmatrix} \alpha^2 & \frac{1}{2}\alpha\beta & 0 & \frac{1}{4}\beta^2 & 0 \\ 0 & \alpha & 0 & \beta & 0 \\ -\frac{3}{2}\alpha^2\beta & -\frac{3}{8}\alpha\beta^2 & \alpha^3 & -\frac{1}{8}\beta^3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{3}{8}\alpha^2\beta^2 & \frac{1}{16}\alpha\beta^3 & -\frac{1}{2}\alpha^3\beta & \frac{1}{64}\beta^4 & \alpha^4 \end{pmatrix}$$

with $\alpha, \lambda \neq 0$, and K consists of all the matrices in H with fourth row a scalar multiple of $(0, 0, 0, 1, 0)$. When $p = 3$ there is a matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

in H , and when $p \neq 3$, there is a matrix

$$A = \begin{pmatrix} -\frac{1}{3} & 0 & 0 & \frac{3}{2} & \frac{2}{27} \\ 0 & 1 & \frac{2}{9} & 3 & -\frac{4}{27} \\ 0 & \frac{1}{2} & \frac{1}{9} & -\frac{3}{2} & \frac{2}{27} \\ 1 & 1 & -\frac{2}{9} & \frac{3}{2} & \frac{2}{27} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{18} & \frac{3}{8} & \frac{1}{54} \end{pmatrix}$$

in H . In both cases H is generated by A and K . (There are $(p^2 - 1)$ possibilities for row 4, and all these possibilities arise as the fourth row in elements of K or AK .)

Group 8.5.10

$$\langle a, b, c, d, e \mid [d, a], [e, a], [d, b], [e, b] = [c, a], [d, c] = [b, a], [e, c], [e, d] \rangle$$

The number of conjugacy classes is $p^5 + 2p^4 - p^3 - p^2$, and the automorphism group has order $(p - 1)(p^2 - 1)(p^2 - p)p^{21}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} * & 0 & 0 & * & * \\ * & \alpha & \beta & * & * \\ * & \gamma & \delta & * & * \\ * & 0 & 0 & * & * \\ * & 0 & 0 & * & * \end{pmatrix}$$

with $\alpha\delta - \beta\gamma \neq 0$. Rows 2 and 3 are arbitrary subject to the condition that $\alpha\delta - \beta\gamma \neq 0$, and rows 2 and 3 then determine rows 1, 4 and 5 up to multiplication by a scalar. (The same scalar for each of the three rows. There is a subgroup of this group of matrices of order $(p - 1)p^6$ consisting of matrices of the form

$$\begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ * & 1 & 0 & * & * \\ * & 0 & 1 & * & * \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix} \quad (\lambda \neq 0),$$

and the full subgroup of $\text{GL}(5, p)$ giving the action of the automorphism group on G/G' is generated by these matrices, together with the matrices

$$\begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (\lambda \neq 0) \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Group 8.5.11

$$\langle a, b, c, d, e \mid [d, a], [e, a], [c, b], [e, b] = [c, a], [d, c] = [b, a], [e, c], [e, d] \rangle$$

The number of conjugacy classes is $p^5 + 2p^4 - p^3 - p^2$, and the automorphism group has order $(p-1)^3 p^{21}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\lambda \begin{pmatrix} \varepsilon\eta & 0 & 0 & 0 & \beta\varepsilon\eta + \varepsilon\zeta \\ \alpha & 1 & \beta & \gamma & \delta \\ -\gamma\varepsilon & 0 & \varepsilon & 0 & -\alpha\varepsilon - \beta\gamma\varepsilon \\ \zeta & 0 & 0 & \eta & \theta \\ 0 & 0 & 0 & 0 & \varepsilon^2\eta \end{pmatrix}$$

with $\lambda, \varepsilon, \eta \neq 0$.

Group 8.5.12

$$\langle a, b, c, d, e \mid [e, a], [c, b], [d, b], [e, c], [e, d], [d, c] = [b, a], [e, b] = [c, a] \rangle$$

The number of conjugacy classes is $p^5 + 2p^4 - p^3 - p^2$, and the automorphism group has order $(p-1)^3 p^{19}$.

