Groups of order p^8 and exponent p

Michael Vaughan-Lee

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Abstract

We prove that for $p > 7$ there are

$$p^4 + 2p^3 + 20p^2 + 147p + (3p + 29)\gcd(p - 1, 3) + 5\gcd(p - 1, 4) + 1246$$

groups of order p^8 with exponent p. If P is a group of order p^8 and exponent p, and if P has class $c > 1$ then P is a descendant of $P/\gamma_c(P)$. For each group of exponent p with order less than p^8 we calculate the number of descendants of order p^8 with exponent p. In all but one case we are able to obtain a complete and irredundant list of the descendants. But in the case of the three generator class two group of order p^6 and exponent p ($p > 3$), while we are able to calculate the number of descendants of order p^8, we have not been able to obtain a list of the descendants. Most of the calculations were carried out in nilpotent Lie algebras over GF(p), and the group results and group presentations are obtained with the Lazard correspondence.

1 Introduction

The classification of groups of order p^k for small k has a long history dating back to the end of the nineteenth century. The groups of order p^2 were classified by Netto [13] in 1882. The groups of order p^3 were independently determined by Cole and Glover [4], Hölder [12] and Young [20] in 1893. The groups of order p^4 were determined by Hölder [12] and Young [20]. The groups of order p^5 were classified by Bagnera [1] in 1898. However it was not until 2004 that Newman, O'Brien and Vaughan-Lee [14] classified the groups of order p^6. (There were several attempts at classifying the groups of order p^6 during the 20^{th} century, but they all suffered from errors.) The groups of order p^7 were classified by O'Brien and Vaughan-Lee [17] in 2005. The 56,092 groups of order 2^8 were determined by O'Brien [16], and the 10,494,213 groups of order 2^9 were determined by Besche, Eick and O'Brien [7]. They also proved that there are 49,487,365,422 groups of order 2^{10}. Databases of the groups of order p^k ($k \leq 7$) and of the groups of order 2^8 and 2^9 are included in the "Small Groups" libraries in the algebraic computation systems GAP [9] and MAGMA [3]. The classification of the groups of order p^7 is based on a classification of the nilpotent Lie rings of order p^7, and a database of the nilpotent Lie rings of order p^k for $k \leq 7$ is available in a GAP package [19]. I have recently computed a list of all 1,396,077 groups of order 3^8, and the intention is to make these groups available through a GAP package.

The table below gives the number of groups of order p^n for $n \leq 5$.

	$p = 2$	$p = 3$	$p \geq 5$
p	1	1	1
p^2	2	2	2
p^3	5	5	5
p^4	14	15	15
p^5	51	67	$2p + 61 + 2\gcd(p-1,3) + \gcd(p-1,4)$

There are 267 groups of order 2^6 and 504 groups of order 3^6. For $p \geq 5$ the number of groups of order p^6 is

$$3p^2 + 39p + 344 + 24\gcd(p-1,3) + 11\gcd(p-1,4) + 2\gcd(p-1,5).$$

The numbers of groups of order 2^7, 3^7, 5^7 are respectively 2328, 9310, 34297. For $p > 5$ the number of groups of order p^7 is

$$\begin{aligned}
&3p^5 + 12p^4 + 44p^3 + 170p^2 + 707p + 2455 \\
&+(4p^2 + 44p + 291)\gcd(p-1,3) + (p^2 + 19p + 135)\gcd(p-1,4) \\
&+(3p + 31)\gcd(p-1,5) + 4\gcd(p-1,7) + 5\gcd(p-1,8) \\
&+\gcd(p-1,9).
\end{aligned}$$

The functions giving the numbers of groups of order p^k for $k \leq 7$ are *polynomial on residue classes* (PORC). (A function $f(p)$ is PORC if there is a finite set of polynomials in p, $g_1(p)$, $g_2(p)$, ..., $g_k(p)$, and a positive integer N, such that for any prime p

$$f(p) = g_i(p)$$

for some i ($1 \leq i \leq k$), with the choice of i depending on the residue class of p modulo N.) Graham Higman [10] proved that for fixed n the number of groups of order p^n, $f(p^n)$, is bounded by a polynomial in p, and he conjectured that (for fixed n) $f(p^n)$ is PORC — this is his famous PORC conjecture. So Higman's conjecture has been proved correct for $n \leq 7$. However du Sautoy and Vaughan-Lee [6] give an example which throws some doubt on whether Higman's conjecture holds for $n = 10$.

The classification of groups of order p^8 appears to be an extraordinarily difficult problem, but we are able to make some progress with the groups of order p^8 and exponent p.

Theorem 1 *For $p > 7$ the number of groups of order p^8 and exponent p is*

$$p^4 + 2p^3 + 20p^2 + 147p + (3p + 29)\gcd(p-1,3) + 5\gcd(p-1,4) + 1246.$$

Note that this function is also PORC. Our proof of this result uses the methods described in [14] and [17].

I will give details of the methods of proof below, but comment here that all the proofs are traditional "hand" proofs, albeit with machine assistance with linear algebra and with adding, multiplying, and factoring polynomials. However the proofs involve a case by case analysis of hundreds of different cases, and although most of the cases are straightforward enough it is virtually impossible to avoid the occasional slip or transcription error. So as far as possible the results and the group presentations have been computer checked for small primes to eliminate errors. For example, I was able to use Eamonn O'Brien's p-group generation algorithm [15] in Magma to check the numbers of two generator groups of order p^8 and exponent p for $11 \leq p \leq 31$, and I was able to use the StandardPresentation function in Magma to check that my group presentations generated the right number of groups. (Since there are two generator groups of order p^8 and class 7, the Lazard correspondence only applies for $p > 7$.)

Finally, I would like to record my thanks to Roger Heath-Brown and to Tony Scholl. The calculations threw up several problems in number theory, most of which I was able to solve myself. But two of the problems were beyond my competence, and I am very grateful to Roger and Tony for providing me with solutions.

2 The Lie algebra generation algorithm

The main tool used in calculating the nilpotent Lie algebras of dimension 8 is the Lie algebra generation algorithm, which is an adaptation to Lie algebras of Eamonn O'Brien's p-group generation algorithm [15].

Let P be a p-group. The p-group generation algorithm uses the lower p-central series, defined recursively by $\mathcal{P}_1(P) = P$ and $\mathcal{P}_{i+1}(P) = [\mathcal{P}_i(P), P]\mathcal{P}_i(P)^p$ for $i \geq 1$. The p-class of P is the length of this series. Each p-group P, apart from the elementary abelian ones, is an *immediate descendant* of the quotient P/R where R is the last non-trivial term of the lower p-central series of P. Thus all the groups with order p^8, except the elementary abelian one, are immediate descendants of groups with order p^k for $k < 8$. All of the immediate descendants of P are quotients of a certain extension of P; the isomorphism problem for these descendants is equivalent to the problem of determining orbits of certain subgroups of this extension under an action of the automorphism group of P. Not all p-groups have immediate descendants, those that do are *capable*.

