Baldwin, E., and D. Swinarski. (2008) "A Geometric Invariant Theory Construction of Moduli Spaces of Stable Maps,"
International Mathematics Research Papers, Vol. 2008, Article ID rpn004, 104 pages.
doi:10.1093/imrp/rpn004

A Geometric Invariant Theory Construction of Moduli Spaces of Stable Maps

Elizabeth Baldwin[1] and David Swinarski[2]

[1]Mathematical Institute, University of Oxford, 24-29 St Giles', Oxford, OX1 3LB, UK, and [2]Mathematics Department, Columbia University, Room 509, MC 4406, 2990 Broadway, New York, NY 10027, USA

Correspondence to be sent to: baldwin@maths.ox.ac.uk

We construct the moduli spaces of stable maps, $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$, via geometric invariant theory (GIT). This construction is only valid over Spec $\mathbb{C}$, but a special case is a GIT presentation of the moduli space of stable curves of genus g with n marked points, $\overline{\mathcal{M}}_{g,n}$; this is valid over Spec $\mathbb{Z}$. In another paper by the first author, a small part of the argument is replaced, making the result valid in far greater generality. Our method follows the one used in the case $n = 0$ by Gieseker in [9], 1982, *Lectures on Moduli of Curves* to construct $\overline{\mathcal{M}}_g$, though our proof that the semistable set is nonempty is entirely different.

1 Introduction

This paper gives a geometric invariant theory (GIT) construction of the Kontsevich–Manin moduli spaces of stable maps $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$, for any values of (g, n, d) such that smooth stable maps exist. From this we derive a GIT construction of all such moduli spaces of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$, where X is a projective variety and β is a discrete invariant, understood as the homology class of the stable maps. Although the first part of the construction closely follows Gieseker's construction in [9] of the moduli spaces

Received August 27, 2007; Revised March 22, 2008; Accepted March 12, 2008
Communicated by Prof. Dragos Oprea

of stable curves $\overline{\mathcal{M}}_g$, our proof that there exist GIT-semistable n-pointed maps uses an entirely different approach.

Some results of this paper are valid over Spec $\mathbb{Z}$. The GIT construction of $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ is in fact only presented in this paper over $\mathbb{C}$, though it can be extended to work much more generally (see [3]). However, a special case of what we prove here is a GIT construction of the moduli spaces of n-pointed curves, $\overline{\mathcal{M}}_{g,n}$, which works over Spec $\mathbb{Z}$. A GIT construction of $\overline{\mathcal{M}}_{g,n}$ does not seem to have been published previously for $n > 0$.

When constructing moduli spaces via GIT, one usually writes down a parameter space of the desired objects together with some extra structure, and then takes a quotient. In our case, following the construction of [8], this extra structure involves an embedding of the domain curve in projective space. Given an n-pointed stable map $f : (C, x_1, \ldots, x_n) \to \mathbf{P}^r$, we define the natural ample line bundle

$$\mathcal{L} := \omega_C(x_1 + \cdots + x_n) \otimes f^*\mathcal{O}_{\mathbf{P}^r}(c)$$

on C, where c is a sufficiently large positive integer, as shall be discussed in Section 2.4. Choose a sufficiently large so that $\mathcal{L}^a$ is very ample. We fix a vector space of dimension $h^0(C, \mathcal{L}^a)$ and denote it by W. A choice of isomorphism $W \cong H^0(C, \mathcal{L}^a)$ induces an embedding $(C, x_1, \ldots, x_n) \subset \mathbf{P}(W)$ and the graph of the map f is a curve $(C, x_1, \ldots, x_n) \subset \mathbf{P}(W) \times \mathbf{P}^r$.

For our parameter space with extra structure, then, we start with the Hilbert scheme $\mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r) \times \prod_n (\mathbf{P}(W) \times \mathbf{P}^r)$, where the final factors represent the marked points. There is a projective subscheme, I, the incidence subscheme where the n points lie on the curve. This is in fact the Hilbert scheme of n-pointed curves in $\mathbf{P}(W) \times \mathbf{P}^r$. We identify a locally closed subscheme $J \subset I$, corresponding to stable maps which have been embedded as described above. This subscheme is identified by Fulton and Pandharipande; they remark ([8], Remark 2.4) that $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ is a quotient of J by the action of $SL(W)$, and should be presentable via GIT, though they follow a different method.

The main theorems of this paper are stated at the beginning of Section 6. The GIT quotient $\overline{J}//_L SL(W)$ is isomorphic to $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ over $\mathbb{C}$, for a narrow but nonempty range of linearizations L; if we set $r = d = 0$, we obtain $\overline{\mathcal{M}}_{g,n}$ over $\mathbb{Z}$. We prove these results for $n = 0$ by generalizing Gieseker's technique, and then use induction on n.

As alternative constructions of these spaces exist, it is natural to ask why one would go to the (considerable) trouble of constructing them via GIT, especially since the construction of this paper depends on the construction of $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ as a coarse moduli space given in [8]. However, this paper paves the way for a construction independent of [8], over a much more general base, laid out in [3]. The potential stability theorem laid out here (Theorem 5.19) is more generally applicable; in this form it is also an important ingredient in GIT constructions of moduli spaces of stable curves and stable maps with weighted marked points [27], which have been constructed by other methods ([13], [21], [1], [5]). The original motivation behind this construction was to use it as a tool for studying $\overline{\mathcal{M}}_{g,n}$, by constructing that as a GIT quotient of a subscheme of $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$; see [4]. Also, once one has a space constructed via GIT, one may vary the defining linearization to obtain birational transformations of the quotient. Such methods may be relevant to study maps arising from the minimal model program for $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$; cf. [14, 15].

The layout of this paper is as follows: Section 2 is a brief review of background material on the theory of moduli, geometric invariant theory, and stable curves and maps. Much of the material in this section is standard. However, we need to extend some of the theory of variation of GIT. Thaddeus [29] and Dolgachev and Hu [7] have a beautiful picture of the way in which GIT quotients vary with linearization. Unfortunately, these results are only proved for projective varieties, sometimes with the extra condition of normality. In addition, the results of [7] are only given over $\mathbb{C}$. We wish to make use of small parts of this theory in the setting of projective schemes over a field k, and so we make the elementary extensions necessary in Section 2.3.

Let us summarize the material we shall need from the theory of variation of GIT. We work in the real vector space of "virtual linearizations" generated by G-linearized line bundles. One may extend the definitions of stability and semistability to virtual linearizations. We take the cone within this space spanned by ample linearizations. Now, suppose a convex region within this ample cone has the property that no virtual linearization in it defines a strictly semistable point. Variation of GIT tells us that all virtual linearizations in the convex region define the same semistable set.

In Section 2.4 we review some basic facts about stable curves and stable maps.

Our construction begins in Section 3, where we define the scheme J described above, and prove that there exists a family $\mathcal{C} \to J$ with the local universal property for the moduli problem of stable maps. Here we also lay out in detail the strategy for the rest of the paper.

Our aim is to show that for some range of virtual linearizations, GIT semistability implies GIT stability and $\overline{J}^{ss} = J$. However, it will be sufficient for us to show that the semistable set $\overline{J}^{ss}$ is nonempty and contained in J. For, by definition, elements of J have finite stabilizer groups, and so all GIT semistable points will be GIT stable points if $\overline{J}^{ss} \subseteq J$. An argument involving the construction of $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ from [8] allows us to conclude that if $\overline{J}^{ss}$ is a nonempty subset of J, then the quotient $\overline{J}//SL(W)$ must be the entirety of $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$.

As this argument uses the construction of [8], which is only given over Spec $\mathbb{C}$, we can only claim to have constructed $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ over Spec $\mathbb{C}$. However, this is only a shortcut which we use for brevity in this paper. An alternative argument is presented by the first author in [3], which allows us to conclude from $\emptyset \neq \overline{J}^{ss} \subseteq J$ that $\overline{J}//SL(W)$ is $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ over a more general base.

Within this paper, in the special case where $r = d = 0$, we obtain $\overline{\mathcal{M}}_{g,n}$. Gieseker's construction of $\overline{\mathcal{M}}_g$ in [9] works over Spec $\mathbb{Z}$ (although there it is only stated to work over any algebraically closed field). We may use induction to show that the same is true for our GIT presentation of $\overline{\mathcal{M}}_{g,n}$.

In Section 4, we describe the range of virtual linearizations and general GIT setup to be used. The longest part of the paper follows. In Section 5, we gradually refine our choice of virtual linearization so that GIT semistability of an n-pointed map implies that it is "potentially stable." The definition of precisely what is meant by this, and the corresponding theorem, can be found in Section 5.5. With this description of possible semistable curves, we are able to show in Section 5.6 that GIT semistable curves in $\overline{J}$ are indeed in J, at least for a carefully defined range of virtual linearizations. All that is left is to prove nonemptiness of the semistable set.

A further important fact may be deduced at this stage. We have a range of virtual linearizations, which is a convex set in the vector space described above. For this range, semistability is equivalent to stability. It follows that the semistable set is the same for the whole of our range. Thus, nonemptiness need only be proved for one such virtual linearization.

In Section 6, we complete the construction by proving this nonemptiness. This is done by induction on the number n of marked points. Section 6 is therefore divided into two parts: the base case and the inductive step. In Section 6.2, we follow the methods of Gieseker and show that smooth maps are GIT semistable when $n = 0$. This gives us the required nonemptiness.

The inductive step follows a more novel approach, and is laid out in Section 6.3. Given a moduli stable map of genus g with n marked points, we attach an elliptic curve at the location of one of the markings to obtain a new stable map of genus $(g+1)$, with $(n-1)$ marked points. Induction tells us that this has a GIT semistable model, so we have verification of the numerical criterion for GIT semistability for this map. This implies GIT semistability of the original stable map for a virtual linearization within the specified range. We use the constancy of the GIT quotient for the whole of the range to deduce the result.

As we talk here about spaces of maps from curves of differing genera and numbers of marked points, it is necessary to extend the notation J to $J_{g,n,d}$ to specify which space we refer to. The crucial result can then be summarized as

$$\overline{J}^{ss}_{g+1,n-1,d} = J_{g+1,n-1,d} \Longrightarrow \overline{J}^{ss}_{g,n,d} = J_{g,n,d}.$$

In the special case of genus 0 curves, this induction constructs the moduli space $\overline{\mathcal{M}}_{0,n}$ for every $n \geq 3$; the base case for the induction in this case is $\overline{\mathcal{M}}_{n,0}$.

In [28], Swinarski gave a GIT construction of $\overline{\mathcal{M}}_{g,0}(X, \beta)$, the moduli spaces of stable maps without marked points. Baldwin extended this in [2] to marked points. This paper brings together the results from those two theses. Finally, we note that Parker has recently given a very different GIT construction of $\overline{\mathcal{M}}_{0,n}(\mathbf{P}^r, d)$ as a quotient of $\overline{\mathcal{M}}_{0,n}(\mathbf{P}^r \times \mathbf{P}^1, (d, 1))$ in [23].

2 Background Material

There is a certain amount of background material which we must review. Almost all of this section is standard, although in Section 2.3 we must extend some results on variation of geometric invariant theory, to work for arbitrary schemes over a base of any characteristic.