The action of the automorphism group on G/G' is given by a group H of matrices in $\text{GL}(5, p)$ where H is generated by matrices of the form

$$\lambda \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & \alpha^{-1}\beta & 0 & 0 \\ 0 & 0 & 0 & \alpha^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (\alpha, \beta, \lambda \neq 0) \quad \text{and} \quad \begin{pmatrix} 1 & 0 & * & 0 & 2\gamma \\ 0 & 1 & \gamma & 0 & * \\ 0 & 0 & 1 & 0 & 0 \\ \gamma & 0 & * & 1 & \gamma^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Group 8.5.13

$$\langle a, b, c, d, e \mid [d, a], [e, a], [c, b], [e, b][d, b]^\omega = [c, a], [d, c] = [b, a], [e, c], [e, d] \rangle$$

The number of conjugacy classes is $p^5 + 2p^4 - p^3 - p^2$, and the automorphism group has order $2(p-1)^2(p^2-1)p^{18}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\lambda \begin{pmatrix} \alpha & 0 & 0 & \omega\beta & \beta \\ -\varepsilon & 1 & 0 & -\delta & \gamma \\ \delta & 0 & 1 & \omega\varepsilon & \varepsilon \\ \beta & 0 & 0 & \alpha & \zeta \\ 0 & 0 & 0 & 0 & \alpha - \omega\zeta \end{pmatrix} \quad \text{and} \quad \lambda \begin{pmatrix} \alpha & 0 & 0 & \omega\beta & \beta \\ \varepsilon & -1 & 0 & \delta & -\gamma \\ \delta & 0 & 1 & \omega\varepsilon & \varepsilon \\ -\beta & 0 & 0 & -\alpha & -\zeta \\ 0 & 0 & 0 & 0 & -\alpha + \omega\zeta \end{pmatrix},$$

where α and β are not both zero and $\zeta \neq \alpha\omega^{-1}$, $\lambda \neq 0$.

Group 8.5.14

$$\langle a, b, c, d, e \mid [c, b], [d, b], [e, c], [e, d], [d, c] = [b, a], [e, b] = [c, a], [e, a] = [d, a]^\omega \rangle$$

The number of conjugacy classes is $p^5 + 2p^4 - p^3 - p^2$, and the automorphism group has order $2(p-1)(p^2-1)p^{17}$.

The action of the automorphism group on G/G' is given by a subgroup $H = KL$ of $\text{GL}(5, p)$, where K is a group of matrices of order $2(p-1)(p^2-1)$ consisting of matrices

$$\begin{pmatrix} \lambda(\alpha^2 + \omega\beta^2) & 0 & 0 & 2\omega\lambda\alpha\beta & 2\lambda\alpha\beta \\ 0 & \alpha & \beta & 0 & 0 \\ 0 & \omega\beta & \alpha & 0 & 0 \\ \lambda\alpha\beta & 0 & 0 & \lambda\alpha^2 & \lambda\beta^2 \\ \omega\lambda\alpha\beta & 0 & 0 & \omega^2\lambda\beta^2 & \lambda\alpha^2 \end{pmatrix}$$

and

$$\begin{pmatrix} \lambda(\alpha^2 + \omega\beta^2) & 0 & 0 & 2\omega\lambda\alpha\beta & 2\lambda\alpha\beta \\ 0 & \alpha & \beta & 0 & 0 \\ 0 & -\omega\beta & -\alpha & 0 & 0 \\ -\lambda\alpha\beta & 0 & 0 & -\lambda\alpha^2 & -\lambda\beta^2 \\ -\omega\lambda\alpha\beta & 0 & 0 & -\omega^2\lambda\beta^2 & -\lambda\alpha^2 \end{pmatrix},$$

with $\lambda \neq 0$ and α, β not both zero, and where L is a normal subgroup of H of order p^2 consisting of matrices

$$\begin{pmatrix} 1 & \omega\alpha & \beta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 \\ 0 & \omega\beta & 0 & 0 & 1 \end{pmatrix}.$$

Group 8.5.15

$$\langle a, b, c, d, e \mid [d, a], [e, a], [c, b], [d, b], [e, c], [d, c] = [b, a], [e, b] = [c, a] \rangle$$

The number of conjugacy classes is $p^5 + 2p^4 - p^3 - p^2$, and the automorphism group has order $2(p-1)^3p^{17}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\lambda \begin{pmatrix} 1 & \alpha & \beta & 0 & 0 \\ 0 & \gamma\delta^{-1} & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & -\beta\delta^{-1} & 0 & \delta^{-1} & 0 \\ 0 & 0 & -\alpha\delta & 0 & \delta \end{pmatrix} \text{ and } \lambda \begin{pmatrix} 1 & \alpha & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & \gamma\delta^{-1} & 0 & 0 & 0 \\ 0 & 0 & -\beta\delta & 0 & \delta \\ 0 & -\beta\delta^{-1} & 0 & \delta^{-1} & 0 \end{pmatrix},$$

with $\gamma, \delta, \lambda \neq 0$.

Group 8.5.16

$$\langle a, b, c, d, e \mid [d, a], [e, a], [c, b], [d, b], [e, b] = [c, a], [e, c], [e, d] \rangle$$

The number of conjugacy classes is $p^5 + 3p^4 - 2p^3 - 2p^2 + p$, and the automorphism group has order $(p-1)^4 p^{19}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & \alpha^{-1}\beta\delta & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \varepsilon \\ -\theta & 1 & 0 & 0 & \zeta \\ 0 & 0 & 1 & \eta & \theta \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

with $\alpha, \beta, \gamma, \delta \neq 0$.