As described in [17], the classification of the groups of order p^7 is based on a classification of the nilpotent Lie rings of order p^7. The Lie ring generation algorithm was developed in analogy with the p-group generation algorithm, and for each nilpotent Lie ring L of order p^k with $k < 7$, the immediate descendants of L of order p^7 were computed. The groups of order p^7 were obtained from the Lie rings of order p^7 via the Lazard correspondence, using the Baker-Campbell-Hausdorff formula.

For groups of exponent p the Lazard correspondence gives a correspondence between nilpotent Lie algebras over $\mathrm{GF}(p)$ and groups of exponent p. If M is a nilpotent

Lie algebra of class $c > 1$ then M is an immediate descendant of M/M^c. The Lie algebra generation algorithm gives a method of computing the immediate descendants of a nilpotent Lie algebra L.

The algorithm is as follows. Let L be a nilpotent Lie algebra of class c, and let M be the covering algebra of L. Thus M is the largest Lie algebra with an ideal I satisfying

1. $I \leq \zeta(M) \cap M^2$,
2. $M/I \cong L$.

Here, $\zeta(M)$ is the centre of M. We call I the multiplier of L. The condition $I \leq \zeta(M)$ ensures that M has class at most $c+1$, and the condition $I \leq M^2$ ensures that $\dim(M/M^2) = \dim(L/L^2)$, so that L and M have the same generator number. The nucleus of M is M^{c+1}, which can be trivial. An allowable subspace of I is a proper subspace $S < I$ such that $S + M^{c+1} = I$. The immediate descendants of L are the quotients M/S where S is an allowable subspace. Note that if $M^{c+1} = \{0\}$ then there are no allowable subspaces, and hence no immediate descendants. In this case we say that L is terminal. If L has immediate descendants then we say that L is capable. The automorphism group of L acts on the subspaces of I, and two quotient algebras M/S, M/T are isomorphic if and only if S and T are in the same orbit under this action. To compute the descendants of L we need to compute the action of the automorphism group of L on the allowable subspaces of I.

Except in four cases mentioned below, I used the Lie algebra generation algorithm to calculate the immediate descendants of dimension 8 over GF(p) of all the nilpotent Lie algebras of dimension less than 8. (These calculations are valid for all $p > 3$.) I then applied the Baker-Campbell-Hausdorff formula to "translate" these presentations into group presentations for groups of order p^8 and exponent p. The four cases where I was unable to use the Lie algebra generation algorithm were in calculating the descendants of the abelian Lie algebras of dimension 4, 5 and 6, and in computing the immediate descendants of the three generator Lie algebra of class two and dimension 6. In these four cases I was able to use ideas introduced by Higman [11] to calculate the number of immediate descendants of dimension 8, and in all but one case I was also able to obtain a list of the descendants.

3 Graham Higman's PORC theory

Higman [11] proved that the number of groups of order p^n with p-class 2 is PORC (for any fixed n). (Higman uses the term Φ-class 2.) Evseev [8] has extended Higman's result to the more general class of p-groups in which the derived group is elementary abelian and central. Higman and Evseev obtained these results as an application of a very general theorem of Higman's about the action of the general linear group on vector spaces. However this theorem takes a page to state, and is given in such generality that it is hard to see what is going on. So I will just describe the theorem as it applies to computing Lie algebras of class two and to computing the descendants of the three generator Lie algebra of class 2 and dimension 6.

First consider the computation of class two Lie algebras L over $\mathrm{GF}(p)$, where L/L^2 has dimension r. If we let M be the free r generator Lie algebra over $\mathrm{GF}(p)$ of class two, then M/M^2 has dimension r, and M^2 has dimension $\frac{r(r-1)}{2}$. Every Lie algebra L over $\mathrm{GF}(p)$ of class two with $\dim(L/L^2) = r$ can be expressed in the form $L = M/S$ for some subspace $S \leq M^2$. The group $\mathrm{GL}(r, p)$ acts on the subspaces of M^2 via its natural action on M/M^2, and two Lie algebras M/S and M/T are isomorphic if and only if S and T are in the same orbit under this action. So we can obtain a complete and irredundant set of r generator Lie algebras of class two and dimension $r + s$ by computing a set of orbit representatives for the subspaces of M^2 of codimension s. Note that this is just a special case of the Lie algebra generation algorithm described above. Higman's general theorem implies that for any given r and s the function $g(r, s)$ giving the number of orbits of subspaces of M^2 of codimension s is PORC. In [18] I showed how Higman's theory can be turned into a practical algorithm for computing $g(r, s)$ for moderate values of r, and in that paper I used the algorithm to compute the number of p-class two groups of order p^8. Using the same algorithm it is easy to compute $g(r, s)$ for $r + s = 8$. Clearly $g(r, 8 - r) = 0$ for $r < 4$, and

$$g(4, 4) = 4,\ g(5, 3) = 22,\ g(6, 2) = 14,\ g(7, 1) = 3.$$

It is easy enough to write down presentations for the three 7 generator Lie algebras of class two and dimension 8, as well as for the corresponding groups. I was also able to write down presentations for the four, five and six generator Lie algebras of class two and dimension 8.

First consider the four generator Lie algebras. As mentioned above, these correspond to orbits of subspaces of M^2 of codimension 4 (where M is the free class two Lie algebra of rank 4). In this case $\dim M^2 = 6$, and by duality the orbits of subspaces of codimension 4 correspond to orbits of subspaces of dimension 4, which in turn correspond to the 4 four generator Lie algebras of class two and dimension 6. Now the four generator Lie algebras of class two and dimension 6 are known, so that in effect the orbits of subspaces of M^2 of dimension 4 are known. Using duality we can find representatives for the orbits of subspaces of codimension 4, and hence write down presentations for the four generator Lie algebras of class two and dimension 8. As a final check I showed that the four presentations obtained in this way defined different Lie algebras by counting the numbers of elements with centralizers of dimension 6.

The five generator class two Lie algebras of dimension 8 correspond to orbits of subspaces of codimension 3 in M^2, where here M is the free class two Lie algebra of rank 5. By duality, these orbits correspond to the orbits of subspaces of dimension 3, and representatives for the 22 orbits were found by Brahana [2]. There is a slightly different set of representatives for the 22 orbits in Annalisa Copetti's Masters thesis [5], and I used those orbit representatives to construct the corresponding Lie algebras of dimension 8. (I am grateful to Mike Newman for drawing my attention to these two references.)