2.1 Moduli and quotients

We shall take the definitions of coarse and fine moduli spaces to be standard. However, our construction will rely on families which have the following property, which ensures that an orbit space quotient of their base is a coarse moduli space.

Definition 2.1 ([22], p. 37)**.** Given a moduli problem, a family $\mathcal{X} \to S$ is said to have the *local universal property* if, for any other family $\mathcal{X}' \to S'$ and any $s \in S'$, there exists a neighborhood U of s in S' and a morphism $\phi : U \to S$ such that $\phi^*\mathcal{X} \sim \mathcal{X}'|_U$. □

Suppose that we also have a group action on the base space S such that orbits correspond to equivalence classes for the moduli problem. Some sort of quotient seems a good candidate as a moduli space, but unfortunately in most cases the naive quotient will not exist as a scheme. What we require instead is a *categorical quotient* ([20], Definition 0.5). We need one additional definition.

Definition 2.2. A categorical quotient (Y, ϕ) of a scheme X by a group G is an *orbit space* if the geometric fibers of ϕ are precisely the orbits of the geometric points of X. □

By definition, a categorical quotient (Y, ϕ) is unique up to isomorphism, and ϕ is a surjective morphism. Now we see that these definitions are enough to provide coarse moduli spaces, as formalized in the following proposition.

Proposition 2.3 ([22], Proposition 2.13)**.** Suppose that the family $\mathcal{X} \to S$ has the local universal property for some moduli problem, and that the algebraic group G acts on S, with the property that $\mathcal{X}_s \sim \mathcal{X}_t$ if and only if $G \cdot s = G \cdot t$. Then

(i) any coarse moduli space is a categorical quotient of S by G;
(ii) a categorical quotient of S by G is a coarse moduli space if and only if it is an orbit space.

□

2.2 Geometric invariant theory

Geometric invariant theory (GIT) is a method to construct categorical quotients. Details of the theory may be found in [20], and the results are extended over more general base in [25] and [26]. More gentle introductions may be found in [22] and [19]. We state here the key concepts.

Recall that a geometric invariant theory quotient depends not only on an algebraic group action on a projective scheme X, but also on a *linearization* of that action ([20], Definition 1.6), which is a lifting of the group action to a line bundle on X.

Line bundles together with linearizations of the action of G form a group, which we denote by $\mathrm{Pic}^G(X)$. An L-linearized action of G on X induces an action of G on the space of sections of L^r, where r is any positive integer. If L is ample, then the quotient scheme we obtain is

$$X/\!/_L G := \mathrm{Proj} \bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n})^G. \tag{1}$$

This is a categorical quotient of an open subset of X, but not necessarily of the whole of X. The rational map $X \dashrightarrow X/\!/_L G$ is only defined at those $x \in X$ where there exists a section $s \in H^0(X, L^{\otimes n})^G$ such that $s(x) \neq 0$. We must identify this open subscheme, and also discover to what extent the categorical quotient is an orbit space. Accordingly we make extra definitions. In the following, we may work with schemes defined over any universally Japanese ring, and in particular over any field or over $\mathbb{Z}$.

Definition 2.4 (cf. [20], Definition 1.7, and [26], Proposition 7 and Remark 9)**.** Let G be a reductive algebraic group, with an L-linear action on the projective scheme X.

(i) A geometric point $x \in X$ is *semistable* (with respect to L and σ) if there exists $s \in H^0(X, L^{\otimes n})^G$ for some $n \geq 0$, such that $s(x) \neq 0$ and the subset X_s is affine. The open subset of X whose geometric points are the semistable points is denoted $X^{ss}_\sigma(L)$.

(ii) A geometric point $x \in X$ is *stable* (with respect to L and σ) if there exists $s \in H^0(X, L^{\otimes n})^G$ for some $n \geq 0$, such that $s(x) \neq 0$ and the subset X_s is affine, the action of G on X_s is closed, and the stabilizer G_x of x is 0-dimensional. The open subset of X whose geometric points are the stable points is denoted $X^s_\sigma(L)$.

(iii) A point $x \in X$ which is semistable but not stable is called *strictly semistable*.

□

In particular, we shall use the following.

Corollary 2.5 ([20], p. 10)**.** If G_x is finite for all $x \in X^{ss}(L)$, then $X^{ss}(L) = X^s(L)$. □

Now the main theorem of GIT is as follows.

Theorem 2.6 ([20], Theorem 1.10, [26], Theorem 4 and Remark 9)**.** Let X be a projective scheme, and G a reductive algebraic group with an L-linear action on X.

(i) A categorical quotient $(X/\!/_L G, \phi)$ of $X^{ss}(L)$ by G exists.

(ii) There is an open subset Y^s of $X/\!/_L G$ such that $\phi^{-1}(Y^s) = X^s(L)$ and (Y^s, ϕ) is an orbit space of $X^s(L)$.

□

This may all be summarized in the diagram

$$\begin{array}{ccccc} X^s(L) & \overset{open}{\subseteq} & X^{ss}(L) & \overset{open}{\subseteq} & X \\ \downarrow & & \downarrow & & \\ X^s(L)/G & \overset{open}{\subseteq} & X/\!/_L G. & & \end{array}$$

Stability and semistability are difficult to prove directly; fortunately the analysis is made much easier by utilizing one-parameter subgroups (1-PSs) of G, i.e. homomorphisms $\lambda : \mathbb{G}_m \to G$. This is the so-called Hilbert–Mumford numerical criterion. It is not used by Seshadri in [26]; although these techniques probably do work for schemes over $\mathbb{Z}$, we shall only need them to apply GIT over a fixed base field, k.

In the following we use the conventions of Gieseker in [9], which are equivalent to but different from those of [20]. Note that throughout this paper, we shall use Grothendieck's convention that if V is a vector space, then $\mathbf{P}(V)$ is the collection of equivalence classes (under scalar action) of the nonzero elements of the dual space $V^\vee$.

Let $\lambda : \mathbb{G}_m \to G$ be a 1-PS of G. Set $x_\infty := \lim_{t\to 0} \lambda(t^{-1}) \cdot x$. The group $\lambda(\mathbb{G}_m)$ acts on the fiber L_{x_∞} via some character $t \mapsto t^R$. Then set

$$\mu^L(x,\lambda) := R.$$

From this perspective, one may see clearly that the map $L \mapsto \mu^L(x,\lambda)$ is a group homomorphism $\mathrm{Pic}^G(X) \to \mathbb{Z}$.

For ample line bundles we have an alternative view. Suppose L is very ample, and consider X as embedded in $\mathbf{P}(H^0(X,L)) =: \mathbf{P}$. We have an induced action of $\lambda(\mathbb{G}_m)$ on $H^0(X,L)$. Pick a basis $\{e_0,\ldots,e_N\}$ of $H^0(X,L)$ such that for some $r_0 \le \cdots \le r_N \in \mathbb{Z}$,

$$\lambda(t)e_i = t^{r_i} e_i \text{ for all } t \in \mathbb{G}_m.$$

If $\{e_0^\vee,\ldots,e_N^\vee\}$ is the dual basis for $H^0(X,L)^\vee$, then the action of $\lambda(t)$ on $H^0(X,L)^\vee$ is given by the weights $-r_0,\ldots,-r_N$.

A point $x \in X$ is represented by some nonzero $\hat{x} = \sum_{i=0}^N x_i e_i^\vee \in H^0(X,L)^\vee$. Let

$$R' := \min\{r_i | x_i \neq 0\} = -\max\{-r_i | x_i \neq 0\}.$$

Then $-R'$ is the maximum of the weights for $\hat{x}$, and so x_∞ is represented by $\hat{x}_\infty := \sum_{r_i=R'} x_i e_i^\vee$. The fiber L_{x_∞} is spanned by $\{e_i(x_\infty) | 0 \le i \le N\}$, but the nonzero part of this set is $\{e_i(x_\infty) | r_i = R'\}$, where by definition $\lambda(\mathbb{G}_m)$ acts via the character $t \mapsto t^{R'}$. Thus

$$\mu^L(x,\lambda) = R' = \min\{r_i | x_i \neq 0\}.$$

We shall refer to the set $\{r_i : x_i \neq 0\}$ as the λ-weights of x. The crucial property is that, for ample linearizations, semistability may be characterized in terms of these minimal weights.

Theorem 2.7 ([21], Theorem 2.1)**.** Let k be a field. Let G be a reductive algebraic group scheme over k, with an L-linear action on the projective scheme X (defined over k), where L is ample. Then

$$x \in X^{ss}(L) \iff \mu^L(x, \lambda) \leq 0 \text{ for all 1-PS } \lambda \neq 0,$$
$$x \in X^{s}(L) \iff \mu^L(x, \lambda) < 0 \text{ for all 1-PS } \lambda \neq 0.$$

□

2.3 Variation of GIT

The semistable set depends on the choice of linearization of the group action. The nature of this relationship is explored in the papers of Thaddeus [29] and Dolgachev and Hu [7] on the variation of GIT. Unfortunately for us, these papers deal only with GIT quotients of projective varieties, sometimes requiring the extra condition of normality. We wish to present $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ as a GIT quotient $\overline{J}//_L SL(W)$, where the scheme $\overline{J}$ will be defined in Section 3.1; as we already know that $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ is in general neither reduced nor irreducible, we cannot expect $\overline{J}$ to have either of these properties.

It seems likely that much of the theory of variation of GIT extends to general projective schemes. We shall here extend the small part that we shall need to use; it is easier to prove nonemptiness of the semistable set $\overline{J}^{ss}$ if we have a certain amount of freedom in the precise choice of linearization. We do not need the full picture of "walls and chambers" as defined by Dolgachev and Hu in [7] and Thaddeus in [29]; developing this theory for general projective schemes would take more work, so we shall not do so here. We shall follow the methods of [7], though we depart from their precise conventions.

We shall assume that X is a projective scheme over a field k. This is more convenient than working over $\mathbb{Z}$ and shall be sufficient for our final results. We further specify for all of the following that the character group $\mathrm{Hom}(G, \mathbb{G}_m)$ is trivial; then there is at most one G-linearization for any line bundle ([20], Proposition 1.4). In particular, this holds for $G = SL(W)$.

We shall write $\mathrm{Pic}^G(X)_{\mathbb{R}}$ for the vector space $\mathrm{Pic}^G(X) \otimes \mathbb{R}$. We shall refer to general elements of $\mathrm{Pic}^G(X)_{\mathbb{R}}$ as "virtual linearizations" of the group action, and denote them with a lower case l to distinguish them from true linearizations, L.

We shall review the construction of the crucial function $M^\bullet(x) : \mathrm{Pic}^G(X)_\mathbb{R} \to \mathbb{R}$. As the map $\mu^\bullet(x,\lambda) : \mathrm{Pic}^G(X) \to \mathbb{Z}$ is a group homomorphism, it may be naturally extended to

$$\mu^\bullet(x,\lambda) : \mathrm{Pic}^G(X)_\mathbb{R} \to \mathbb{Z} \otimes \mathbb{R} = \mathbb{R}.$$

The numerical criterion applies for ample linearizations, so we shall be most interested in their convex cone.