Group 8.5.17

$$\langle a, b, c, d, e \mid [e, b] = [c, a], [d, c] = [b, a][d, a]^{-1}, [e, a], [c, b], [d, b], [e, c], [e, d] \rangle$$

The number of conjugacy classes is $p^5 + 3p^4 - 2p^3 - 2p^2 + p$, and the automorphism group has order $2(p-1)^3 p^{18}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\lambda \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \alpha^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1-\gamma & 0 & \varepsilon \\ \beta & \gamma & \beta & 0 & \delta \\ 0 & 0 & \gamma & 0 & -\beta \\ \beta & 0 & \beta + \beta\gamma - \gamma\varepsilon & \gamma & \beta\varepsilon - \beta^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\lambda \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \alpha^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1+\gamma & 0 & \varepsilon \\ 0 & \gamma & -\beta\gamma & -\gamma & \delta \\ -1 & 0 & -1 & 0 & \beta \\ \beta + \varepsilon & 0 & \beta + \varepsilon - \beta\gamma & -\gamma & -\beta\varepsilon - \beta^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

with $\alpha, \gamma, \lambda \neq 0$.

Group 8.5.18

$$\langle a, b, c, d, e \mid [e, c][e, b], [c, a], [d, a], [e, a], [c, b], [d, b], [e, d] \rangle$$

The number of conjugacy classes is $p^5 + 4p^4 - 3p^3 - 3p^2 + 2p$, and the automorphism group has order $6(p-1)^4 p^{18}$.

The action of the automorphism group on G/G' is generate by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \varepsilon & 1 & 0 & 0 & \zeta \\ 0 & 0 & 1 & \eta & -\zeta \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} (\alpha, \beta, \gamma, \delta \neq 0),$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here the matrices of the first kind above form a normal subgroup of order $(p-1)^4 p^3$.

Group 8.5.19

$$\langle a, b, c, d, e \mid [d, a], [e, a], [c, b], [d, b], [e, b], [d, c], [e, c] \rangle$$

The number of conjugacy classes is $2p^5 + 2p^4 - 3p^3 - p^2 + p$, and the automorphism group has order $(p-1)(p^2-1)^2(p^2-p)^2 p^{17}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} * & * & * & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix}.$$

Group 8.5.20

$$\langle a, b, c, d, e \mid [d, c] = [b, a], [c, a], [d, a], [e, a], [d, b], [e, b], [e, d] \rangle$$

The number of conjugacy classes is $2p^5 + p^4 - 2p^3$, and the automorphism group has order $(p-1)^4 p^{23}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(5, p)$ of the form

$$\begin{pmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & \alpha\beta\gamma^{-1} & 0 \\ 0 & 0 & 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \varepsilon & 0 \\ * & 1 & 0 & * & * \\ * & \varepsilon & 1 & * & * \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & * & 1 \end{pmatrix},$$

with $\alpha, \beta, \gamma, \delta \neq 0$.

Group 8.5.21

$$\langle a, b, c, d, e \mid [d, c] = [b, a], [c, a], [d, a], [e, a], [c, b], [d, b], [e, d] \rangle$$

The number of conjugacy classes is $2p^5 + p^4 - 2p^3$, and the automorphism group has order $(p-1)^2(p^2-1)(p^2-p)p^{20}$.

The action of the automorphism group on G/G' is given by the subgroup of $\text{GL}(5, p)$ generated by matrices of the form

$$\begin{pmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & \alpha\beta\gamma^{-1} & 0 \\ 0 & 0 & 0 & 0 & \delta \end{pmatrix} \quad (\alpha, \beta, \gamma, \delta \neq 0)$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & \varepsilon & 0 \\ -\varepsilon & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 & 0 \\ * & 0 & 0 & * & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ \varepsilon & 0 & -1 & \zeta & 0 \\ * & 1 & -1 & \varepsilon + \zeta & 0 \\ -1 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & * & 1 \end{pmatrix}$$

(If we let λ, μ, ν, ξ be the $(1, 1)$, $(1, 4)$, $(4, 1)$, $(4, 4)$ entries in a general matrix in this group of matrices, then $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$ takes all values in $\text{GL}(2, p)$.)

Group 8.5.22

$$\langle a, b, c, d, e \mid [d, c] = [b, a], [c, a], [d, a], [e, a], [c, b], [d, b], [e, b] \rangle$$

The number of conjugacy classes is $2p^5 + p^4 - 2p^3$, and the automorphism group has order $(p^2-1)^2(p^2-p)^2p^{17}$.