I found presentations for the 14 six generator Lie algebras of class two and dimension 8 by more or less randomly generating lots of presentation until I found 14 non-isomorphic Lie algebras. By counting the numbers of elements with breadth 1 (i.e. with centralizers of codimension 1), and by taking the dimension of the centre into account, I was able to distinguish between these 14 algebras, except for two algebras with $p^4 - p^2$ elements with breadth 1 and with centres of dimension 2. But these two algebras cannot be isomorphic since in one of them the elements with breadth 1 all commute, and in the other algebra they do not all commute.

I was able to calculate the number of immediate descendants of dimension 8 of the three generator Lie algebra L of class two and dimension 6 as follows. The Lie algebra L is the free class two Lie algebra of rank 3, and its immediate descendants are quotients M/S of the free class three Lie algebra M of rank 3 by subspaces S of codimension 2 in M^3. As above, there is an action of $\mathrm{GL}(3,p)$ on the subspaces of M^3 via its natural action on M/M^2, and two quotients M/S and M/T are isomorphic if and only if S and T are in the same orbit under this action. I proved that the number of orbits of subspaces of codimension 2 is

$$p^4 + p^3 + 10p^2 + 82p + (3p+19)\gcd(p-1,3) + 3\gcd(p-1,4) + 522$$

for all $p > 3$. However I have made no attempt to find a list of the descendants.

4 Two worked examples

There are two important differences between applications of the Lie algebra generation algorithm to calculating the descendants of an abelian Lie algebra L (as described in the last section) and applications to calculating the descendants of a general nilpotent Lie algebra L. The first difference is that in general the automorphism group of L may be quite hard to compute, but when L is abelian of dimension r then the automorphism group is just $\mathrm{GL}(r,p)$. The second important difference is that when L is abelian all the subspaces of the multiplier I are allowable, whereas this is not the case for general L.

One of the things that makes the calculation of the nilpotent Lie algebras of dimension 8 feasible is that the automorphism groups of the nilpotent Lie algebras of dimension less than 8 can be described in a uniform way. For example, the automorphism group of

$$L = \langle a,b,c \mid \ ca - bab,\ cb,\ \text{class } 3\rangle$$

consists of all maps sending a,b,c to

$$\alpha a + \beta b + \gamma c + d,\ \lambda b + \mu c + e,\ \lambda^2 c + f$$

where $\alpha,\beta,\gamma,\lambda,\mu \in \mathrm{GF}(p)$, $\alpha,\lambda \neq 0$, and where $d,e \in L^2$, $f \in L^3$. Occasionally the automorphism group depends on the residue class of p modulo some integer N. For example the automorphism group of

$$\langle a,b,c \mid cb,\ bac,\ caa - bab,\ cac - baa,\ \text{class } 3\rangle$$

consists of all maps sending a, b, c to

$$\alpha a + d,\ \alpha\beta b + \gamma ba + e,\ \alpha\beta^2 c + \delta ca + f$$

or to

$$\alpha a + d,\ \alpha\beta c + \gamma ca + e,\ \alpha\beta^2 b + \delta ba + f$$

where $\alpha, \beta, \gamma, \delta \in \mathrm{GF}(p)$, $\alpha \neq 0$, $\beta^3 = 1$, and where $d \in L^2$, $e, f \in L^3$. So in this case the automorphism group depends on $p \bmod 3$. Since the number of descendants depends on the action of the automorphism group, having a uniform description for the automorphism group makes it more likely that there will be a uniform description of the descendants. In [6] Marcus du Sautoy and I investigated a class two Lie algebra of dimension 9 on 6 generators a, b, c, d, e, f satisfying the relations

$$da = eb,\ db = ea = fc,\ dc = fa,$$

with all other Lie products of the generators zero. It turns out that the automorphism group of this Lie algebra takes one of two forms (both with a uniform description), where the choice of automorphism group depends on whether or not the polynomial $y^8 + 360y^4 - 48$ has a root in $\mathrm{GF}(p)$. Furthermore, the primes for which $y^8 + 360y^4 - 48$ has a root cannot be described in terms of residue class conditions. The number of descendants of dimension 10 of this Lie algebra depend on which of the two automorphism groups apply, and so the number of descendants of dimension 10 is *not* PORC. Of course there are likely to be other Lie algebras of class 2 and dimension 9 with a non-PORC number of descendants of dimension 10, so that the grand total of class 3 Lie algebras of dimension 10 may be PORC (with all the bad behaviour over individual Lie algebras cancelling out). But our example does show that the automorphism groups of class two Lie algebras of dimension 9 can exhibit non-uniform behaviour which does not occur in the automorphism groups of nilpotent Lie algebras of dimension 7 or less.

We illustrate the Lie algebra generation algorithm by applying it to two three generator Lie algebras of dimension 7 and class 3.

First we consider the Lie algebra

$$L = \langle a, b, c \mid cb,\ bac,\ caa - bab,\ cac - baa,\ \text{class } 3\rangle$$

with automorphism group depending on $p \bmod 3$, as described above. The nucleus of the covering algebra has dimension 1, and is spanned by $babb$. So the descendants of dimension 8 have presentations of the form

$$\langle a, b, c \mid cb - xbabb,\ bac - ybabb,\ caa - bab - zbabb,\ cac - baa - tbabb,\ \text{class } 4\rangle$$

for some $x, y, z, t \in \mathrm{GF}(p)$. If we let $c' = c + \lambda ca + \mu db$, then a, b, c' satisfy the relations

$$c'b = (x + \lambda y + \mu)babb,\ bac' = (y + \lambda)babb,\ c'aa - bab = zbabb,\ c'ac' - baa = tbabb$$

so we can assume that $x = y = 0$. (I prefer to work with presentations rather than with allowable subspaces, but the two approaches are equivalent.) So we consider presentations of the form

$$\langle a, b, c \,|\, cb,\ bac,\ caa - bab - zbabb,\ cac - baa - tbabb,\ \text{class } 4\rangle.$$

We now restrict ourselves to automorphisms which preserve the relations $cb = bac = 0$. If we apply the automorphisms, and rewrite the relations in terms of the images of a, b, c under the automorphisms, then the first type of automorphism changes (z, t) to

$$\left(\alpha^{-1}\beta^2 z, \alpha^{-1}\beta t\right),$$

and the second type changes (z, t) to

$$\left(\alpha^{-1}\beta^2 t, \alpha^{-1}\beta z\right).$$

So we can take (z, t) to be $(0, 0)$, $(0, 1)$, or $(1, t)$ where t ranges over a set of representatives for the equivalence classes of non-zero elements of $\mathrm{GF}(p)$ under the equivalence relation defined by $t \sim \frac{1}{t}$ and $t \sim \beta t$ for all β such that $\beta^3 = 1$. Hence there are $\frac{p+5}{2}$ descendants when $p = 2 \bmod 3$, and $\frac{p+17}{6}$ descendants when $p = 1 \bmod 3$.