Definition 2.8. The G-linearized ample cone $\mathbf{A}^G(X)_\mathbb{R}$ is the convex cone in $\mathrm{Pic}^G(X)_\mathbb{R}$ spanned by ample line bundles possessing a G-linearization. □

Let T be a maximal torus of G, and let $W = N_G(T)/T$ be its Weyl group. Let $\mathcal{X}_*(G)$ be the set of nontrivial one-parameter subgroups of G. Note that $\mathcal{X}_*(G) = \bigcup_{g \in G} \mathcal{X}_*(gTg^{-1})$. If $\dim T = n$, then we can identify $\mathcal{X}_*(T) \otimes \mathbb{R}$ with $\mathbb{R}^n$. Let $\|\cdot\|$ be a W-invariant Euclidean norm on $\mathbb{R}^n$. Then for any λ in $\mathcal{X}_*(G)$, define $\|\lambda\| := \|g\lambda g^{-1}\|$ where $g\lambda g^{-1} \in \mathcal{X}_*(T)$. For any 1-PS $\lambda \neq 0$, any $x \in X$, and virtual linearization $l \in \mathrm{Pic}^G(X)_\mathbb{R}$, we may set

$$\bar{\mu}^l(x,\lambda) := \frac{\mu^l(x,\lambda)}{\|\lambda\|}.$$

Now our crucial function may be defined as follows.

Definition 2.9. The function $M^\bullet(x) : \mathrm{Pic}^G(X)_\mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is defined as

$$M^l(x) := \sup_{\lambda \in \mathcal{X}_*(G)} \bar{\mu}^l(x,\lambda).$$ □

It is a result of Mumford that if L is an ample line bundle, then $M^L(x)$ is finite ([20], Proposition 2.17); recall that [20] treats GIT over an arbitrary base field k). We observe that $M^\bullet(x)$ is a positively homogeneous lower convex function on $\mathrm{Pic}^G(X)_\mathbb{R}$. Thus $M^\bullet(x)$ is finite-valued on the whole of $\mathbf{A}^G(X)_\mathbb{R}$.

The numerical criterion may be expressed in terms of $M^L(x)$. We use $M^\bullet(x)$ to extend naturally the definitions of stability and semistability to virtual linearizations $l \in \mathbf{A}^G(X)_\mathbb{R}$.

Definition 2.10. Let $l \in \mathbf{A}^G(X)_\mathbb{R}$. Then

$$X^{ss}(l) := \{x \in X : M^l(x) \leq 0\},$$
$$X^s(l) := \{x \in X : M^l(x) < 0\}.$$

□

Using lower convexity we may prove the following lemma.

Lemma 2.11. Suppose $l \in \mathbf{A}^G(X)_{\mathbb{R}}$. If x is semistable with respect to $l_1, \ldots, l_k \in \mathrm{Pic}^G(X)_{\mathbb{R}}$, then it is semistable with respect to all virtual linearizations in the convex hull of $l_1, \ldots, l_k$. □

Now we may prove the result that we shall need.

Proposition 2.12 (cf. [7], Theorem 3.3.2)**.** Suppose $\mathbf{H} \subset \mathbf{A}^G(X)_{\mathbb{R}}$ is a convex region satisfying $X^{ss}(l) = X^s(l)$ for all $l \in \mathbf{H}$. It follows that $X^s(l) = X^{ss}(l) = X^{ss}(l') = X^s(l')$ for all $l, l' \in \mathbf{H}$. □

Proof. Let $x \in X$ be arbitrary. It follows from the assumptions that the function $M^\bullet(x)$ is nonzero in $\mathbf{H}$. Let $l, l' \in \mathbf{H}$ and let V be the vector subspace of $\mathrm{Pic}^G(X)_{\mathbb{R}}$ spanned by l and l'. This has a basis consisting either of l and l', or just of l; use this basis to define a norm and hence a topology on V. Now, since $M^\bullet(x)$ is positively homogeneous lower convex, the restriction

$$M^\bullet(x) : V \cap \mathbf{A}^G(X)_{\mathbb{R}} \to \mathbb{R}$$

is a continuous function. Let L be the line between l and l'. Then $L \subset V \cap \mathbf{H} \subset V \cap \mathbf{A}^G(X)_{\mathbb{R}}$, so $M^\bullet(x)$ is nonzero and continuous on L. Thus, it does not change sign; either $x \in X^s(l'')$ for every $l'' \in L$ or $x \notin X^{ss}(l'')$ for every $l'' \in L$. This holds for all $x \in X$, and so in particular, $X^s(l) = X^s(l')$. ■

Remark. In [29] and [7], the authors use the fact that algebraically equivalent line bundles give rise to the same semistable sets, and so work in the Néron–Severi group

$$\mathrm{NS}^G(X) := \frac{\mathrm{Pic}^G(X)}{\mathrm{Pic}^G_0(X)}.$$

The advantage is that in many cases, this is known to be a finitely generated abelian group, and so $NS^G(X) \otimes \mathbb{R}$ is a finite-dimensional vector space. However, this finite generation does not appear to have been proved in sufficient generality for our purposes. We could show ([2], Proposition 1.3.4) that the group homomorphism $\mu^\bullet(x, \lambda) : \mathrm{Pic}^G(X) \to \mathbb{Z}$ descends to $\mathrm{NS}^G(X)$, but as we would be left with a possibly infinite-dimensional vector space, we have not troubled with the extra definitions and results.

2.4 Stable curves and stable maps

We now turn our attention to the specific objects that we shall study: the coarse moduli spaces of stable curves and of stable maps.

The moduli space $\overline{\mathcal{M}}_{g,n}$ of Deligne–Mumford stable pointed curves is by now very well known. We shall not rehearse all the definitions here and instead simply cite Knudsen's work [17]; lots of background and context is given in [30]. The only terminology we shall use which may not be completely standard is the following: a *prestable curve* is a connected reduced projective curve whose singularities (if there are any) are nodes.

The moduli spaces of stable maps, $\overline{\mathcal{M}}_{g,n}(X,\beta)$ parametrize isomorphism classes of certain maps from pointed nodal curves to X (this will be made precise below). They were introduced as a tool for calculating Gromov–Witten invariants, which are used in enumerative geometry and quantum cohomology.

Fix a projective scheme X. The discrete invariant β may intuitively be understood as the class of the pushforward $f_*[C] \in H_*(X;\mathbb{Z})$. In this paper, we shall only complete the construction of moduli of stable maps over $\mathbb{C}$, and so there is no harm in taking this as the definition of β. For the case of more general schemes X, see ([6], Definition 2.1).

We may define our moduli problem.

Definition 2.13.

(i) A *stable map* of genus g, degree d, and homology class β is a map $f: (C, x_1, \ldots, x_n) \to X$, where C is an n-pointed prestable curve of genus g, the homology class $f_*[C] = \beta$, and the following stability conditions are satisfied: if C' is a nonsingular rational component of C and C' is mapped to a point by f, then C' must have at least three special points (either marked points or nodes); if C' is a component of arithmetic genus 1 and C' is mapped to a point by f, then C' must contain at least one special point. (Note that since we require the domain curves C to be connected, the stability condition on genus 1 components is automatically satisfied except in $\overline{\mathcal{M}}_{1,0}(X,0)$, which is empty).

(ii) A *family of stable maps*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & X \\ \varphi \downarrow\uparrow \sigma_i & & \\ S & & \end{array}$$

is a family $(\mathcal{X} \xrightarrow{\varphi} S, \sigma_1, \ldots, \sigma_n)$ of pointed prestable curves together with a morphism $f : \mathcal{X} \to X$ such that $f_*[\mathcal{X}] = \beta$, and satisfying that $f|_{\mathcal{X}_s} : (\mathcal{X}_s, \sigma_1(s), \ldots, \sigma_n(s)) \to X$ is a stable map for each $s \in S$.

(iv) Two families $(\mathcal{X} \xrightarrow{\varphi} S, \sigma_1, \ldots, \sigma_n, f)$ and $(\mathcal{X}' \xrightarrow{\varphi'} S, \sigma'_1 \ldots, \sigma'_n, f')$ of stable maps are *equivalent* if there is an isomorphism $\tau : \mathcal{X} \cong \mathcal{X}'$ over S, compatible with sections, such that $f' \circ \tau = f$. □

Note that $\overline{\mathcal{M}}_{g,n}(X, 0)$ is simply $\overline{\mathcal{M}}_{g,n} \times X$. In this sense, the Kontsevich spaces generalize the moduli spaces of stable curves. However, although there is an open subscheme $\mathcal{M}_{g,n}(X, \beta)$ corresponding to maps from smooth curves, in general it is not dense in the moduli space of stable maps—in general, $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is reducible and has components corresponding entirely to nodal maps.

In addition, it is very important to note that the domain of a stable map is not necessarily a stable curve! It may have rational components with fewer than three special points (though such components cannot be collapsed by f). The dualizing sheaf may not be ample, even after twisting by the marked points. We use a sheaf that provides an extra twist to all components which are not collapsed,

$$\mathcal{L} := \omega_C(x_1 + \cdots + x_n) \otimes f^*\mathcal{O}_{\mathbf{P}^r}(c),$$

where c is a positive integer, whose magnitude we will discuss below.

In the special case where $X = \mathbf{P}^r$, we may fix an isomorphism $H_2(X; \mathbb{Z}) \cong \mathbb{Z}$ and denote β by an integer $d \geq 0$. For smooth stable maps to exist, we require $2g - 2 + n + 3d > 0$; we shall only consider these cases.

Remark on the magnitude of c. We require $\mathcal{L}$ to be ample on a nodal map if and only if the map is stable. This is certainly true if $c \geq 3$, as then $\mathcal{L}$ is positive on all rational components which are not collapsed by the map or have at least three special points. However, unless we are in the case $g = n = 0$, all rational components have at least one special point, and so $c \geq 2$ will suffice for us. If $g = n = 0$, we in addition ensure that $\mathcal{L}$ is positive on irreducible curves (which now have no special points); this holds when $cd \geq 3$, so it is only in the case $(g, n, d) = (0, 0, 1)$ that we require $c \geq 3$.

We shall construct $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ by GIT. A corollary is a GIT construction of $\overline{\mathcal{M}}_{g,n}(X, \beta)$. An existing construction (not by GIT) is crucial to our proof.

Theorem 2.14 ([8], Theorem 1). Let X be a projective algebraic scheme over $\mathbb{C}$, and let $\beta \in H_2(X; \mathbb{Z})^+$. There exists a projective, coarse moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$. □

In [28], Swinarski gave a GIT construction of $\overline{\mathcal{M}}_{g,0}(X,\beta)$, the moduli spaces of stable maps without marked points. Baldwin extended this in [2] to marked points; this seemingly innocent extension turns out to be very difficult in GIT, because finding a linearization with the required properties becomes much more subtle. This paper brings together the results from those two theses.