The action of the automorphism group on G/G' is given by the subgroup of $\text{GL}(5, p)$ consisting of matrices of the form

$$\begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 \\ 0 & 0 & * & * & 0 \\ 0 & 0 & * & * & * \end{pmatrix},$$

with the restriction that if we let A be the elements of $\text{GL}(2, p)$ in positions $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$ and if we let B be the element of $\text{GL}(2, p)$ in positions $(3, 3)$, $(3, 4)$, $(4, 3)$, $(4, 4)$, then $\det A = \det B$.

8.6 Six generator groups

For all these groups we take the generators to be a, b, c, d, e, f , and we just give the relations, with the class two and exponent p conditions understood.

Group 8.6.1

$$[c, b], [d, a], [d, b], [d, c], [e, a], [e, b], [e, c], \\ [e, d], [f, a], [f, b], [f, c], [f, d], [f, e]$$

The number of conjugacy classes is $2p^6 - p^4$ and the order of the automorphism group is $(p-1)(p^2-1)(p^2-p)(p^3-1)(p^3-p)(p^3-p^2)p^{23}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(6, p)$ of the form

$$\begin{pmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{pmatrix}.$$

Group 8.6.2

$$[c, b], [d, a], [d, b] = [b, a], [d, c], [e, a], [e, b], [e, c], \\ [e, d], [f, a], [f, b], [f, c], [f, d], [f, e]$$

The number of conjugacy classes is $p^6 + 2p^5 - p^4 - 2p^3 + p^2$ and the order of the automorphism group is $2(p^2-1)^3(p^2-p)^3p^{20}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(6, p)$ of the form

$$\begin{pmatrix} -\alpha & \beta & -\gamma & -\alpha + \delta & * & * \\ \varepsilon & 0 & \zeta & \varepsilon & * & * \\ 0 & \eta & 0 & \theta & * & * \\ \alpha & 0 & \gamma & \alpha & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix}$$

with $(\alpha\zeta - \gamma\varepsilon)(\beta\theta - \delta\eta) \neq 0$ and

$$\begin{pmatrix} \alpha & -\beta & \gamma & \alpha - \delta & * & * \\ 0 & \varepsilon & 0 & \zeta & * & * \\ \eta & 0 & \theta & \eta & * & * \\ 0 & \beta & 0 & \delta & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix}$$

with $(\alpha\theta - \gamma\eta)(\beta\zeta - \delta\varepsilon) \neq 0$.

Group 8.6.3

$$\begin{aligned} [c, b], [d, a], [d, b] &= [c, a], [d, c], [e, a], [e, b], [e, c], \\ [e, d], [f, a], [f, b], [f, c], [f, d], [f, e] \end{aligned}$$

The number of conjugacy classes is $p^6 + p^5 - p^3$ and the order of the automorphism group is $(p-1)(p^2-1)^2(p^2-p)^2p^{24}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(6, p)$ of the form

$$\begin{pmatrix} \alpha & \beta & * & * & * & * \\ \gamma & \delta & * & * & * & * \\ 0 & 0 & \lambda\delta & -\lambda\gamma & * & * \\ 0 & 0 & -\lambda\beta & \lambda\alpha & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix}$$

with $\lambda(\alpha\delta - \beta\gamma) \neq 0$.

Group 8.6.4

$$\begin{aligned} [c, b], [d, a], [d, b] &= [c, a], [d, c] = [b, a]^\omega, [e, a], [e, b], [e, c], \\ [e, d], [f, a], [f, b], [f, c], [f, d], [f, e] \end{aligned}$$

The number of conjugacy classes is $p^6 + p^4 - p^2$ and the order of the automorphism group is $2(p^4-1)(p^4-p^2)(p^2-1)(p^2-p)p^{20}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(6, p)$ of the form

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & * & * \\ \varepsilon & \zeta & \eta & \theta & * & * \\ \omega\theta & -\omega\eta & -\zeta & \varepsilon & * & * \\ -\omega\delta & \omega\gamma & \beta & -\alpha & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & * & * \\ \varepsilon & \zeta & \eta & \theta & * & * \\ -\omega\theta & \omega\eta & \zeta & -\varepsilon & * & * \\ \omega\delta & -\omega\gamma & -\beta & \alpha & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix},$$

where $(\alpha, \beta, \gamma, \delta)$ can be any 4-vector other than zero ($p^4 - 1$ possibilities), and where $(\varepsilon, \zeta, \eta, \theta)$ can be any 4-vector which is *not* in the linear span of $(\alpha, \beta, \gamma, \delta)$ and $(\omega\delta, -\omega\gamma, -\beta, \alpha)$ ($p^4 - p^2$ possibilities).

Group 8.6.5

$$[c, b], [d, a], [d, b], [d, c], [e, a], [e, b], [e, c], \\ [e, d] = [b, a], [f, a], [f, b], [f, c], [f, d], [f, e]$$

The number of conjugacy classes is $p^6 + p^5 - p^3$ and the order of the automorphism group is $(p-1)^3(p^2-1)(p^2-p)p^{22}$.