As a second example, we consider the algebra

$$L = \langle a, b, c \,|\, cb,\ bab,\ bac,\ cac,\ \text{class } 3\rangle.$$

The automorphism group consists of all maps

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & \mu \\ 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} \qquad (*)$$

where $\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & \mu \\ 0 & \nu & \xi \end{pmatrix}$ is a non-singular matrix in $\mathrm{GL}(3, p)$, and $d, e, f \in L^2$. If M is the covering algebra of L, and if $I \leq M$ is the multiplier, then the descendants of L of dimension 8 have the form M/S for some allowable subspace S of codimension 1 in I. The nucleus of M has dimension 3, and is spanned by $baaa$, $baac$, $caaa$. So if S is an allowable subspace of codimension 1 in I, then

$$\dim(S \cap \langle baaa,\ baac,\ caaa\rangle) = 2.$$

The automorphisms described above map

$$\begin{pmatrix} baaa \\ baac \\ caaa \end{pmatrix} \to \begin{pmatrix} \alpha^3\lambda & -\alpha^2\beta\mu + \alpha^2\gamma\lambda & \alpha^3\mu \\ 0 & \alpha^2\lambda\xi - \alpha^2\mu\nu & 0 \\ \alpha^3\nu & -\alpha^2\beta\xi + \alpha^2\gamma\nu & \alpha^3\xi \end{pmatrix} \begin{pmatrix} baaa \\ baac \\ caaa \end{pmatrix},$$

and so we see that automorphism group has two orbits on subspaces of the nucleus of dimension 2. Orbit representatives for the two orbits are $\langle baaa, baac\rangle$ and $\langle baaa, caaa\rangle$. So we may suppose that $\langle baaa, baac\rangle \leq S$ or that $\langle baaa, caaa\rangle \leq S$.

First consider the case when $\langle baaa, baac\rangle \le S$. Then M/S has a presentation of the form

$$\langle a,b,c \,|\, cb - xcaaa,\ bab - ycaaa,\ bac - zcaaa,\ cac - tcaaa,\ baaa,\ baac,\ \text{class } 4\rangle,$$

with $x,y,z,t \in \mathrm{GF}(p)$. We now restrict ourselves to automorphisms preserving the relations $baaa = baac = 0$. This means that we have to restrict ourselves to automorphisms (*) with $\mu = 0$. If we apply one of these automorphisms

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & 0 \\ 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

where $e = \zeta ba + \eta ca$ modulo L^3 and $f = \rho ba + \sigma ca$ modulo L^3 then (x,y,z,t) changes to

$$\alpha^{-3}\xi^{-1}(x\lambda\xi + y(\lambda\rho - \zeta\nu) + z(\lambda\sigma - \zeta\xi - \eta\nu) - t\eta\xi,\ y\alpha\lambda^2,\ y\alpha\lambda\nu + z\alpha\lambda\xi,\ y\alpha\nu^2 + 2z\alpha\nu\xi + t\alpha\xi^2).$$

So we can take $y = 0$ or 1. If $y = 1$ then we need $\xi = \alpha^{-2}\lambda^2$, and we can take $x = z = 0$ and take $t = 0, 1$ or ω where ω is any element of $\mathrm{GF}(p)$ which is not a square. If $y = 0$ then we can take $z = 0$ or 1. If $y = 0$ and $z = 1$ then we can take $x = t = 0$. If $y = z = 0$ then we can take $t = 0$ or 1. If $y = z = t = 0$ then we can take $x = 0$ or 1, and if $y = z = 0$, $t = 1$ then we can take $x = 0$. So L has 7 descendants of dimension 8 satisfying the relations $baaa = baac = 0$.

Similarly, if $\langle baaa, caaa\rangle \le S$ then M/S has a presentation of the form

$$\langle a,b,c \,|\, cb - xbaac,\ bab - ybaac,\ bac - zbaac,\ cac - tbaac,\ baaa,\ caaa,\ \text{class } 4\rangle,$$

with $x,y,z,t \in \mathrm{GF}(p)$. To preserve the relations $baaa = caaa = 0$ we need to restrict ourselves to automorphisms (*) with $\beta = \gamma = 0$. However it is sufficient to consider automorphisms of the form

$$a \to a,\ b \to b + \delta ca,\ c \to c + \varepsilon ba + \zeta ca + \eta caa.$$

Then

$$(x,y,z,t) \to (x + \varepsilon y + \zeta z - \delta t - \delta\varepsilon - \eta, y - 2\delta, z - \zeta, t + 2\varepsilon),$$

and so we can take $x = y = z = t = 0$. So L has just one descendant of dimension 8 satisfying the relations $baaa = caaa = 0$.

Putting these two calculations together we see that L has 8 descendants of dimension 8.

5 Summary of results

There follows below a complete list of the nilpotent Lie algebras over $\mathrm{GF}(p)$ with immediate descendants of dimension 8, together with the numbers of those descendants. These results are valid for all $p > 3$. In all except for one algebra we have a complete and irredundant list of the descendants. In the case when p is greater than the class of the descendants, I used the Lazard correspondence and the Baker-Campbell-Hausdorff formula to produce a complete list of the corresponding groups. There are MAGMA programs to generate the groups in a file named "p8expp.tar" on my website `http://users.ox.ac.uk/~vlee/PORC/` for those who are interested.

The Lie algebra presentations have a standard format with a set of generators, a set of relators, and a specified nilpotency class. I denote the Lie product of a and b by ab, rather than the more usual $[a, b]$, and I use a left-normed convention so that baa denotes $(ba)a$. Many of the presentations involve a parameter ω denoting a primitive element in $\mathrm{GF}(p)$. For consistency, I assume that for a given value of p the same value of ω is used throughout. A few of the presentations also involve a parameter λ, and one presentation involves two parameters λ, μ. These parameters take values in $\mathrm{GF}(p)$. For example

$$\langle a, b \,|\, baa - babb,\ babba - \lambda babbbb,\ \text{class } 6\rangle$$

is a family of p distinct algebras (one for each value of $\lambda \in \mathrm{GF}(p)$), and

$$\langle a, b, c \,|\, ca + baa,\ cb,\ baab - baaa,\ babb - \lambda baaa,\ \text{class } 4\rangle\ (\lambda \neq 0, 1)$$

is a family of $p-2$ distinct algebras. In the case of parametrized families of Lie algebras like this, the number of descendants given is the total number of descendants of all the algebras in the family.