We gather together a few more facts that have been proven about these spaces. Most of the progress has been made in the case $g = 0$. If X is a nonsingular convex projective variety, e.g. $\mathbf{P}^r$, then $\overline{\mathcal{M}}_{0,n}(X,\beta)$ is an orbifold projective variety; when nonempty, it has the "expected dimension." However, moduli spaces for stable maps of higher genera have fewer such nice properties. Kim and Pandharipande have shown in [16] that, if X is a homogeneous space G/P, where P is a parabolic subgroup of a connected complex semisimple algebraic group G, then $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is connected. Little more can be said even when $X = \mathbf{P}^r$; the spaces $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r,d)$ are in general reducible, nonreduced, and singular. Further, Vakil has shown in [31] that every singularity of finite type over $\mathbb{Z}$ appears in one of the moduli spaces $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r,d)$.

3 Constructing $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r,d)$: Core Definitions and Strategy

We use the standard isomorphism $H_2(\mathbf{P}^r) \cong \mathbb{Z}$ throughout. For any (g,n,d) such that stable maps exist, we write $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r,d)$ for the coarse moduli space of stable maps of degree d from n-pointed genus g curves into $\mathbf{P}^r$, as defined in Section 2.4. We wish to construct this moduli space via geometric invariant theory.

The structure of the main theorem of this paper is given in this section; we shall summarize it briefly here. We shall define a subscheme J of a Hilbert scheme, such that J is the base for a locally universal family of stable maps. A group G acts on J such that orbits of the action correspond to isomorphism classes in the family, and hence an orbit space of J by G, if it exists, will be precisely $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r,d)$.

The group action extends to the projective scheme $\overline{J}$, which is the closure of J in the relevant Hilbert scheme. Given any linearization L of this action, we may form a GIT quotient $\overline{J}/\!/_L G$. Such a quotient is a categorical quotient of the semistable set $\overline{J}^{ss}(L)$, and is in addition an orbit space if all semistable points are stable. Thus, if we can show that there exists a linearization L of the action of G on $\overline{J}$ such that

$$\overline{J}^{ss}(L) = \overline{J}^{s}(L) = J,$$

then we will have proved that $\overline{J}/\!/_L G \cong \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r,d)$.

3.1 The schemes I and J

We start by defining the scheme desired, $J = J_{g,n,d}$. Note that all the quantities and spaces defined in the following depend on (g, n, d), but that we shall only decorate them with subscripts when it is necessary to make the distinction.

Notation. Given a morphism $f : (C, x_1, \dots x_n) \to \mathbf{P}^r$, where C is nodal, write

$$\mathcal{L} := \omega_C(x_1 + \cdots + x_n) \otimes f^*(\mathcal{O}_{\mathbf{P}^r}(c)).$$

where c is a positive integer satisfying $c \geq 2$ unless $(g, n, d) = (0, 0, 1)$, in which case we require $c \geq 3$, as discussed in Section 2.4. Then $\mathcal{L}$ is ample on C if and only if C is a stable map. If $a \geq 3$, then $\mathcal{L}^a$ is very ample and $h^1(C, \mathcal{L}^a) = 0$. However, larger values of a will be required for us to complete our GIT construction; it is shown in [24] that cusps are GIT stable for $a = 3$. We shall assume for now that $a \geq 5$, although it will become apparent that further refinements are needed in some cases. Define

$$e := \deg(\mathcal{L}^a) = a(2g - 2 + n + cd),$$

so $h^0(C, \mathcal{L}^a) = e - g + 1$. We will work a lot with projective space of dimension $e - g$, so it is convenient to define

$$N := e - g.$$

Note that a corollary of our assumptions is that $e \geq ag$. If $g \geq 2$, then this follows from the inequality $2g - 2 + n + cd \geq 2g - 2 \geq g$. If $g \leq 1$, we see $e = a(2g - 2 + n + cd) \geq a \geq ag$. If $g = 0$, it will be more useful to estimate $e \geq a$. In any case, it follows that $N \geq 4$, since $a \geq 5$.

Let W be a vector space over k of dimension $N + 1$. Then an isomorphism $W \cong H^0(C, \mathcal{L}^a)$ induces an embedding $C \hookrightarrow \mathbf{P}(W)$ (recall that our convention is that $\mathbf{P}(V)$ is the set of equivalence classes of nonzero linear forms on V). Now the graph of f is an n-pointed nodal curve $(C, x_1, \dots, x_n) \subset \mathbf{P}(W) \times \mathbf{P}^r$, of bidegree (e, d). Its Hilbert polynomial is

$$P(m, \hat{m}) := em + d\hat{m} - g + 1.$$

Let $\mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r)$ be the Hilbert scheme of curves in $\mathbf{P}(W) \times \mathbf{P}^r$ with Hilbert polynomial $P(m, \hat{m})$. We append n extra factors of $\mathbf{P}(W) \times \mathbf{P}^r$ to give the locations of the marked points. Thus, given a stable map $f : (C, x_1, \ldots, x_n) \to \mathbf{P}^r$ and a choice of isomorphism $W \cong H^0(C, \mathcal{L}^a)$, we obtain an associated point in $\mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r) \times (\mathbf{P}(W) \times \mathbf{P}^r)^{\times n}$.

We write $\mathcal{C} \xrightarrow{\varphi} \mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r)$ for the universal family. This may be extended as

$$\left(\varphi, 1_{\prod_{i=1}^n \mathbf{P}(W) \times \mathbf{P}^r}\right) : \mathcal{C} \times \prod_{i=0}^{n} (\mathbf{P}(W) \times \mathbf{P}^r) \to \mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r) \times \prod_{i=0}^{n} (\mathbf{P}(W) \times \mathbf{P}^r). \tag{2}$$

Definition 3.1 ([8], p. 58)**.** The scheme

$$I \subset \mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r) \times \prod_{i=1}^{n} (\mathbf{P}(W) \times \mathbf{P}^r)$$

is the closed incidence subscheme consisting of $(h, x_1, \ldots, x_n) \subset \mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r) \times (\mathbf{P}(W) \times \mathbf{P}^r)^{\times n}$ such that $x_1, \ldots, x_n$ lie on $\mathcal{C}_h$. □

We restrict the family (2) over I. This restriction is the universal family of n-pointed curves in $\mathbf{P}(W) \times \mathbf{P}^r$, possessing n sections $\sigma_1, \ldots, \sigma_n$, giving the marked points. Next we consider the subscheme of I corresponding to a-canonically embedded stable maps.

Definition 3.2 ([8], p. 58)**.** The scheme $J \subset I$ is the locally closed subscheme consisting of those $(h, x_1, \ldots, x_n) \in I$ such that

(i) $(\mathcal{C}_h, x_1, \ldots, x_n)$ is *prestable*, i.e. $\mathcal{C}_h$ is projective, connected, reduced, and nodal, and $x_1, \ldots, x_n$ are nonsingular, distinct points on $\mathcal{C}_h$;

(ii) the projection map $\mathcal{C}_h \to \mathbf{P}(W)$ is a nondegenerate embedding;

(iii) $(\mathcal{O}_{\mathbf{P}(W)}(1) \otimes \mathcal{O}_{\mathbf{P}^r}(1))|_{\mathcal{C}_h}$ and $(\omega^a_{\mathcal{C}_h}(ax_1 + \cdots + ax_n) \otimes \mathcal{O}_{\mathbf{P}^r}(ca + 1))|_{\mathcal{C}_h}$ are isomorphic.

We denote by $\overline{J}$ the closure of J in I. □

Following [8], we abuse notation and write $\mathcal{O}_{\mathbf{P}(W)}(1) \otimes \mathcal{O}_{\mathbf{P}^r}(1)$ to denote $p_1^*\mathcal{O}_{\mathbf{P}(W)}(1) \otimes p_2^*\mathcal{O}_{\mathbf{P}^r}(1)$, where p_W and p_r are the projections of $\mathbf{P}(W) \times \mathbf{P}^r$ to its corresponding factors.

That J is indeed a locally closed subscheme is verified in ([8], p. 58). As I is a projective scheme, it follows that this is also the case for $\overline{J}$. If $(C, x_1, \ldots, x_n) \subset \mathbf{P}(W) \times \mathbf{P}^r$

is an n-pointed curve, we shall say that it is represented in J if $C = \mathcal{C}_h \subset \mathbf{P}(W) \times \mathbf{P}^r$ and $(h, x_1, \ldots, x_n) \in J$.

We restrict our universal family over J, denoting it $(\tilde{\varphi}^J : \tilde{\mathcal{C}}^J \to J, \sigma_1, \ldots, \sigma_n, p_r)$, where $p_r : \tilde{\mathcal{C}}^J \to \mathbf{P}^r$ is a projection from the universal family (a subscheme of $\mathbf{P}(W) \times \mathbf{P}^r \times J$) to $\mathbf{P}^r$. The reader may easily check that this is a family of stable maps.

We extend the natural action of $SL(W)$ on $\mathbf{P}(W)$ to $\mathbf{P}(W) \times \mathbf{P}^r$ by defining it to be trivial on the second factor. This induces an action on $\mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r)$, which we extend "diagonally" on $\mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r) \times (\mathbf{P}(W) \times \mathbf{P}^r)^{\times n}$. Note that I and J are invariant under this action. Our main object of study is the $SL(W)$ action on $\overline{J}$, but we shall approach this by studying the $SL(W)$ action on I.

As W is a vector space over k, we have so far only defined I and J as schemes over the field k. By replacing each use of $\mathbf{P}(W)$ with $\mathbf{P}^N$, we may define all our schemes over $\mathbb{Z}$. However, we shall continue to use W for notational convenience.

3.2 Strategy for the construction of $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$

Our aim is to apply Proposition 2.3 to the family $(\tilde{\varphi}^J : \tilde{\mathcal{C}}^J \to J, \sigma_1, \ldots, \sigma_n, p_r)$. Therefore, we check in this section that it has the local universal property, and that orbits correspond to isomorphism classes. We will first need the following lemma.

Lemma 3.3. Suppose $(\pi : \mathcal{X} \to S, \sigma_1, \ldots, \sigma_n, f)$ is a family of stable n-pointed maps to $\mathbf{P}^r$. Then $\pi_*(\omega^a_{\mathcal{X}/S}(a\sigma_1(S) + \cdots + a\sigma_n(S)) \otimes f^*(\mathcal{O}_{\mathbf{P}^r}(ca)))$ is a locally free $\mathcal{O}_S$-module, where $\sigma_i(S)$ denotes the divisor defined by the image of σ_i in $\mathcal{X}$. □

Proof. The morphism $\pi : \mathcal{X} \to S$ is proper and flat, so the $\mathcal{O}_{\mathcal{X}}$-module $\omega^a_{\mathcal{X}_s}(a\sigma_1(s) + \cdots + a\sigma_n(s)) \otimes f_s^*(\mathcal{O}_{\mathbf{P}^r}(ca))$ is flat over S for all s in S. Recall that we have chosen a, so that

$$H^1\left(\mathcal{X}_s, \omega^a_{\mathcal{X}_s}(a\sigma_1(s) + \cdots + a\sigma_n(s)) \otimes f_s^*(\mathcal{O}_{\mathbf{P}^r}(ca))\right) = 0$$

for all $s \in S$. Since $\mathcal{X}_s$ is a curve, all higher cohomology groups are also zero. The hypotheses of [10], Corollary III.7.9.9 are met, and we conclude that

$$R^0\pi_*\left(\omega^a_{\mathcal{X}/S}(a\sigma_1(S) + \cdots + a\sigma_n(S)) \otimes f^*(\mathcal{O}_{\mathbf{P}^r}(ca))\right)$$

is locally free. ■

Proposition 3.4.