The action of the automorphism group on G/G' is given by matrices in $\text{GL}(6, p)$ of the form

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & * \\ 0 & \alpha^{-1}(\lambda\xi - \mu\nu) & 0 & 0 & 0 & * \\ 0 & \zeta & \eta & 0 & 0 & * \\ 0 & \alpha^{-1}(-\delta\mu + \varepsilon\lambda) & 0 & \lambda & \mu & * \\ 0 & \alpha^{-1}(-\delta\xi + \varepsilon\nu) & 0 & \nu & \xi & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

with $\alpha, \eta, \lambda\xi - \mu\nu \neq 0$.

Group 8.6.6

$$[c, b], [d, a], [d, b] = [c, a], [d, c], [e, a], [e, b], [e, c], \\ [e, d] = [b, a], [f, a], [f, b], [f, c], [f, d], [f, e]$$

The number of conjugacy classes is $p^6 + p^4 - p^2$ and the order of the automorphism group is $(p-1)^2(p^2-1)(p^2-p)p^{21}$.

The action of the automorphism group on G/G' is given by a subgroup H of $\text{GL}(6, p)$, where the matrices in H have the form

$$\begin{pmatrix} \alpha & * & * & \beta & * & * \\ 0 & * & * & 0 & * & * \\ 0 & * & * & 0 & * & * \\ \gamma & * & * & \delta & * & * \\ 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

As we run through the elements of H , $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ takes all values in $\text{GL}(2, p)$. If we take generators $\begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ for $\text{GL}(2, p)$ then we obtain the following generating matrices for H :

$$\begin{pmatrix} \omega & \alpha & \beta & 0 & \gamma & * \\ 0 & \omega\lambda & 0 & 0 & 0 & * \\ 0 & 0 & \lambda & 0 & 0 & * \\ 0 & \omega^{-1}\gamma & -\omega^{-1}\alpha & 1 & \delta & * \\ 0 & 0 & 0 & 0 & \omega^2\lambda & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix} \quad (\lambda \neq 0)$$

and

$$\begin{pmatrix} -1 & \alpha & \beta & 1 & \gamma & * \\ 0 & \lambda & 0 & 0 & 2\lambda & * \\ 0 & 0 & 0 & 0 & \lambda & * \\ -1 & \delta & \beta - \delta & 0 & \delta - \alpha & * \\ 0 & -\lambda & \lambda & 0 & -\lambda & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix} (\lambda \neq 0).$$

Group 8.6.7

$$\begin{aligned} &[c, a], [c, b], [d, a], [d, b], [e, a], [e, b], [e, c], \\ &[e, d], [f, a], [f, b], [f, c], [f, d], [f, e] = [b, a] \end{aligned}$$

The number of conjugacy classes is $p^6 + p^5 - p^4 + p^3 - 2p + 1$ and the order of the automorphism group is $(p^4 - 1)(p^4 - p^3)(p^2 - 1)^2(p^2 - p)p^{13}$.

The action of the automorphism group on G/G' is given by a subgroup H of $\text{GL}(6, p)$, where the matrices in H have the form

$$\begin{pmatrix} * & * & 0 & 0 & * & * \\ * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \\ * & * & 0 & 0 & * & * \\ * & * & 0 & 0 & * & * \end{pmatrix}.$$

The first row takes on all possible $p^4 - 1$ non-zero values of the form shown. Let the first row correspond to an element $g_1 = a^*b^*e^*f^*$. The centralizer of g_1 has index p in G , and the second row must correspond to an element $g_2 = a^*b^*e^*f^*$ outside the centralizer of g_1 ($p^4 - p^3$ possibilities). The fifth row must correspond to a non-trivial element $g_5 = a^*b^*e^*f^*$ which centralizes g_1 and g_2 ($p^2 - 1$ possibilities). The sixth row must correspond to an element $g_6 = a^*b^*e^*f^*$ which centralizes g_1 and g_2 but does not centralize g_5 ($p^2 - p$ possibilities), but we require $[g_6, g_5] = [g_2, g_1]$ and this reduces the number of choices for g_6 to p . The third and fourth rows correspond to non-commuting elements of the form c^*d^* ($(p^2 - 1)(p^2 - p)$ possibilities).

Group 8.6.8

$$\begin{aligned} &[c, a], [c, b], [d, a], [d, b], [e, a], [e, b], [e, c], \\ &[e, d], [f, a], [f, b], [f, c], [f, d], [f, e] = [b, a][d, c] \end{aligned}$$

The number of conjugacy classes is $p^6 + 3p^3 - 2p^2 - 3p + 2$ and the order of the automorphism group is $6(p^2 - 1)^3(p^2 - p)p^{14}$.