5.1 Two generator Lie algebras

There are $p+21$ two generator nilpotent Lie algebras over $\mathrm{GF}(p)$ which have immediate descendants of dimension 8. We list these Lie algebras below, grouping them by dimension and class. For $p > 3$ the number of two generator nilpotent Lie algebras of dimension 8 over $\mathrm{GF}(p)$ is

$$5p^2 + 22p + 2\gcd(p-1, 3) + 76.$$

Dimension 5, class 3	Descendants of dimension 8
$\langle a, b \,|\, \text{class } 3\rangle$	1

Dimension 6, class 4	Descendants of dimension 8
$\langle a, b \,|\, baaa,\ baab,\ \text{class } 4\rangle$	$2p + 7$
$\langle a, b \,|\, baab,\ baaa + babb,\ \text{class } 4\rangle$	$\frac{1}{2}(p^2 + 3p + \gcd(p-1, 3) + 11)$
$\langle a, b \,|\, baab,\ babb + \omega baaa,\ \text{class } 4\rangle$	$\frac{1}{2}(p^2 + p - \gcd(p-1, 3) + 7)$

Dimension 7, class 4	Descendants of dimension 8
$\langle a, b \,|\, baaa,\ \text{class } 4\rangle$	$4p + 12$
$\langle a, b \,|\, baaa - babb,\ \text{class } 4\rangle$	$p^2 + 3p + \gcd(p-1, 3) + 9$
$\langle a, b \,|\, baaa - \omega babb,\ \text{class } 4\rangle$	$p^2 + p + 5 - \gcd(p-1, 3)$

Dimension 7, class 5	Descendants of dimension 8
$\langle a, b \mid baa, \text{ class } 5\rangle$	3
$\langle a, b \mid baa - babbb, \text{ class } 5\rangle$	3
$\langle a, b \mid baa - babb, \text{ class } 5\rangle$	5
$\langle a, b \mid baaa, baab, babba, \text{ class } 5\rangle$	$p + 5$
$\langle a, b \mid baaa, babba, baab - babbb, \text{ class } 5\rangle$	$p^2 + p$
$\langle a, b \mid baab, babba, baaa - babbb, \text{ class } 5\rangle$	$p^2 + p + 2 + \gcd(p - 1, 3)$

Dimension 7, class 6	Descendants of dimension 8
$\langle a, b \mid baa, babba, \text{ class } 6\rangle$	4
$\langle a, b \mid baa - babbbb, babba, \text{ class } 6\rangle$	4
$\langle a, b \mid baa, babba - babbbb, \text{ class } 6\rangle$	1
$\langle a, b \mid baa - babbb, babba, \text{ class } 6\rangle$	$p + 1$
$\langle a, b \mid baa - babbb - babbbb, babba, \text{ class } 6\rangle$	$p + 1$
$\langle a, b \mid baa - babbb, babba - babbbb, \text{ class } 6\rangle$	1
$\langle a, b \mid baa - babb, babba - \lambda babbbb, \text{ class } 6\rangle$ $(\lambda \neq 1)$	$3p + 1$
$\langle a, b \mid baa - babb, babba - babbbb, \text{ class } 6\rangle$	$2 + \gcd(p - 1, 3)$
$\langle a, b \mid baa - babb - babbbb, babba - babbbb, \text{ class } 6\rangle$	p
$\langle a, b \mid baa - babb - \omega babbbb, babba - babbbb, \text{ class } 6\rangle$	p

5.2 Three generator Lie algebras

The total number of three generator nilpotent Lie algebras of dimension 8 over GF(p) for $p > 3$ is

$$p^4 + p^3 + 10p^2 + 82p + (3p + 19)\gcd(p - 1, 3) + 3\gcd(p - 1, 4) + 522.$$

There is only one three algebra of dimension 5 or less with immediate descendants of dimension 8.

Dimension 5, class 2	Descendants of dimension 8
$\langle a, b, c \mid cb = 0, \text{ class } 2\rangle$	28

There is only one three generator algebra of dimension 6 and class 2, and I was able to compute the number of immediate descendants of dimension 8 using the method introduced by Graham Higman in his PORC paper. But I do not have presentations for the descendants.

Dimension 6, class 2	Descendants of dimension 8
$\langle a, b, c \mid \text{ class } 2\rangle$	$p^4 + p^3 + 5p^2 + 13p + (2p + 3)\gcd(p - 1, 3) + \gcd(p - 1, 4) + 26$

There are 10 three generator algebras of dimension 6 and class 3, but only 4 have immediate descendants of dimension 8. These four algebras have a total of $2p + 32$ immediate descendants of dimension 8.

Dimension 6, class 3	Descendants of dimension 8
$\langle a,b,c \mid\ ca,\ cb,\ \text{class } 3\rangle$	9
$\langle a,b,c \mid\ ca-bab,\ cb,\ \text{class } 3\rangle$	11
$\langle a,b,c \mid\ ca-baa,\ cb,\ \text{class } 3\rangle$	$p+8+(\gcd(p-1,3)-1)/2$
$\langle a,b,c \mid\ ca-bab,\ cb-\omega baa,\ \text{class } 3\rangle$	$p+4-(\gcd(p-1,3)-1)/2$

There are 5 three generator algebras of dimension 6 and class 4, but only 4 have immediate descendants of dimension 8. These four algebras have a total of

$$2p+25+\gcd(p-1,3)+\gcd(p-1,4)$$

immediate descendants of dimension 8.

Dimension 6, class 4	Descendants of dimension 8
$\langle a,b,c \mid\ baa,\ ca-babb,\ cb,\ \text{class } 4\rangle$	6
$\langle a,b,c \mid\ baa-babb,\ ca-babb,\ cb,\ \text{class } 4\rangle$	$p+2+\gcd(p-1,3)$
$\langle a,b,c \mid\ baa,\ ca,\ cb,\ \text{class } 4\rangle$	$10+\gcd(p-1,4)$
$\langle a,b,c \mid\ baa-babb,\ ca,\ cb,\ \text{class } 4\rangle$	$p+7$

There are $p+27+\gcd(p-1,3)$ three generator algebras of dimension 7 and class 3, and 26 of then have immediate descendants of dimension 8, including one which only arises when $p=1 \bmod 3$ and one which only arises when $p=2 \bmod 3$. The total number of descendants of dimension 8 is $3p^2+27p+8\gcd(p-1,3)+179$.