(i) $(\tilde{\varphi}^J : \tilde{\mathcal{C}}^J \to J, \sigma_1, \ldots, \sigma_n, p_r)$ has the local universal property for the moduli problem $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$.

(ii) For all $(h, x_1, \ldots, x_n) \in J$ and $(h', x_1', \ldots x_n') \in J$, the maps $(\mathcal{C}_h, x_1, \ldots, x_n, p_r|_{\mathcal{C}_h}) \cong (\mathcal{C}_{h'}, x_1', \ldots, x_n', p_r|_{\mathcal{C}_{h'}})$ if and only if $(h, x_1, \ldots, x_n)$ and $(h', x_1', \ldots x_n')$ lie in the same orbit under the action of $SL(W)$.

□

Proof. (i) Suppose

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathbf{P}^r \\ \pi \downarrow\uparrow \sigma_i & & \\ S & & \end{array}$$

is an n-pointed family of stable maps to $\mathbf{P}^r$. For any $s_0 \in S$, we seek an open neighborhood $V \ni s_0$ and a morphism $V \xrightarrow{\psi} J$ such that $\psi^*(\tilde{\mathcal{C}}^J) \sim_{\text{fam}} \mathcal{X}|_V$.

Pick a basis for

$$H^0\big(\mathcal{X}_{s_0}, \omega^a_{\mathcal{X}_{s_0}}(a\sigma_1(s_0) + \cdots + a\sigma_n(s_0)) \otimes f^*_{s_0}(\mathcal{O}_{\mathbf{P}^r}(ca))\big),$$

and a basis for W. We showed that $\pi_*(\omega^a_{\mathcal{X}/S}(a\sigma_1(S) + \cdots a\sigma_n(S)) \otimes f^*(\mathcal{O}_{\mathbf{P}^r}(ca)))$ is locally free; it will be free on a sufficiently small neighborhood $V \ni s_0$. Then by ([12], III.12.11(b)) and Lemma 3.3, there is an induced basis of

$$H^0\big(\mathcal{X}_s, \omega^a_{\mathcal{X}_s}(a\sigma_1(s) + \cdots + a\sigma_n(s)) \otimes f^*_s(\mathcal{O}_{\mathbf{P}^r}(ca))\big)$$

for each $s \in V$. With our basis for W, then, this defines a map $\mathcal{X}_s \xrightarrow{\iota_s} \mathbf{P}(W)$ for each $s \in V$. These fit together as a morphism $\iota : \mathcal{X}|_V \to \mathbf{P}(W)$.

We have given $\mathcal{X}|_V \to V$ the structure of a family of n-pointed curves in $\mathbf{P}(W) \times \mathbf{P}^r$ parametrized by V,

$$\begin{array}{ccc} \mathcal{X}|_V & \xrightarrow{(\iota, f)} & \mathbf{P}(W) \times \mathbf{P}^r. \\ \pi_{\mathcal{X}_V} \downarrow\uparrow \sigma_i|_V & & \\ V & & \end{array}$$

By the universal properties of I, there is a unique morphism $\psi : V \to I$ such that $\mathcal{X}|_V \sim_{\text{fam}} \psi^*(\tilde{\mathcal{C}})$. Finally, observe that $\psi(V) \subset J$.

(ii) If $(h, x_1, \ldots, x_n)$ and $(h', x'_1, \ldots, x'_n)$ are in the same orbit of the action of $SL(W)$ on J, it is immediate that as stable maps, $(\mathcal{C}_h, x_1, \ldots, x_n, p_r|_{\mathcal{C}_h}) \cong (\mathcal{C}_{h'}, x'_1, \ldots, x'_n, p_r|_{\mathcal{C}|_{h'}})$. Conversely, suppose we are given an isomorphism $\tau : (\mathcal{C}_h, x_1, \ldots, x_n) \cong (\mathcal{C}_{h'}, x'_1, \ldots, x'_n)$; we wish to extend τ to be an automorphism of projective space $\mathbf{P}(W)$. This is possible because the curves $\mathcal{C}_h$ and $\mathcal{C}_{h'}$ embed nondegenerately in $\mathbf{P}(W)$ via $\mathcal{L}^a_{\mathcal{C}_h}$ and $\mathcal{L}^a_{\mathcal{C}_{h'}}$ respectively. As $\mathcal{L}^a_{\mathcal{C}_h}$ and $\mathcal{L}^a_{\mathcal{C}_{h'}}$ are canonically defined on isomorphic stable maps, we have an isomorphism $\mathcal{L}^a_{\mathcal{C}_h} \cong \tau^*(\mathcal{L}^a_{\mathcal{C}_{h'}})$, which induces an isomorphism on the spaces of global sections, from which we obtain an automorphism of $\mathbf{P}(W)$. ■

In the following GIT construction, we will first seek a linearization L such that $\overline{J}^{ss}(L) \subseteq J$. This has many useful implications, which are explored in the following proposition. Note that in the proof of part (iii) below, we use the existing construction of $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ over $\mathbb{C}$, given by Fulton and Pandharipande in [8]. This is not necessary (see [3]), but for brevity we take this shortcut.

Proposition 3.5.

(i) If there exists a linearization L of the action of $SL(W)$ on $\overline{J}$ such that $\overline{J}^{ss}(L) = \overline{J}^{s}(L) = J$, then $\overline{J}//_L SL(W) \cong \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$.

(ii) Suppose there exists a linearization L such that $\overline{J}^{ss}(L) \subseteq J$. Then $\overline{J}^{ss}(L) = \overline{J}^{s}(L)$.

(iii) There exists a map $j : J \to \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$, which is an orbit space for the $SL(W)$ action, and in particular a categorical quotient. The morphism j is universally closed. □

Proof. (i) If $\overline{J}^{ss}(L) = J$, then $\overline{J}//_L SL(W)$ is a categorical quotient of J, and if $\overline{J}^{ss}(L) = \overline{J}^{s}(L)$, the quotient is an orbit space. The result follows from Proposition 3.4 and Proposition 2.3.

(ii) Every point of J corresponds to a moduli stable map, and so has finite stabilizer. The result follows from Corollary 2.5.

(iii) $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ is a coarse moduli space ([8], Theorem 1). By Proposition 3.4, J carries a local universal family, and $SL(W)$ acts on J such that orbits of the group action correspond to equivalence classes of stable maps. The existence of the orbit space morphism $j : J \to \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ follows by Proposition 2.3. Universal closure of j is a consequence of ([8], Proposition 6). ■

The next theorem shows that if a linearization exists for which $\overline{J}^{ss}(L) \subseteq J$ and the semistable set is nonempty, then it yields the desired results.

Theorem 3.6. Suppose that, for some linearization L of the $SL(W)$-action, $\emptyset \neq \overline{J}^{ss}(L) \subseteq J$. Then, over $\mathbb{C}$,

(i) $\overline{J}//_L SL(W) \cong \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$;
(ii) $\overline{J}^{ss}(L) = \overline{J}^{s}(L) = J$. □

Proof. (i) Write $\overline{J}^{ss}$ for $\overline{J}^{ss}(L)$; this is an open subset of $\overline{J}$, and so $J - \overline{J}^{ss}$ is closed in J. From Proposition 3.5, we have a closed morphism $j : J \to \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$. Therefore $j(J - \overline{J}^{ss})$ is closed in $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$. However, $\overline{J}^{ss}$ is $SL(W)$-invariant, and j is an orbit space morphism, so $j(\overline{J}^{ss})$ and $j(J - \overline{J}^{ss})$ are disjoint subsets of $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$; an orbit space morphism is surjective, so these subsets make up the whole of $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$. It follows that $j(\overline{J}^{ss}) = \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d) - j(J - \overline{J}^{ss})$, and is thus an open subset.

Now since j is an orbit space and $\overline{J}^{ss}$ is $SL(W)$-invariant, it follows that the inverse image $j^{-1} j(\overline{J}^{ss}) = \overline{J}^{ss}$. The property of being a categorical quotient is local on the base, and we showed that $j(\overline{J}^{ss})$ is open in $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$, so the restriction $j|_{\overline{J}^{ss}} : \overline{J}^{ss} = j^{-1} j(\overline{J}^{ss}) \to j(\overline{J}^{ss})$ is a categorical quotient of $\overline{J}^{ss}$ for the $SL(W)$ action. However, so is $\overline{J}//SL(W)$. A categorical quotient is unique up to isomorphism, so

$$\overline{J}//SL(W) \cong j(\overline{J}^{ss}).$$

The scheme $\overline{J}$ is projective, so by construction $\overline{J}//SL(W)$ is projective. Thus $j(\overline{J}^{ss})$ is also projective, hence closed as a subset of $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$.

We have shown that $j(\overline{J}^{ss})$ is open and closed as a subset of $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$. It must be a union of connected components of $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$. However, Pandharipande and Kim have shown (main theorem, [16]) that $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ is connected over $\mathbb{C}$. We assumed that $\overline{J}^{ss} \neq \emptyset$. Hence

$$\overline{J}//SL(W) \cong \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d).$$

(ii) We showed that $\overline{J}^{ss} = \overline{J}^{s}$ in Proposition 3.5. We saw in the proof of (i) that $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d) = j(\overline{J}^{ss}) \sqcup j(J - \overline{J}^{ss})$ and that $j(\overline{J}^{ss}) = \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$. It follows that $j(J - \overline{J}^{ss}) = \emptyset$, whence $\overline{J}^{ss} = J$. ■

Moreover, proving that $\overline{J}^{ss}(L) = \overline{J}^{s}(L) = J$ is adequate to provide a GIT construction of the moduli spaces $\overline{\mathcal{M}}_{g,n}(X, \beta)$, where X is a general projective variety. The proof of this corollary follows the lines of ([8], Lemma 8), and is given in detail as ([2], Corollary 3.2.8).

Corollary 3.7 ([2], Corollary 3.2.8, cf. [8], Lemma 8). Let X be a projective variety defined over $\mathbb{C}$, with a fixed embedding to projective space $X \stackrel{\iota}{\hookrightarrow} \mathbf{P}^r$, and let $\beta \in H_2(X; \mathbb{Z})^+$. Let g and n be non-negative integers. If $\beta = 0$, then suppose in addition that $2g - 2 + n \geq 1$. Let $\iota_*(\beta) = d \in H_2(\mathbf{P}^r; \mathbb{Z})^+$. Let J be the scheme from Definition 3.2 corresponding to the moduli space $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$. Suppose, in the language of Theorem 3.6, that there exists a linearization L such that $\overline{J}^{ss}(L) = \overline{J}^{s}(L) = J$.