The action of the automorphism group on G/G' is given by a subgroup H of $\text{GL}(6, p)$. The group H has a subgroup K of order $(p^2 - 1)^3(p^2 - p)p^2$ consisting of matrices of the form

$$\begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix},$$

where the determinants of the three 2×2 blocks are equal. The three blocks “can be permuted around” in 6 ways, and the group H is generated by K and the matrices

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Group 8.6.9

$$\begin{aligned} [c, b], [d, a], [d, b] &= [c, a], [d, c], [e, a], [e, b], [e, c], \\ [e, d], [f, a], [f, b], [f, c], [f, d], [f, e] &= [b, a] \end{aligned}$$

The number of conjugacy classes is $p^6 + 2p^3 - p^2 - 2p + 1$ and the order of the automorphism group is $(p^2 - 1)^2(p^2 - p)^2p^{15}$.

The action of the automorphism group on G/G' is given by the subgroup of $\text{GL}(6, p)$ consisting of matrices

$$\begin{pmatrix} \alpha & \beta & \varepsilon & \zeta & 0 & 0 \\ \gamma & \delta & \eta & \theta & 0 & 0 \\ 0 & 0 & \lambda\delta & -\lambda\alpha & 0 & 0 \\ 0 & 0 & -\lambda\beta & \lambda\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho & \sigma \\ 0 & 0 & 0 & 0 & \tau & \varphi \end{pmatrix}$$

with $\alpha\delta - \beta\gamma = \rho\varphi - \sigma\tau \neq 0$, $\lambda \neq 0$, $\alpha\eta + \beta\theta = \gamma\varepsilon + \delta\zeta$.

Group 8.6.10

$$\begin{aligned} [c, b], [d, a], [d, b] &= [c, a], [d, c], [e, a], [e, b], [e, c], \\ [e, d], [f, a], [f, b], [f, c], [f, d], [f, e] &= [c, a] \end{aligned}$$

The number of conjugacy classes is $p^6 + p^5 - p^4 + p^2 - p$ and the order of the automorphism group is $(p^2 - 1)^2(p^2 - p)^2p^{20}$.

The action of the automorphism group on G/G' is given by the subgroup H of $\text{GL}(6, p)$ where the matrices in H have the form

$$\begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{pmatrix}.$$

There is a subgroup of H of order p^8 consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 & \alpha & \beta & \gamma & \delta \\ 0 & 1 & \varepsilon & \zeta & \eta & \theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \delta & \theta & 1 & 0 \\ 0 & 0 & -\gamma & -\eta & 0 & 1 \end{pmatrix}.$$

The group H is generated by these matrices together with matrices of the form

$$\begin{pmatrix} \alpha & \beta & 0 & 0 & 0 & 0 \\ \gamma & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda\delta & -\lambda\gamma & 0 & 0 \\ 0 & 0 & -\lambda\beta & \lambda\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho & \sigma \\ 0 & 0 & 0 & 0 & \tau & \varphi \end{pmatrix}$$

where $\rho\varphi - \sigma\tau = \lambda(\alpha\delta - \beta\gamma) \neq 0$.

Group 8.6.11

$$\begin{aligned} [c, b], [d, a], [d, b] &= [c, a], [d, c] = [b, a]^\omega, [e, a], [e, b], [e, c], \\ [e, d], [f, a], [f, b], [f, c], [f, d], [f, e] &= [b, a] \end{aligned}$$

The number of conjugacy classes is $p^6 + p^3 - p$ and the order of the automorphism group is $2(p^4 - 1)(p^3 - p^2)(p^2 - 1)p^{13}$.

The action of the automorphism group on G/G' is given by the matrices in $\text{GL}(6, p)$ with the form

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & 0 & 0 \\ \varepsilon & \zeta & \eta & \theta & 0 & 0 \\ \omega\theta & -\omega\eta & -\zeta & \varepsilon & 0 & 0 \\ -\omega\delta & \omega\gamma & \beta & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & 0 & \nu & \xi \end{pmatrix} \text{ and } \begin{pmatrix} \alpha & \beta & \gamma & \delta & 0 & 0 \\ \varepsilon & \zeta & \eta & \theta & 0 & 0 \\ -\omega\theta & \omega\eta & \zeta & -\varepsilon & 0 & 0 \\ \omega\delta & -\omega\gamma & -\beta & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & 0 & \nu & \xi \end{pmatrix}$$

where $\alpha\eta + \beta\theta = \gamma\varepsilon + \delta\zeta$ and $\lambda\xi - \mu\nu = \alpha\zeta - \beta\varepsilon + \omega(\gamma\theta - \delta\eta) \neq 0$. Note that there are $p^4 - 1$ choices for row 1, and that once row 1 has been fixed there are $p^3 - p^2$ choices for row 2.

Group 8.6.12

$$[c, b], [d, a], [d, b], [d, c], [e, a], [e, b], [e, c], \\ [e, d] = [b, a], [f, a], [f, b], [f, c], [f, d] = [c, a], [f, e]$$

The number of conjugacy classes is $p^6 + p^4 - p^2$ and the order of the automorphism group is $(p^2 - 1)^2(p^2 - p)^2p^{18}$.