Dimension 7, class 3	Descendants of dimension 8
$\langle a,b,c \mid cb,\ caa,\ cab,\ cac,\ \text{class } 3\rangle$	$p+23$
$\langle a,b,c \mid cb-baa,\ caa,\ cab,\ cac,\ \text{class } 3\rangle$	$p^2+4p+11+\gcd(p-1,3)$
$\langle a,b,c \mid cb,\ bab,\ bac,\ cac,\ \text{class } 3\rangle$	8
$\langle a,b,c \mid cb-baa,\ bab,\ bac,\ cac,\ \text{class } 3\rangle$	p^2+5p+3
$\langle a,b,c \mid cb,\ bac,\ caa-bab,\ cac,\ \text{class } 3\rangle$	$p+10$
$\langle a,b,c \mid cb-baa,\ bac,\ caa-bab,\ cac,\ \text{class } 3\rangle$	$2p+3$
$\langle a,b,c \mid cb,\ baa,\ bac,\ cac,\ \text{class } 3\rangle$	17
$\langle a,b,c \mid cb-caa,\ baa,\ bac,\ cac,\ \text{class } 3\rangle$	$5p+7+3\gcd(p-1,3)$
$\langle a,b,c \mid cb,\ bac,\ caa,\ cac-bab,\ \text{class } 3\rangle$	$\frac{p+9}{2}$
$\langle a,b,c \mid cb,\ bac,\ caa,\ cac-\omega bab,\ \text{class } 3\rangle$	$\frac{p+9}{2}$
$\langle a,b,c \mid cb-baa,\ bac,\ caa,\ cac-bab,\ \text{class } 3\rangle$	$\frac{p(p+1)}{2}+1$
$\langle a,b,c \mid cb-baa,\ bac,\ caa,\ cac-\omega bab,\ \text{class } 3\rangle$	$\frac{p(p+1)}{2}+1$
$\langle a,b,c \mid cb,\ bac,\ caa-baa,\ cac+bab,\ \text{class } 3\rangle$	9

In the next three tables we save space in the presentations by only specifying the relators. The generators a,b,c and the class are to be understood.

Dimension 7, class 3	Descendants of dimension 8
$cb - baa, bac, caa - baa, cac + bab$	$p + 4$
cb, baa, bac, caa	$p + 10 + 3\gcd(p - 1, 3)$
$cb, bac, caa, cac - baa$	$p + 7$
$cb, bac, caa - bab, cac - baa$	$(4p - (p - 1)\gcd(p - 1, 3) + 14)/6$
$cb, bac, caa - \omega bab, cac - baa\,(p = 1 \bmod 3)$	$\frac{p+8}{3}$
cb, baa, caa, cac	14
$cb, baa, caa - bab, cac$	$3 + \gcd(p - 1, 3)$
$cb, bab - baa, caa, cac$	10
$cb, baa, caa, cac - \omega bab$	$p + 8 - \gcd(p - 1, 3)$
$cb, baa, caa - bac, cac - \omega bab$	$\frac{(p+1)\gcd(p-1,3)+12}{6}$
$cb, baa, caa - kbab - bac, cac - \omega bab\,(p = 2 \bmod 3)$	$\frac{p+4}{3}$
$bab,\ caa,\ cab,\ cac,\ cba,\ cbb,\ cbc$	$2p + 14$
$bab - baa, bac, caa - baa, cab, cac + baa, cbb + baa, cbc - baa$	2

In the second from last of these algebras, k is chosen so that it is not a value of

$$\frac{\lambda(\lambda^2 + 3\omega\mu^2)}{\mu(3\lambda^2 + \omega\mu^2)}.$$

There are $4p+33$ three generator algebras of class 4 and dimension 7, and $3p+28$ of these are capable. The total number of descendants of dimension 8 of these algebras is

$$2p^2 + 31p + (p + 5)\gcd(p - 1, 3) + \gcd(p - 1, 4) + 196.$$

Dimension 7, class 4	Descendants of dimension 8
$ca,\ cb,\ baaa,\ baab$	11
$ca,\ cb,\ baaa,\ baab$	$9+\gcd(p-1,3)$
$ca,\ cb,\ baab,\ babb+baaa$	$p+15$
$ca,\ cb,\ baab,\ babb+\omega baaa$	$p+5$
$ca+bab,\ cb,\ baaa,\ baab$	8
$ca+bab,\ cb,\ baab,\ babb$	$p+8$
$ca+bab,\ cb-baaa,\ baab,\ babb$	$2p+6$
$ca+bab,\ cb,\ baab,\ babb+\frac{1}{2}baaa$	$\frac{p+5}{2}$
$ca+bab,\ cb,\ baab,\ babb+\frac{\omega}{2}baaa$	$\frac{p+3}{2}$
$ca+bab,\ cb,\ baaa,\ babb$	8
$ca+baa,\ cb,\ baaa,\ baab$	$p+12+\gcd(p-1,3)$
$ca+baa-babb,\ cb,\ baaa,\ baab$	$p^2+4p+p\gcd(p-1,3)+1$
$ca+baa,\ cb,\ baab-baaa,\ babb-baaa$	$p+5$
$ca+baa,\ cb,\ baab,\ babb-baaa$	$p+1+\gcd(p-1,4)$
$ca+baa,\ cb,\ baab,\ babb-\omega baaa$	$p+3$
$ca+baa,\ cb,\ baaa,\ babb$	3
$ca+baa,\ cb,\ baaa,\ babb-baab$	$p+3$
$ca+baa,\ cb,\ baab-baaa,\ babb-\lambda baaa\,(\lambda\neq 0,1)$	$\frac{1}{2}p^2+\frac{1}{2}p-4$
$ca+bab,\ cb+\omega baa,\ baab,\ babb$	$p+5$
$ca+bab,\ cb+\omega bab,\ baab-\lambda baaa,\ babb-\mu baaa$	$\frac{1}{2}p^2+\frac{3}{2}p+2$

In this last algebra above the parameters (λ,μ) take the following values:

1. $(0,-\omega)$,
2. (λ,ω) where $\lambda^2-\omega$ is a square (all the corresponding algebras are isomorphic),
3. (λ,ω) where $\lambda^2-\omega$ is not a square (all the corresponding algebras are isomorphic),
4. $(\lambda,0)$ where $1\leq\lambda\leq(p-1)/2$ (giving $(p-1)/2$ different algebras),
5. (λ,μ) where $\lambda^2-\mu$ is not a square, $\mu\neq\omega$, $\lambda\neq 0$ if $\mu=-\omega$; these parameters give $(p-3)/2$ different algebras with two pairs (λ,μ), (λ',μ') giving isomorphic algebras if $(\lambda,\mu)=(\lambda',\mu')$ or if

$$(\lambda',\mu')=\left(\frac{r^2\lambda+r(\omega+\mu)+\omega\lambda}{r^2+2r\lambda+\mu},\frac{r^2\mu+2r\omega\lambda+\omega^2}{r^2+2r\lambda+\mu}\right)$$

for some $r\in\mathrm{GF}(p)$.