Then there exists a closed subscheme $J_{X,\beta}$ of J such that

$$\overline{J}_{X,\beta} /\!/_{L|_{\overline{J}_{X,\beta}}} SL(W) \cong \overline{\mathcal{M}}_{g,n}(X, \beta),$$

where $\overline{J}_{X,\beta}$ is the closure of $J_{X,\beta}$ in $\overline{J}$. □

In Section 5 we shall prove that $\overline{J}^{ss}(L) \subseteq J$, for a suitable range of linearizations L. Nonemptiness of $\overline{J}^{ss}(L)$ is dealt with in Section 6. This uses induction on the number n of marked points. For $n = 0$, we show that $\overline{J}^{ss}$ is nonempty by showing that smooth maps are stable, following Gieseker. We then apply Theorem 3.6 to see that all moduli stable maps have a GIT semistable model. However, the inductive step follows a different route, and in fact we are able to prove that $\overline{J}^{ss} = J$ directly, and then apply Proposition 3.5, which unlike Theorem 3.6 does not depend on the construction of [8].

The generality of our proof thus is limited by the base case. The GIT quotient $\overline{J}/\!/SL(W)$ may be defined over quite general base schemes. Indeed, the Hilbert scheme $\mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r)$ is projective over Spec $\mathbb{Z}$, so our $\overline{J}$ is also, and thus we obtain a projective quotient $\overline{J}/\!/SL(W)$ over Spec $\mathbb{Z}$. However, Theorem 3.6 depends on the cited results of Fulton, Kim, and Pandharipande that a projective scheme $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ exists, is a coarse moduli space for this moduli problem, and is connected; the relevant papers [8] and [16] present their results only over $\mathbb{C}$; we can only claim to have constructed $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ over Spec $\mathbb{C}$.

In the special case of $\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n}(\mathbf{P}^0, 0)$, our base case is Gieseker's construction of $\overline{\mathcal{M}}_g$, which does indeed work over Spec $\mathbb{Z}$; our construction of $\overline{\mathcal{M}}_{g,n}$ thus works in this generality.

A modification of our argument is given in [3], making our construction work independently from that of [8], and over the more general base Spec $\mathbb{Z}[p_1^{-1} \cdots p_j^{-1}]$ where $p_1, \ldots, p_j$ are all prime numbers less than or equal to the degree d. However, it is not clear whether our quotient, or an open set thereof, coarsely represents a desirable functor over $\mathbb{Z}$. If the characteristic is less than the degree, a stable map may itself be an inseparable morphism. The proof that the moduli functor is separated then fails ([6], Lemma 4.2). Such maps can be left out if one wishes to obtain a Deligne–Mumford stack (cf. [31], p. 2); this however will not be proper, and so our projective quotient cannot be its coarse moduli space.

Gieseker's construction of $\overline{\mathcal{M}}_g$ begins analogously to ours. However, the end of his argument is different from the proof of Theorem 3.6 in two ways. Gieseker shows directly that all smooth curves are GIT semistable, and then uses a deformation argument and semistable replacement to show that all Deligne–Mumford stable curves have $SL(W)$-semistable Hilbert points. However, the proof that smooth curves are GIT semistable does not provide a good enough inequality to prove stability of n-pointed curves or maps; hence our inductive argument. Further, not all stable maps can be smoothed (the smooth locus is not in general dense in $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$), so that aspect of the argument has also needed modification.

4 The Range of Linearizations to be Used

Now we shall define linearizations of the action of $SL(W)$ on $\overline{J}$ and I, which were defined in Definitions 3.1 and 3.2. Most of our analysis of GIT semistability is valid for general curves in I. Accordingly, it makes sense to prove results in this greater generality, so we shall define linearizations on I and restrict them to $\overline{J}$. If L is a linearization of the group action on I, then $\overline{J}^{ss}(L|_{\overline{J}}) = \overline{J} \cap I^{ss}(L)$ by ([20], Theorem 1.19).

4.1 The linearizations $L_{m,\hat{m},m'}$ and their hull $\mathrm{H}_M(I)$

Let $(h, x_1, \ldots, x_n) \in I \subset \mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r) \times (\mathbf{P}(W) \times \mathbf{P}^r)^{\times n}$. It is natural to start by defining line bundles on the separate factors, namely $\mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r)$ and n copies of $\mathbf{P}(W) \times \mathbf{P}^r$, and then take the tensor product of their pullbacks.

Notation. This notation will be used throughout the following:

$$Z_{m,\hat{m}} := H^0(\mathbf{P}(W) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W)}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m}))$$

$$P(m, \hat{m}) := em + d\hat{m} - g + 1;$$

$P(m, \hat{m})$ is the Hilbert polynomial of a genus g curve $C \subset \mathbf{P}(W) \times \mathbf{P}^r$, of bidegree (e, d).

We first define our line bundles on $\mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r)$. Let $h \in \mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r)$. If m and $\hat{m}$ are sufficiently large, then $h^1(\mathcal{C}_h, \mathcal{O}_{\mathbf{P}(W)}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})|_{\mathcal{C}_h}) = 0$ and the restriction map

$$\hat{\rho}^{C}_{m,\hat{m}} : H^0(\mathbf{P}(W) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W)}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})) \to H^0\big(\mathcal{C}_h, \mathcal{O}_{\mathbf{P}(W)}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})|_{\mathcal{C}_h}\big)$$

is surjective (cf. Grothendieck's "Uniform m Lemma," [11], 1.11). The Hilbert polynomial $P(m, \hat{m})$ will be equal to $h^0(\mathcal{C}_h, \mathcal{O}_{\mathbf{P}(W)}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})|_{\mathcal{C}_h})$, so that $\bigwedge^{P(m,\hat{m})} \hat{\rho}^{C}_{m,\hat{m}}$ gives a point of

$$\mathbf{P}\left(\bigwedge^{P(m,\hat{m})} H^0(\mathbf{P}(W) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W)}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m}))\right) = \mathbf{P}\left(\bigwedge^{P(m,\hat{m})} Z_{m,\hat{m}}\right).$$

By the "Uniform m Lemma" again, for sufficiently large $m, \hat{m}$, say $m, \hat{m} \geq m_3$, the Hilbert embedding

$$\begin{aligned} \hat{H}_{m,\hat{m}} &: \mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r) \to \mathbf{P}\left(\bigwedge^{P(m,\hat{m})} Z_{m,\hat{m}}\right) \\ \hat{H}_{m,\hat{m}} &: h \mapsto \left[\bigwedge^{P(m,\hat{m})} \hat{\rho}^{C}_{m,\hat{m}}\right] \end{aligned} \tag{3}$$

is a closed immersion (see Proposition 4.6).

Definition 4.1. Let the setup be as above, and let $m, \hat{m} \geq m_3$. The line bundle $L_{m,\hat{m}}$ on $\mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r)$ is defined to be the pullback of the hyperplane line bundle $\mathcal{O}_{\mathbf{P}(\bigwedge^{P(m,\hat{m})} Z_{m,\hat{m}})}(1)$ via the Hilbert embedding

$$\hat{H}_{m,\hat{m}} : \mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r) \hookrightarrow \mathbf{P}\left(\bigwedge^{P(m,\hat{m})} Z_{m,\hat{m}}\right). \qquad \square$$

Recall that $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^0, 0) = \overline{\mathcal{M}}_{g,n}$. Whenever we write "assume $m, \hat{m} \geq m_3$," one should bear in mind that $\hat{m}$ may be set to zero in the case $r = d = 0$.

We identify $L_{m,\hat{m}}$ with its pullback to $\mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r) \times (\mathbf{P}(W) \times \mathbf{P}^r)^{\times n}$. Now, for $i = 1, \ldots, n$, let

$$p_i : \mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r) \times (\mathbf{P}(W) \times \mathbf{P}^r)^{\times n} \to \mathbf{P}(W) \times \mathbf{P}^r$$

be projection to the ith such factor. Then, for choices $m_1', \hat{m}_1', \ldots, m_n', \hat{m}_n' \in \mathbb{Z}$, we may define n line bundles on the product $\mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r) \times (\mathbf{P}(W) \times \mathbf{P}^r)^{\times n}$,

$$p_i^*(\mathcal{O}_{\mathbf{P}(W)}(m_i') \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m}_i')).$$

The integers $\hat{m}_1', \ldots, \hat{m}_n'$ will in fact turn out to be irrelevant to our following work. We shall assume that they are all positive, but suppress them in notation to make things more readable.

Definition 4.2. If $m, \hat{m} \geq m_3$ and $m_1', \hat{m}_1', \ldots, m_n', \hat{m}_n' \geq 1$, then we define the very ample line bundle on I,

$$L_{m,\hat{m},m_1',\ldots,m_n'} := \left(L_{m,\hat{m}} \otimes \bigotimes_{i=1}^{n} p_i^*(\mathcal{O}_{\mathbf{P}(W)}(m_i') \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m}_i')) \right) \Bigg|_I . \tag{4}$$

If $m_1' = \cdots = m_n' = m'$, then we write this as $L_{m,\hat{m},m'}$. □

These line bundles each possess a unique $SL(W)$-action linearizing the action on I, which will be described in Section 4.2. Our aim is to show that

$$\overline{J}//_{L_{m,\hat{m},m'}} SL(W) \cong \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d),$$

for a suitable range of choices of $m, \hat{m}, m'$. However, in order to prove that $\overline{J}^{ss}(L_{m,\hat{m},m'})$ has the desired properties, we shall make use of the theory of variation of GIT (summarized in Section 2.3). It is therefore necessary to prove results, not just for certain $L_{m,\hat{m},m'}$ but for all virtual linearizations lying within the convex hull of this range in $\mathrm{Pic}^{SL(W)}(I)_{\mathbb{R}}$ or $\mathrm{Pic}^{SL(W)}(\overline{J})_{\mathbb{R}}$. To make this precise, let $M \subset \mathbb{N}^3$ be a set such that, for every $(m, \hat{m}, m') \in M$, we have $m, \hat{m} \geq m_3$ and $m' \geq 1$.

Definition 4.3. We define $\mathbf{H}_M(I)$ to be the convex hull in $\mathrm{Pic}^{SL(W)}(I)_{\mathbb{R}}$ of

$$\{L_{m,\hat{m},m'} : (m, \hat{m}, m') \in M\}.$$

We define $\mathbf{H}_M(\overline{J}) \subseteq \mathrm{Pic}^{SL(W)}(\overline{J})_{\mathbb{R}}$ by taking the convex hull of the restrictions of the line bundles to $\overline{J}$. □

As each $L_{m,\hat{m},m'}$ possesses a unique lift of the action of $SL(W)$, there is an induced group action on each $l \in \mathbf{H}_M(I)$ or $\mathbf{H}_M(\overline{J})$.

4.2 The action of $SL(W)$ for linearizations $L_{m,\hat{m},m'}$

We shall describe the $SL(W)$ action on $L_{m,\hat{m},m'}$; recall its definition from line (4) above. The linearization of the action of $SL(W)$ on the last factors is easy to describe. The action of $SL(W)$ on $\mathcal{O}_{\mathbf{P}(W)}(1)$ is induced from the natural action on W (recall that $H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) \cong W$). The trivial action on $\mathbf{P}^r$ lifts to a trivial action on $\mathcal{O}_{\mathbf{P}^r}(1)$. Thus we have an induced action on each $\mathcal{O}_{\mathbf{P}(W)}(m_i') \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m}_i')$, which may be pulled back along p_i and restricted to the invariant subscheme I.