The action of the automorphism group on G/G' is given by a subgroup H of $\text{GL}(6, p)$, where the elements of have the form

$$\begin{pmatrix} \alpha & * & * & \beta & * & * \\ 0 & * & * & 0 & * & * \\ 0 & * & * & 0 & * & * \\ \gamma & * & * & \delta & * & * \\ 0 & * & * & 0 & * & * \\ 0 & * & * & 0 & * & * \end{pmatrix}$$

where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ takes all values in $\text{GL}(2, p)$. There is a subgroup of H of order p^6 consisting of all elements of the form

$$\begin{pmatrix} 1 & \varepsilon & \zeta & 0 & \eta & \theta \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \eta & \theta & 1 & \lambda & \mu \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and H is generated by this subgroup together with elements

$$\begin{pmatrix} \alpha & 0 & 0 & \beta & 0 & 0 \\ 0 & \delta\lambda & \delta\mu & 0 & -\gamma\lambda & -\gamma\mu \\ 0 & \delta\nu & \delta\xi & 0 & -\gamma\nu & -\gamma\xi \\ \gamma & 0 & 0 & \delta & 0 & 0 \\ 0 & -\beta\lambda & -\beta\mu & 0 & \alpha\lambda & \alpha\mu \\ 0 & -\beta\nu & -\beta\xi & 0 & \alpha\nu & \alpha\xi \end{pmatrix}$$

with $(\alpha\delta - \beta\gamma)(\lambda\xi - \mu\nu) \neq 0$.

Group 8.6.13

$$\begin{aligned} [b, a], [d, a], [e, a][c, a], [f, a], [c, b], [d, b] &= [c, a], [e, b], \\ [f, b][c, a]^2, [d, c], [e, c], [e, d] &= [f, c], [f, d], [f, e] = [c, a][f, c] \end{aligned}$$

The number of conjugacy classes is $p^6 + p^3 - p$ and the order of the automorphism group is $(p-1)(p^2-1)(p^2-p)p^{19}$.

The action of the automorphism group on G/G' is given by a subgroup $H = KL$ of $\text{GL}(6, p)$, where K is a subgroup of H and L is a normal subgroup of H . The subgroup K is the set of all matrices of the form

$$\begin{pmatrix} \alpha & \beta & 0 & 0 & 0 & 0 \\ \gamma & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2\delta & -\lambda^2\gamma & 0 & 0 \\ 0 & 0 & -\lambda^2\beta & \lambda^2\alpha & 0 & 0 \\ 0 & 0 & (\lambda - \lambda^2)\delta & (2\lambda + \lambda^2)\gamma & \lambda\delta & \lambda\gamma \\ 0 & 0 & (\lambda + 2\lambda^2)\beta & 2(\lambda - \lambda^2)\alpha & \lambda\beta & \lambda\alpha \end{pmatrix}$$

with $\lambda(\alpha\delta - \beta\gamma) \neq 0$, and L is the set of all matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \alpha & -2\beta^2 + 2\beta + \varepsilon - \zeta + \gamma\delta & \beta + 1 & \gamma & \beta & \frac{1}{2}\gamma \\ -\beta^2 + 2\beta + \varepsilon - \zeta + \frac{1}{2}\gamma\delta & \frac{1}{2}\delta - \frac{1}{2}\eta & \delta & 2\beta + 1 & \delta & \beta \\ -\alpha - \frac{1}{2}\gamma & \varepsilon & -\beta & -\gamma & 1 - \beta & -\frac{1}{2}\gamma \\ \zeta & \eta & -2\delta & -4\beta & -2\delta & 1 - 2\beta \end{pmatrix}.$$

Group 8.6.14

$$\begin{aligned} [c, b], [d, a], [d, b] &= [c, a], [d, c], [e, a], [e, b], [e, c], \\ [e, d] &= [b, a], [f, a], [f, b] = [c, a]^m, [f, c] = [b, a], [f, d], [f, e] = [c, a], \end{aligned}$$

where $x^3 - mx + 1$ is irreducible over $\text{GF}(p)$. (Different choices of m give isomorphic groups.)

The number of conjugacy classes is $p^6 + p^2 - 1$ and the order of the automorphism group is $3(p^6 - 1)(p - 1)p^{15}$.

Note that since $x^3 - mx + 1$ is irreducible over $\text{GF}(p)$ its discriminant $4m^3 - 27$ must be a square, and we let $u^2 = 4m^3 - 27$.