Dimension 7, class 4	Descendants of dimension 8
$cb,\ baa,\ caa,\ cab,\ cac$	$p+20$
$cb,\ baa-babb,\ caa,\ cab,\ cac$	$p+19+\gcd(p-1,3)$
$cb,\ baa,\ caa,\ cab-babb,\ cac$	$4p+2$
$cb,\ baa,\ caa-babb,\ cab,\ cac$	$4+2\gcd(p-1,3)$
$cb,\ baa,\ caa-babb,\ cab-babb,\ cac$	$p+1$
$cb,\ bab,\ bac,\ caa,\ cac$	12
$cb,\ bab-baaa,\ bac,\ caa,\ cac$	8
$cb-baaa,\ bab,\ bac,\ caa,\ cac$	$2p+7$
$cb-baaa,\ bab-baaa,\ bac,\ caa,\ cac$	$2p+3$
$cb,\ bab,\ bac-baaa,\ caa,\ cac$	1
$cb-baa,\ bab,\ bac-\lambda baaa,\ caa,\ cac$	$2p+13$
$cb-baa,\ bab-baaa,\ bac+\frac{1}{2}baaa,\ caa,\ cac$	1

There are $20+\gcd(p-1,3)$ three generator algebras of class 5 and dimension 7, and 13 of these algebras are capable and they have a total of

$$7p+36+2\gcd(p-1,3)$$

descendants of dimension 8.

Dimension 7, class 5	Descendants of dimension 8
$\langle a,b,c\,\vert\, ca+babb,\ cb,\ baa,\ babba,\ \text{class } 5\rangle$	3
$\langle a,b,c\,\vert\, ca+babb-babbb,\ cb,\ baa,\ babba,\ \text{class } 5\rangle$	3
$\langle a,b,c\,\vert\, ca+babb,\ cb,\ baa-babb,\ babba,\ \text{class } 5\rangle$	$p+2$
$\langle a,b,c\,\vert\, ca,\ cb,\ baa,\ babba,\ \text{class } 5\rangle$	6
$\langle a,b,c\,\vert\, ca,\ cb,\ baa-babbb,\ babba,\ \text{class } 5\rangle$	6
$\langle a,b,c\,\vert\, ca-babbb,\ cb,\ baa,\ babba,\ \text{class } 5\rangle$	3
$\langle a,b,c\,\vert\, ca-babbb,\ cb,\ baa-babbb,\ babba,\ \text{class } 5\rangle$	3
$\langle a,b,c\,\vert\, ca,\ cb,\ baa-babb,\ babba,\ \text{class } 5\rangle$	$2p+4$
$\langle a,b,c\,\vert\, ca-babbb,\ cb,\ baa-babb,\ babba,\ \text{class } 5\rangle$	$2p$
$\langle a,b,c\,\vert\, ca+bab,\ cb,\ baa,\ \text{class } 5\rangle$	$2+\gcd(p-1,3)$
$\langle a,b,c\,\vert\, ca+bab-babbb,\ cb,\ baa,\ \text{class } 5\rangle$	$\frac{p+3}{2}$
$\langle a,b,c\,\vert\, ca+bab-\omega babbb,\ cb,\ baa,\ \text{class } 5\rangle$	$\frac{p+3}{2}$
$\langle a,b,c\,\vert\, ca+bab,\ cb,\ baa-babbb,\ \text{class } 5\rangle$	$p+1+\gcd(p-1,3)$

5.3 Four generator Lie algebras

The total number of nilpotent four generator Lie algebras over GF(p) of dimension 8 for $p>3$ is

$$p^3+5p^2+43p+8\gcd(p-1,3)+2\gcd(p-1,4)+502$$

Dimension 4, class 1	Descendants of dimension 8
$\langle a, b, c, d \mid \text{class } 1\rangle$	4

There are 4 four generator Lie algebras of dimension 6 and class 2 and they all have immediate descendants of dimension 8. The total number of these descendants is

$$p^3 + 3p^2 + 17p + 5\gcd(p-1,3) + 2\gcd(p-1,4) + 211$$

In the table below we just give the relators of the Lie algebras to save space, with the generators a, b, c, d and the class to be understood.

Dimension 6, class 2	Descendants of dimension 8
cb, da, db, dc	$p^3 + 3p^2 + 17p + 4\gcd(p-1,3) + 2\gcd(p-1,4) + 136$
ca, cb, da, db	$28 + \frac{9}{2}(\gcd(p-1,3) - 1)$
cb, da, $db - ca$, dc	32
cb, da, $db - ca$, $dc - \omega ba$	$16 - \frac{7}{2}(\gcd(p-1,3) - 1)$

There are six 4 generator class 2 Lie algebras of dimension 7 and they are all capable. The total number of descendants of dimension 8 is $6p + 2\gcd(p-1,3) + 97$.

Dimension 7, class 2	Descendants of dimension 8
$\langle a, b, c, d \mid da,\ db,\ dc, \text{ class } 2\rangle$	$p + 9 + \gcd(p-1,3)$
$\langle a, b, c, d \mid ca,\ da,\ db, \text{ class } 2\rangle$	$p + 39$
$\langle a, b, c, d \mid ca,\ da,\ dc, \text{ class } 2\rangle$	12
$\langle a, b, c, d \mid ca,\ da,\ dc - ba, \text{ class } 2\rangle$	$p + 15$
$\langle a, b, c, d \mid ba,\ da - cb,\ dc, \text{ class } 2\rangle$	$2p + 15 + \gcd(p-1,3)$
$\langle a, b, c, d \mid da,\ db - \omega ca,\ dc - ba, \text{ class } 2\rangle$	$p + 7$

There are 33 four generator class 3 algebras of dimension 7, and 20 of these are capable. They have a total of $2p^2 + 20p + 152$ descendants of dimension 8.