To describe the $SL(W)$ action on $L_{m,\hat{m}}$, it is easier to talk about the linear action on the projective space $\mathbf{P}(\bigwedge^{P(m,\hat{m})} Z_{m,\hat{m}})$. Indeed, recall our conventions for the numerical criterion, and our definition of the function $\mu^L(x,\lambda)$ as given in Section 2.2; what we shall wish to calculate are the weights of the $SL(W)$ action on the vector space $\bigwedge^{P(m,\hat{m})} Z_{m,\hat{m}}$. These will enable us to verify stability for a point in $\mathbf{P}(\bigwedge^{P(m,\hat{m})} Z_{m,\hat{m}})$, where $SL(W)$ acts with the dual action.

Fix a basis $w_0, \dots, w_N$ for $W = H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1))$ and a basis $f_0, \dots, f_r$ for $H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(1))$. The group $SL(W)$ acts canonically on $H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1))$; the action on $H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(1))$ is trivial.

We describe the $SL(W)$ action on a basis for $\bigwedge^{P(m,\hat{m})} Z_{m,\hat{m}}$. Let $\hat{B}_{m,\hat{m}}$ be a basis for $Z_{m,\hat{m}} \cong H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(m)) \otimes H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(\hat{m}))$ consisting of monomials of bidegree $(m, \hat{m})$, where the degree m part is a monomial in $w_0, \dots, w_N$ and the degree $\hat{m}$ part is a monomial in $f_0, \dots, f_r$. Then if $\hat{M}_i \in \hat{B}_{m,\hat{m}}$ is given by $w_0^{\gamma_0} \cdots w_N^{\gamma_N} f_0^{\Gamma_0} \cdots f_r^{\Gamma_r}$, we define $g.\hat{M}_i := (g.w_0)^{\gamma_0} \cdots (g.w_N)^{\gamma_N} f_0^{\Gamma_0} \cdots f_r^{\Gamma_r}$.

A basis for $\bigwedge^{P(m,\hat{m})} Z_{m,\hat{m}}$ is given by

$$\bigwedge^{P(m,\hat{m})} \hat{B}_{m,\hat{m}} := \left\{ \hat{M}_{i_1} \wedge \cdots \wedge \hat{M}_{i_{P(m,\hat{m})}} \middle| 1 \le i_1 < \cdots < i_{P(m,\hat{m})} \le \dim Z_{m,\hat{m}}, \hat{M}_{i_j} \in \hat{B}_{m,\hat{m}} \right\}. \tag{5}$$

The $SL(W)$ action on this basis is given by

$$g.\left(\hat{M}_{i_1} \wedge \cdots \wedge \hat{M}_{i_{P(m,\hat{m})}}\right) = \left(g.\hat{M}_{i_1}\right) \wedge \cdots \wedge \left(g.\hat{M}_{i_{P(m,\hat{m})}}\right).$$

Terminology. We have defined virtual linearizations of the $SL(W)$ action on the scheme I. We may abuse notation, and say that $(C, x_1, \dots, x_n) \subset \mathbf{P}(W) \times \mathbf{P}^r$ is semistable with respect to a virtual linearization l to mean that $(h, x_1, \ldots, x_n) \in I^{ss}(l)$, where $(C, x_1, \ldots, x_n) = (\mathcal{C}_h, x_1, \ldots, x_n)$.

4.3 The numerical criterion for $L_{m,\hat{m},m'}$

Let λ' be a 1-PS of $SL(W)$. We wish to state the Hilbert–Mumford numerical criterion for our situation. In fact, if we are careful in our analysis then we need only prove results about the semistability of points with respect to linearizations of the form $L_{m,\hat{m},m'}$, as the following key lemma shows.

Lemma 4.4. Fix $(h, x_1, \ldots, x_n) \in I$. Let M be a range of values for $(m, \hat{m}, m')$. Suppose that there exists a one-parameter subgroup λ' of $SL(W)$, such that

$$\mu^{L_{m,\hat{m},m'}}((h, x_1, \ldots, x_n), \lambda') > 0$$

for all $(m, \hat{m}, m') \in M$. Then x is unstable with respect to l for all $l \in \mathbf{H}_M(I)$. □

Proof. Let $l \in \mathbf{H}_M(I)$. Then l is a finite combination,

$$l = L^{\alpha_1}_{m_1,\hat{m}_1,m'_1} \otimes \cdots \otimes L^{\alpha_k}_{m_k,\hat{m}_k,m'_k},$$

where $\alpha_1, \ldots, \alpha_k \in \mathbb{R}_{\geq 0}$ satisfy $\sum \alpha_i = 1$, and where $(m_i, \hat{m}_i, m'_i) \in M$ for $i = 1, \ldots, k$. We know that $\mu^{L_{m_i,\hat{m}_i,m'_i}}((h, x_1, \ldots, x_n), \lambda') > 0$ for $i = 1, \ldots, k$. The map $l' \mapsto \mu^{l'}(x, \lambda')$ is a group homomorphism $\mathrm{Pic}^G(I)_{\mathbb{R}} \to \mathbb{R}$, where $\mathbb{R}$ has its additive structure, so it follows that $\mu^l((h, x_1, \ldots, x_n), \lambda') > 0$. Hence $M^l(h, x_1, \ldots, x_n) > 0$, and so $(h, x_1, \ldots, x_n)$ is unstable with respect to l. ■

Note the necessity of the condition that the destabilizing 1-PS λ' be the same for all $L_{m,\hat{m},m'}$ such that $(m, \hat{m}, m') \in M$.

Recall the definition of $L_{m,\hat{m},m'}$ given in line (4). From this and the functorial nature of $\mu^{\bullet}((C, x_1, \ldots, x_n), \lambda)$, we see

$$\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda) = \mu^{L_{m,\hat{m}}}(C, \lambda) + \sum_{i=1}^{n} \mu^{\mathcal{O}_{\mathbf{P}(W)}(1)}(x_i, \lambda) m'_i. \tag{6}$$

Let us start, then, with $w_0, \dots, w_N$, a basis of $W = H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1))$ diagonalizing the action of λ'. There exist integers $r_0, \dots, r_N$ such that $\lambda'(t)w_i = t^{r_i} w_i$ for all $t \in k^*$ and $0 \le i \le N$.

In the first place, following the conventions set up in Section 2.2,

$$\mu^{\mathcal{O}_{\mathbf{P}(W)}(1)}(x_i, \lambda') = \min\{r_j | w_j(x_i) \neq 0\}.$$

We calculate $\mu^{L_{m,\hat{m}}}(C, \lambda)$. Referring to the notation of the previous subsection, if $\hat{M} := w_0^{\gamma_0} \cdots w_N^{\gamma_N} f_0^{\Gamma_0} \cdots f_r^{\Gamma_r}$, then

$$\lambda'(t)\hat{M} = t^{\sum \gamma_p r_p} \hat{M}.$$

Accordingly, we define the λ'-weight of the monomial $\hat{M}$ to be

$$w_{\lambda'}(\hat{M}) = \sum_{p=0}^{N} \gamma_p r_p.$$

Let $\bigwedge^{P(m,\hat{m})} \hat{B}_{m,\hat{m}}$ be the basis for $\bigwedge^{P(m,\hat{m})} Z_{m,\hat{m}}$ given in line (5). Then the λ' action on this basis is given by

$$\lambda'(t)\big(\hat{M}_{i_1} \wedge \cdots \wedge \hat{M}_{i_{P(m,\hat{m})}}\big) = t^{\theta}\big(\hat{M}_{i_1} \wedge \cdots \wedge \hat{M}_{i_{P(m,\hat{m})}}\big),$$

where $\theta := \sum_{j=1}^{P(m,\hat{m})} w_{\lambda'}\big(\hat{M}_{i_j}\big)$. If we write $\hat{H}_{m,\hat{m}}(h)$ in the basis which is dual to $\bigwedge^{P(m,\hat{m})} \hat{B}_{m,\hat{m}}$,

$$\hat{H}_{m,\hat{m}}(h) = \left[\sum_{1 \le j_1 < \cdots < j_{P(m,\hat{m})}} \hat{\rho}^{C}_{m,\hat{m}}\big(\hat{M}_{j_1} \wedge \cdots \wedge \hat{M}_{j_{P(m,\hat{m})}}\big) \cdot \big(\hat{M}_{j_1} \wedge \cdots \wedge \hat{M}_{j_{P(m,\hat{m})}}\big)^{\vee} \right],$$

so we may calculate

$$\mu^{L_{m,\hat{m}}}(C, \lambda') = \min \left\{ \sum_{j=1}^{P(m,\hat{m})} w_{\lambda'}\big(\hat{M}_{i_j}\big) \right\},$$

where the minimum is taken over all sequences $1 \le i_j < \cdots < i_{P(m,\hat{m})}$ such that $\hat{\rho}^{C}_{m,\hat{m}}(\hat{M}_{i_1} \wedge \cdots \wedge \hat{M}_{i_{P(m,\hat{m})}}) \neq 0$. However, the latter is true precisely when the set $\{\hat{\rho}^{C}_{m,\hat{m}}(\hat{M}_{i_1}), \dots, \hat{\rho}^{C}_{m,\hat{m}}(\hat{M}_{i_{P(m,\hat{m})}})\}$ is a basis for $H^0(C, \mathcal{O}_{\mathbf{P}(W)}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})|_C)$.

Putting this together in equation (6), we may state the numerical criterion: $(h, x_1, \dots, x_n)$ is semistable with respect to $L_{m,\hat{m},m'}$ if and only if $\mu^{L_{m,\hat{m},m'}}((C, x_1, \dots, x_n), \lambda') \leq 0$ for all 1-PS λ', where

$$\mu^{L_{m,\hat{m},m'}}((h, x_1, \dots, x_n), \lambda') = \min \left\{ \sum_{j=1}^{P(m,\hat{m})} w_{\lambda'}\left(\hat{M}_{i_j}\right) + \sum_{l=1}^{n} r_{k_l} m' \right\},$$

and the minimum is taken over all sequences $1 \leq i_1 < \cdots < i_{P(m,\hat{m})}$ such that $\{\hat{\rho}^C_{m,\hat{m}}(\hat{M}_{i_1}), \dots, \hat{\rho}^C_{m,\hat{m}}(\hat{M}_{i_{P(m,\hat{m})}})\}$ is a basis for $H^0(C, \mathcal{O}_{\mathbf{P}(W)}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})|_C)$, and all basis elements w_{k_l} such that $w_{k_l}(x_l) \neq 0$.

In our applications, we will often "naturally" write down torus actions on W which highlight the geometric pathologies we wish to exclude from our quotient space. These will usually be one-parameter subgroups of $GL(W)$ rather than $SL(W)$, as then they may be defined to act trivially on most of the space, which makes it easier to calculate their weights. We may translate these by using "$GL(W)$ version" of the numerical criterion, derived as follows.