The action of the automorphism group on G/G' is given by a subgroup H of $\text{GL}(6, p)$. The first rows of the matrices in H are completely arbitrary, except that they must

be non-zero. The subgroup K of H consisting of those matrices in H with first row $(1, 0, 0, 0, 0, 0)$ has order $3(p-1)p^3$ and is generated by the following matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -m\gamma - \beta & 1 & 0 & -\gamma & 0 & -\alpha \\ \alpha & 0 & 1 & \beta & 0 & \gamma \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\gamma & 0 & 0 & m\beta - \alpha & 1 & -\beta \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2u}(9-u) & \frac{m^2}{u} & 0 & \frac{m}{u}(u-3) & 0 \\ 0 & -\frac{3m}{u} & -\frac{1}{2u}(9+u) & 0 & \frac{2m^2}{u} & 0 \\ \frac{m}{2u}(3-u) & 0 & 0 & \frac{1}{2u}(9-u) & 0 & -\frac{m^2}{u} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{m^2}{u} & 0 & 0 & \frac{3m}{u} & 0 & -\frac{1}{2u}(9+u) \end{pmatrix}.$$

(Here $A^3 = I$.)

The group H is generated by the matrices above, together with $(p^6 - 1)(p^4 - p^3)$ matrices of the form

$$B = \begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \zeta \\ \mu + m\rho & \lambda & m\xi + \sigma & \rho & -\xi & \nu \\ -\nu & \xi & \lambda & -\mu & \sigma & -\rho \\ \zeta & \varepsilon & -\beta & \alpha - m\delta & -\gamma - m\varepsilon & \delta \\ \rho & -\sigma & \xi & \nu - m\mu & \lambda + m\sigma & \mu \\ \delta + m\zeta & -\gamma & -m\beta - \varepsilon & \zeta & \beta & \alpha \end{pmatrix}.$$

Note that the first row of B determines rows 4 and 6, and that the third of B determines rows 2 and 5. The first row can take any non-zero value ($p^6 - 1$ possibilities). Once the first row of B is fixed the entries $\lambda, \mu, \nu, \xi, \rho, \sigma$ from row 3 are required to satisfy two homogeneous linear equations with coefficients determined by the first row. These two linear equations correspond to the requirement that the group elements corresponding to rows 3 and 4 commute. There are p^4 solutions to these two equations, but we also require that the third row lie outside the subspace spanned by rows 1, 4 and 6. Any row 3 lying in this subspace automatically satisfies these two linear equations, since the group elements corresponding to rows 1, 4 and 6 are guaranteed to commute by the choice of rows 4 and 6. So, once row 1 of B is fixed, there are $p^4 - p^3$ choices for row 3 of B , and then all the rows of B are determined.

It follows from all this that H has order $3(p^6 - 1)(p - 1)p^3$.

If we want to obtain explicit generators for H then we only need to find one possible third row of B for any given first row. Then these particular choices of B ,

together with the generators of K will generate H . If one of α, δ, ζ is non-zero, then we can take the third row of B to be

$$(0, \delta^2 + m\zeta\delta - \alpha\zeta, -\alpha^2 + m\delta\alpha + \delta\zeta, 0, \zeta^2 - \alpha\delta, 0),$$

and if one of $\beta, \gamma, \varepsilon$ is non-zero, then we can take the third row of B to be

$$(\gamma^2 + m\varepsilon\gamma - \beta\varepsilon, 0, 0, \gamma\varepsilon + \beta^2, 0, \varepsilon^2 + m\beta\varepsilon + \beta\gamma).$$

Experimentally, it seems that that H is generated by the generators of K together with the single matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & -1 \\ 0 & 0 & -m & 0 & 1 & 0 \end{pmatrix}.$$

8.7 Seven generator groups

Group 8.7.1

$$\langle a, b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle \times \langle f \rangle \times \langle g \rangle.$$

The number of conjugacy classes is $p^7 + p^6 - p^5$, and the automorphism group has order $(p^8 - p^6)(p^8 - p^7)(p^6 - p)(p^6 - p^2)(p^6 - p^3)(p^6 - p^4)(p^6 - p^5)$.

Group 8.7.2

$$\langle a, b \rangle \times_{[b,a]=[d,c]} \langle c, d \rangle \times \langle e \rangle \times \langle f \rangle \times \langle g \rangle.$$

The number of conjugacy classes is $p^7 + p^4 - p^3$, and the automorphism group has order $(p^8 - p^4)(p^8 - p^7)(p^6 - p^4)p^5(p^4 - p)(p^4 - p^2)(p^4 - p^3)$.

Group 8.7.3

$$\langle a, b \rangle \times_{[b,a]=[d,c]=[f,e]} \langle c, d \rangle \times_{[b,a]=[d,c]=[f,e]} \langle e, f \rangle \times \langle g \rangle.$$

The number of conjugacy classes is $p^7 + p^2 - p$, and the automorphism group has order $(p^8 - p^2)(p^8 - p^7)(p^6 - p^2)p^5(p^4 - p^2)p^3(p^2 - p)$.

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