Dimension 7, class 3	Descendants of dimension 8
$\langle a, b, c, d \mid ca,\ cb,\ da,\ db,\ dc,\ \text{class } 3\rangle$	7
$\langle a, b, c, d \mid ca,\ cb - baa,\ da,\ db,\ dc,\ \text{class } 3\rangle$	13
$\langle a, b, c, d \mid ca,\ cb - bab,\ da,\ db,\ dc,\ \text{class } 3\rangle$	$2p + 11$
$\langle a, b, c, d \mid ca - bab,\ cb - \omega baa,\ da,\ db,\ dc,\ \text{class } 3\rangle$	$2p + 5$
$\langle a, b, c, d \mid ca,\ cb - baa,\ da,\ db - bab,\ dc,\ \text{class } 3\rangle$	$3p + 8$
$\langle a, b, c, d \mid ca - bab,\ cb + baa,\ da,\ db - bab,\ dc,\ \text{class } 3\rangle$	$p^2 + 2p + 5$
$\langle a, b, c, d \mid ca - bab,\ cb + \omega baa,\ da,\ db - bab,\ dc,\ \text{class } 3\rangle$	$p^2 + p + 2$
$\langle a, b, c, d \mid ca,\ cb,\ da,\ db,\ dc - baa,\ \text{class } 3\rangle$	4
$\langle a, b, c, d \mid ca - bab,\ cb,\ da,\ db,\ dc - baa,\ \text{class } 3\rangle$	$p + 4$
$\langle a, b, c, d \mid cb,\ da,\ db,\ dc,\ baa,\ bac,\ caa,\ cac,\ \text{class } 3\rangle$	18
$\langle a, b, c, d \mid cb,\ da - bab,\ db,\ dc,\ baa,\ bac,\ caa,\ cac,\ \text{class } 3\rangle$	6
$\langle a, b, c, d \mid cb,\ da,\ db,\ dc,\ bab,\ bac,\ caa,\ cac,\ \text{class } 3\rangle$	16
$\langle a, b, c, d \mid cb,\ da,\ db - baa,\ dc,\ bab,\ bac,\ caa,\ cac,\ \text{class } 3\rangle$	6
$\langle a, b, c, d \mid cb,\ da,\ db,\ dc - baa,\ bab,\ bac,\ caa,\ cac,\ \text{class } 3\rangle$	5
$\langle a, b, c, d \mid cb - baa,\ da,\ db,\ dc,\ bab,\ bac,\ caa,\ cac,\ \text{class } 3\rangle$	$4p + 4$
$\langle a, b, c, d \mid ca,\ cb,\ da,\ db,\ baa,\ dcc,\ dcd,\ \text{class } 3\rangle$	9
$\langle a, b, c, d \mid ca - bab,\ cb,\ da,\ db,\ baa,\ dcc,\ dcd,\ \text{class } 3\rangle$	13
$\langle a, b, c, d \mid cb,\ da,\ db - ca,\ dc - baa,\ bab,\ bac,\ bad,\ \text{class } 3\rangle$	4
$\langle a, b, c, d \mid cb,\ da,\ db - ca,\ dc,\ bab,\ bac,\ bad,\ \text{class } 3\rangle$	$2p + 11$
$\langle a, b, c, d \mid cb - baa,\ da,\ db - ca,\ dc,\ bab,\ bac,\ bad,\ \text{class } 3\rangle$	$3p + 1$

There are 10 four generator Lie algebras of dimension 7 and class 4 and 6 of these are capable. They have a total of $42 + \gcd(p - 1, 3)$ descendants of dimension 8.

Dimension 7, class 4	Descendants of dimension 8
$\langle a, b, c, d \mid ca\,,\ cb,\ da,\ db,\ dc,\ baa,\ \text{class } 4\rangle$	9
$\langle a, b, c, d \mid ca\,,\ cb,\ da,\ db,\ dc,\ baa - babb,\ \text{class } 4\rangle$	6
$\langle a, b, c, d \mid ca - babb\,,\ cb,\ da,\ db,\ dc,\ baa,\ \text{class } 4\rangle$	9
$\langle a, b, c, d \mid ca - babb\,,\ cb,\ da,\ db,\ dc,\ baa - babb,\ \text{class } 4\rangle$	$6 + \gcd(p - 1, 3)$
$\langle a, b, c, d \mid ca + bab,\ cb,\ da,\ db,\ dc,\ baa,\ \text{class } 4\rangle$	10
$\langle a, b, c, d \mid ca + bab,\ cb,\ da - babb,\ db,\ dc,\ baa,\ \text{class } 4\rangle$	2

5.4 Five generator Lie algebras

The total number of nilpotent five generator Lie algebras of dimension 8 over GF(p) for $p > 3$ is 123.

Dimension 5, class 1	Descendants of dimension 8
$\langle a, b, c, d, e \mid \text{class } 1\rangle$	22

There are two five generator Lie algebras of dimension 6 and class 2, but only one is capable.

Dimension 6, class 2	Descendants of dimension 8
$\langle a, b \mid \text{class } 2\rangle \oplus \langle c\rangle \oplus \langle d\rangle \oplus \langle e\rangle$	23

There are 6 five generator nilpotent Lie algebras of class 2 and dimension seven. They are all capable, and have a total of 64 descendants of dimension 8.

Dimension 7, class 2	Descendants of dimension 8
$\langle a, b, c, d, e \mid cb,\ da,\ db,\ dc,\ ea,\ eb,\ ec,\ ed, \text{ class } 2\rangle$	21
$\langle a, b, c, d, e \mid ca,\ cb,\ da,\ db,\ ea,\ eb,\ ec,\ ed, \text{ class } 2\rangle$	11
$\langle a, b, c, d, e \mid cb,\ da,\ db - ca,\ dc,\ ea,\ eb,\ ec,\ ed, \text{ class } 2\rangle$	12
$\langle a, b, c, d, e \mid cb,\ da,\ db - ca,\ dc - \omega ba,\ ea,\ eb,\ ec,\ ed, \text{ class } 2\rangle$	4
$\langle a, b, c, d, e \mid cb,\ da,\ db,\ dc,\ ea,\ eb,\ ec,\ ed - ba, \text{ class } 2\rangle$	12
$\langle a, b, c, d, e \mid cb,\ da,\ db - ca,\ dc,\ ea,\ eb,\ ec,\ ed - ba, \text{ class } 2\rangle$	4

There are 4 five generator nilpotent Lie algebras of class 3 and dimension seven and 2 of these are capable. They have 14 descendants of dimension 8.

Dimension 7, class 3	Descendants of dimension 8
$\langle a, b, c, d, e \mid bab,\ ca,\ cb,\ da,\ db,\ dc,\ ea,\ eb,\ ec,\ ed, \text{ class } 3\rangle$	8
$\langle a, b, c, d, e \mid bab,\ ca,\ cb - baa,\ da,\ db,\ dc,\ ea,\ eb,\ ec,\ ed, \text{ class } 3\rangle$	6

5.5 Six generator Lie algebras

There are 19 nilpotent six generator Lie algebras of dimension 8 over GF(p) for all $p > 3$.

There are 14 six generator class two Lie algebras of dimension 8 over GF(p) for all $p > 2$.

Dimension 6, class 1	Descendants of dimension 8
$\langle a, b, c, d, e, f \mid \text{ class } 1\rangle$	14

There are three six generator Lie algebras of dimension 7 and class 2, but only one is capable. It has 5 descendants of dimension 8.

Dimension 7, class 2	Descendants of dimension 8
$\langle a, b \mid \text{class } 2\rangle \oplus \langle c\rangle \oplus \langle d\rangle \oplus \langle e\rangle \oplus \langle f\rangle$	5

5.6 Seven generator Lie algebras

Dimension 7, class 1	Descendants of dimension 8
$\langle a, b, c, d, e, f, g \mid \text{ class } 1\rangle$	3

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