Comments on "A straightforward solution to Burnside's problem" by Seymour Bachmuth

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For several years now Seymour Bachmuth has been circulating various versions of a paper which claims to prove that the two generator Burnside group of prime-power exponent q, $B(2,q)$, is finite. In 2008 he posted a version of this paper on arXiv:0803.1612. In 2016 he posted another version of the paper on arXiv:1603.08421. In all these papers Seymour constructs a two generator group $F(\mathcal{S}[t,t^{-1}])$, establishes various properties of this group, including the fact that it is solvable, and then claims that $B(2,q)$ is a homomorphic image of $F(\mathcal{S}[t,t^{-1}])$. His group is indeed solvable. So if it were true that $B(2,q)$ is a homomorphic of $F(\mathcal{S}[t,t^{-1}])$ then indeed $B(2,q)$ would be finite (whatever Adjan, Ol'schanskii and Rips might say about the matter). But of course it isn't true, and I have never been able to fathom out why Seymour should think it is true. As far as I can make out, Seymour seems to have an intuition that it is "obvious" that $B(2,q)$ is a homomorphic image of his group $F(\mathcal{S}[t,t^{-1}])$. The various versions of his paper seem mainly to differ in his attempts to explain why this is true, but I have never been able to make any sense of his explanations. In fact his explanations seem to be mostly gobbledygook. (For a year or more we corresponded regularly about this point.) It seems that in this latest version of his paper he has yet another explanation containing four pages of gobbledygook.

In this latest version of his paper, Seymour takes 10 pages of somewhat confusing mathematics to define his group and prove that it is solvable. But in fact the construction of the group takes only a few lines, and you can prove that it is solvable in a page and a half. In fact the normal closure of one of the two generators of his group is nilpotent, and solvability follows immediately from this. (Seymour is aware that the normal closure of one of the two generators is nilpotent, but has not recorded this key fact in any versions of his paper that I have seen.) In Section 1 below I will define his group and prove that it is solvable. In Section 2 I will discuss how to show that Seymour must be wrong without appealing to the various negative solutions of the Burnside problem.

1 The group $F(\mathcal{S}[t, t^{-1}])$

Seymour lets F be the group generated by two 2×2 matrices M_1, M_2T, with entries in the polynomial ring $\mathbb{Z}[x, x^{-1}, y, y^{-1}, t, t^{-1}]$ where

$$M_1 = \begin{pmatrix} 1 & 1-y \\ 0 & x \end{pmatrix}, \ M_2T = \begin{pmatrix} yt & 0 \\ 1-xt & 1 \end{pmatrix}.$$

He lets $\mathcal{R} = \mathbb{Z}[x, x^{-1}, y, y^{-1}]$, and lets $\mathcal{S}$ be the quotient ring $\mathcal{R}/J$ where J is defined below. Then he defines $F(\mathcal{S}[t, t^{-1}])$ to be the group generated by M_1 and M_2T, where the coefficients are taken to lie in $\mathcal{S}[t, t^{-1}]$. To define J, we first let $\sum$ be the ideal of $\mathcal{R}$ generated by $1-x, 1-y$. (Seymour defines $\sum$ to be the augmentation ideal of $\mathcal{R}$, which makes sense if you view $\mathcal{R}$ as the group ring of the free (multiplicative) abelian group of rank 2 generated by x, y.) Then we let $\mathcal{I}(q)$ be the ideal of $\mathcal{R}$ generated by the elements $1 + u + u^2 + \ldots + u^{q-1}$ with u ranging over the elements $x^i y^j$ with $i, j \in \mathbb{Z}$. Finally we set $J = \mathcal{I}(q)\sum$. Seymour appeals to Theorem B in [1] to prove that if $q = p^e$ (where p is prime) then $\sum^{e(p^e-p^{e-1}} \leq \mathcal{I}(q)$, and hence that $\sum^{e(p^e-p^{e-1})+1} \leq J$. Note that if $q = p$ (and $e = 1$) then we have $\sum^p \leq J$.

Theorem 1 *Let K be the normal closure of M_1 in $F(\mathcal{S}[t, t^{-1}])$. Then K is nilpotent of class $e(p^e - p^{e-1})$.*

Proof. We show by induction on k that the elements in $\gamma_k(K)$ can be expressed in the form $I + A$ where I is the 2×2 identity matrix and

$$A = \sum_{i+j=k} (1-x)^i (1-y)^j A_{ij}$$

where the A_{ij} are 2×2 matrices with entries in $\mathcal{S}[t, t^{-1}]$. The theorem follows immediately from this since if $i+j = e(p^e - p^{e-1})+1$ then $(1-x)^i(1-y)^j \in J$ by Theorem B in [1].

First we consider the case $k = 1$. We have

$$M_1 = I + (1-x)\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + (1-y)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$M_1^{-1} = I + (1-x)\begin{pmatrix} 0 & 0 \\ 0 & x^{-1} \end{pmatrix} + (1-y)\begin{pmatrix} 0 & -x^{-1} \\ 0 & 0 \end{pmatrix}.$$

So if $B \in F(\mathcal{S}[t,t^{-1}])$ then

$$B^{-1}M_1B = I + (1-x)B^{-1}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}B + (1-y)B^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}B$$

and

$$B^{-1}M_1^{-1}B = I + (1-x)B^{-1}\begin{pmatrix} 0 & 0 \\ 0 & x^{-1} \end{pmatrix}B + (1-y)B^{-1}\begin{pmatrix} 0 & -x^{-1} \\ 0 & 0 \end{pmatrix}B.$$

It follows that any product of conjugates of M_1 and its inverse can be expressed in the form

$$I + (1-x)A_1 + (1-y)A_2$$

which establishes the case $k = 1$ of our induction.

Now let $C \in \gamma_k(K)$ and let $D \in K$ and consider $[C, D]$. By induction we may assume that $C = I + A$ where

$$A = \sum_{i+j=k} (1-x)^i(1-y)^j A_{ij},$$

and we may assume that $D = I + B$ where

$$B = (1-x)B_1 + (1-y)B_2.$$

We may also assume that $C^{-1} = I + U$ where

$$U = \sum_{i+j=k} (1-x)^i(1-y)^j U_{ij},$$

and we may assume that $D^{-1} = I + V$ where

$$V = (1-x)V_1 + (1-y)V_2.$$

Since $C^{-1}C = D^{-1}D = I$ we have

$$U + A + UA = V + B + VB = 0.$$

So

$$\begin{aligned}[C,D] &= (I+U)(I+V)(I+A)(I+B) \\ &= I + UV + UB + VA + AB\end{aligned}$$

and clearly $UV + UB + VA + AB$ can be expressed in the form

$$\sum_{i+j=k+1} (1-x)^i (1-y)^j W_{ij}$$

as required. The same argument shows that $[C, D]^{-1} = [D, C]$ has the required form, and it follows that every element of $\gamma_{k+1}(K)$ has the required form.

This completes our induction, and proves the theorem. □

The fact that $F(\mathcal{S}[t, t^{-1}])$ is solvable follows immediately from Theorem 1.

2 Refutation?

Seymour says with some justification that very few people claim to understand the negative solutions of the Burnside problem given by Adjan, Ol'schanskii and Rips. He says that since his positive "solution" of the Burnside problem is so short and easy to understand they must just be wrong. Now as well as contradicting the negative solutions of the Burnside problem, Seymour's results also contradict some known facts about the groups $R(2, p)$. (Here $R(2, p)$ is the largest finite quotient of $B(2, p)$.) In particular, if $B(2, p)$ were a homomorphic image of $F(\mathcal{S}[t, t^{-1}])$, then Theorem 1 would imply that one of the generators of $B(2, p)$ would have a normal closure which is nilpotent of class $p - 1$. This, of course, in turn would imply that both of the generators of $B(2, p)$ would have normal closures which were nilpotent of class $p - 1$, so that $B(2, p)$ would have class at most $2p - 2$. This contradicts the known results that $R(2, 5)$ has class 12 and $R(2, 7)$ has class 28. The problem is that these results come from computer calculations using the p-quotient algorithm, and Seymour (of course!) says that the programs must have a bug in them. Some years ago he asked me why nobody had produced a matrix representation or a permutation group representation of $R(2, 5)$, and I took this up as a challenge. Now $R(2, 5)$ has a core free subgroup of index 5^9, and so it is quite easy to produce a faithful permutation representation of degree 5^9 for $R(2, 5)$. But I didn't feel that was very satisfactory. So I looked for a matrix representation and managed to find two 66×66 upper unitriangular matrices over GF(5) which generate a copy of $R(2, 5)$. These matrices can be found on my website `http://users.ox.ac.uk/~vlee/selected.htm`. Seymour (of course!!) said that he couldn't multiply them by hand, and was unable to verify that the group has exponent 5. As mentioned above, $R(2, 5)$

has class 12 and the Sylow p-subgroup of $\mathrm{GL}(n,p)$ has class $n-1$, so any representation of $R(2,5)$ as a subgroup of $\mathrm{GL}(n,5)$ would need n to be at least 13. Nobody is going to multiply 13×13 matrices with entries in GF(5) by hand, but you would think that even Seymour might believe that a computer could multiply 66×66 matrices with entries in GF(5) correctly. If you trust the computer to multiply matrices then it is easy to check that the two matrices I found generate a class 12 group of order 5^{34}. To check that the exponent of the group is 5, you need to make use of a very well understood theorem of Graham Higman [2] which implies that to check that a group of class c has exponent dividing n, then it is only necessary to check that $w^n = 1$ for all words w of length at most c in the generators of the group. So in the case of a two generator group of class 12, to check that the group has exponent 5 it is only necessary to check that $w^5 = 1$ for roughly 2^{13} words w in the two generators. You would need to write a short bit of code to do this, but there would be no need to make use of any complicated computer algorithms.

There is also a completely computer free way of contradicting Seymour's claim. Sanov [3] proves that the relations of weight at most $2p-2$ in the associated Lie ring of $R(2,p)$ all follow from the identity $px = 0$ and the $(p-1)$-Engel identity. Using this result there is a very short hand proof that if $R(2,5)$ is generated by x and y then there are non-trivial commutators in $R(2,5)$ of multiweight $(5,3)$ in x and y, whereas by Theorem 1 all commutators of multiweight $(5,3)$ in the generators of $F(\mathcal{S}[t,t^{-1}])$ are trivial (when $p = 5$). When $p > 5$ there is similarly a very short hand proof that $R(2,p)$ has non-trivial commutators of multiweight $(p,2)$. A complete proof of Sanov's theorem and proofs of these claims can be found on my website `http://users.ox.ac.uk/~vlee/sanov2.pdf`.

References

[1] S. Bachmuth, H. Heilbronn, and H.Y. Mochizuki, *Burnside metabelian groups*, Proc. Royal Soc. (London) **307** (1968), 235–250.

[2] G. Higman, *Some remarks on varieties of groups*, Quart. J. Math. Oxford **(2) 10** (1959), 165–178.

[3] I.N. Sanov, *Establishment of a connection between periodic groups with period a prime number and Lie rings*, Izv. Akad. Nauk SSSR, Ser. Mat. **16** (1952), 23–58.