Given a 1-PS λ of $GL(W)$, we define our "associated 1-PS λ' of $SL(W)$." There is a basis $w_0, \dots, w_N$ of $H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1))$, so that the action of λ is given by $\lambda(t)w_i = t^{r_i} w_i$ where $r_i \in \mathbb{Z}$ (the sum $\sum_{p=0}^{N} r_p$ is not necessarily zero). We obtain a 1-PS λ' of $SL(W)$ by the rule $\lambda'(t)w_i = t^{r'_i} w_i$, where

$$r'_i = (N+1)r_i - \sum_{p=0}^{N} r_p.$$

Note that now $\sum_{p=0}^{N} r'_p = 0$.

We use this relationship to convert our numerical criterion for the λ'-action into one for the λ action. We define the total λ-weight of a monomial in analogy with that defined for a 1-PS of $SL(W)$. Let λ' be the 1-PS of $SL(W)$ arising from the 1-PS λ of $GL(W)$. Then the numerical criterion may be expressed as follows.

Lemma 4.5 (cf. [9], p. 10). Let $(h, x_1, \dots, x_n) \in I$, let λ be a 1-PS of $GL(W)$, and let λ' be the associated 1-PS of $SL(W)$. There exist monomials $\hat{M}_{i_1}, \dots, \hat{M}_{i_{P(m,\hat{m})}}$ in $\hat{B}_{m,\hat{m}}$ such that $\{\hat{\rho}^{\mathcal{C}_h}_{m,\hat{m}}(\hat{M}_{i_1}), \dots, \hat{\rho}^{\mathcal{C}_h}_{m,\hat{m}}(\hat{M}_{i_{P(m,\hat{m})}})\}$ is a basis of $H^0(\mathcal{C}_h, \mathcal{O}_{\mathbf{P}(W)}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})|_{\mathcal{C}_h})$, and there exist basis elements $w_{k_1}, \dots w_{k_n}$ for the $SL(W)$ action such that $w_{k_l}(x_l) \neq 0$, with

$$\mu^{L_{m,\hat{m},m'}}((C, x_1, \dots, x_n), \lambda') = \sum_{j=1}^{P(m,\hat{m})} w_{\lambda'}\left(\hat{M}_{i_j}\right) + \sum_{l=1}^{n} r'_{k_l} m' \tag{7}$$

$$= \left(\sum_{j=1}^{P(m,\hat{m})} w_\lambda\left(\hat{M}_{i_j}\right) + \sum_{l=0}^{n} w_\lambda(w_{k_l}) m' \right) (e - g + 1)$$

$$- (mP(m,\hat{m}) + nm') \sum_{p=0}^{N} w_\lambda(w_p); \tag{8}$$

moreover, this choice of monomials minimizes the left-hand side of equation (8). □

In the course of the construction, we progressively place constraints on the set M. In particular, for $(m, \hat{m}, m') \in M$, we shall be concerned with the values of the ratios $\frac{\hat{m}}{m}$ and $\frac{m'}{m^2}$. It may appear surprising at first that m' varies with m^2 and not with m. Note, however, that both terms on the right-hand side of equation (8) have terms of order $mP(m, \hat{m}) = em^2 + dm\hat{m} - (g - 1)m$ and terms of order m', so in fact it is quite natural that m' is proportional to m^2.

4.4 Fundamental constants and notation

We shall now fix some notation for the whole of this paper. The morphisms $p_W : \mathbf{P}(W) \times \mathbf{P}^r \to \mathbf{P}(W)$ and $p_r : \mathbf{P}(W) \times \mathbf{P}^r \to \mathbf{P}^r$ are projection onto the first and second factors, respectively. Let $C \xrightarrow{\iota} \mathbf{P}(W) \times \mathbf{P}^r$ be the inclusion. We define

$$L_W := \iota^* p_W^* \mathcal{O}_{\mathbf{P}(W)}(1),$$
$$L_r := \iota^* p_r^* \mathcal{O}_{\mathbf{P}^r}(1).$$

The following well-known facts are analogous to those given by Gieseker.

Proposition 4.6 (cf. [9], p. 25)**.** Let $C \subset \mathbf{P}(W) \times \mathbf{P}^r$ have genus g and bidegree (e, d). There exist positive integers m_1, m_2, m_3, q_1, q_2, q_3, μ_1, and μ_2 satisfying the following properties.

(i) For all $m, \hat{m} > m_1$, $H^1(C, L_W^m) = H^1(C, L_r^{\hat{m}}) = H^1(C, L_W^m \otimes L_r^{\hat{m}}) = 0$. Also $H^1(\bar{C}, \bar{L}_W^m) = H^1(\bar{C}, \bar{L}_r^{\hat{m}}) = H^1(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}}) = 0$ and the three restriction maps

$$H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(m)) \to H^0\big(p_W(C), \mathcal{O}_{p_W(C)}(m)\big),$$
$$H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(\hat{m})) \to H^0\big(p_r(C), \mathcal{O}_{p_r(C)}(\hat{m})\big),$$
$$H^0(\mathbf{P}(W) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W)}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})) \to H^0\big(C, L_W^m \otimes L_r^{\hat{m}}\big)$$

are surjective.

(ii) $\mathcal{I}_C^{q_1} = 0$, where $\mathcal{I}_C$ is the sheaf of nilpotents in $\mathcal{O}_C$.

(iii) $h^0(C, \mathcal{I}_C) \leq q_2$.

(iv) For every complete subcurve $\tilde{C}$ of C, $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}) \leq q_3$ and $q_3 \geq q_1$.

(v) $\mu_1 > \mu_2$ and for every point $P \in C$ and for all integers $x \geq 0$,

$$\dim \frac{\mathcal{O}_{C,P}}{m_{C,P}^x} \leq \mu_1 x + \mu_2,$$

where $\mathcal{O}_{C,P}$ is the local ring of C at P and $m_{C,P}$ is the maximal ideal of $\mathcal{O}_{C,P}$.

(vi) For every subcurve $\tilde{C}$ of C, for every point $P \in C$, and for all integers i such that $m_2 \leq i < m$, the cohomology $H^1(\tilde{C}, \mathcal{I}_P^{m-i} \otimes L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}}) = 0$, where $\mathcal{I}_P$ is the ideal subsheaf of $\mathcal{O}_{\tilde{C}}$ defining P.

(vii) For all integers $m, \hat{m} \geq m_3$, the map

$$\begin{aligned} h &\mapsto \hat{H}_{m,\hat{m}}(h) \\ \mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r) &\to \mathbf{P}\left(\bigwedge^{P(m,\hat{m})} H^0(\mathbf{P}(W) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W)}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})) \right) \end{aligned}$$

is a closed immersion. □

In addition, we define a constant not used by Gieseker.

$$\begin{aligned} \bar{g} := \min\{0, g_{\bar{Y}} \mid \bar{Y} &\text{ is the normalization of a complete subcurve } Y \text{ contained} \\ &\text{in a connected fiber } \mathcal{C}_h \text{ for some } h \in \mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r)\}. \end{aligned}$$

Y need not be a proper subcurve. The maximum number of irreducible components of Y is $e+d$, as each must have positive degree. Hence a lower bound for $\bar{g}$ is given by $-(e+d)+1$. One would expect $\bar{g}$ to be negative for most (g, n, d), but we have stipulated in particular that $\bar{g} \leq 0$ as this will be convenient in our calculations.

5 GIT Semistable Maps Represented in $\overline{J}$ are Moduli Stable

We now embark on our proof that $\overline{J}//SL(W)$ is isomorphic to $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$. Recall Proposition 3.5; our first goal is to show that $\overline{J}^{ss} \subseteq J$. We achieve this in this section. In Sections 5.2 to 5.5 we work with the locus of semistable points in I. Over the course of results 5.3–5.19, we find a range M of values for $(m, \hat{m}, m')$, such that if $(C, x_1, \ldots, x_n) \subset \mathbf{P}(W) \times \mathbf{P}^r$ is semistable with respect to $l \in \mathbf{H}_M(I)$, then $(C, x_1, \ldots, x_n)$ must be close to being a moduli stable map; this "potential stability" is defined formally in Definition 5.20.

For this section, it is only necessary to work over a field; if we prove that equality $\overline{J}^{ss}(L) = \overline{J}^{s}(L) = J$ holds over any field k, then equality over $\mathbb{Z}$ follows (see the proof of Theorem 6.3, given at the end of Section 6.3).

In Section 5.6 we finally restrict attention to $\overline{J}$, and greatly refine the range M. Now the results of Sections 5.2–5.5, together with an application of the valuative criterion of properness, show us that if $l \in \mathbf{H}_M(\overline{J})$ then $\overline{J}^{ss}(l) \subseteq J$, as required.

5.1 General strategy for this section

The strategy for proving many of the results in this section is similar, so we outline it here in detail and refer back to this subsection as needed.

The proofs will be by contradiction. We will suppose that $(C, x_1, \ldots, x_n) \subset \mathbf{P}(W) \times \mathbf{P}^r$ is connected and $SL(W)$-semistable with respect to a given virtual linearization l, and also that C has some "undesirable" geometric property. We will then exhibit a 1-PS λ of $GL(W)$ which we claim is destabilizing. The 1-PS will all have a special form: they will give rise to a two- or three-stage weighted filtration $0 \subset W_0 \subset W_1 \subseteq W_2 := H^0(p_W(C), L_W)$. (Recall that L_W denotes $\iota^* p_W^* \mathcal{O}_{\mathbf{P}(W)}(1)$ and L_r denotes $\iota^* p_r^* \mathcal{O}_{\mathbf{P}^r}(1)$.) We may choose a basis $w_0, \ldots, w_N$ diagonalizing the λ action and adapted to this filtration. Let $N_j := \dim W_j$, and let r_j be the weight of the basis elements corresponding to the stage W_j. That is, λ acts as follows:

$$\begin{aligned}
\lambda(t)w_i &= t^{r_0} w_i,\ t \in \mathbb{C}^*,\ \ 0 \le i \le N_0 - 1\\
\lambda(t)w_i &= t^{r_1} w_i,\ t \in \mathbb{C}^*,\ N_0 \le i \le N_1 - 1\\
\lambda(t)w_i &= t^{r_2} w_i,\ t \in \mathbb{C}^*,\ N_1 \le i \le N_2 - 1.
\end{aligned}$$

For our purposes, to specify the λ action, it is enough to specify W_0 and W_1, and the weights r_0, r_1, r_2.

We shall find a lower bound for $\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda')$ by filtering the vector space $H^0(C, L_W^m \otimes L_r^{\hat{m}})$ according to the weight with which λ acts. The filtration is constructed in the same way every time; we describe it now. Assume $r_0 \geq 0$ (this will be the case in all our applications). Let R be a positive integer such that $\sum_{i=0}^{N} r_i \leq R$. For $0 \leq p \leq m$, let $\Omega_p^{m,\hat{m}}$ be the subspace of $H^0(\mathbf{P}(W) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W)}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m}))$ spanned by monomials of weight less than or equal to p. Let

$$\overline{\Omega}_p := \hat{\rho}^C_{m,\hat{m}}(\Omega_p) \subset H^0\big(C, L_W^m \otimes L_r^{\hat{m}}\big).$$

We have a filtration of $H^0(C, L_W^m \otimes L_r^{\hat{m}})$ in order of increasing weight,

$$0 \subseteq \overline{\Omega}_0^{m,\hat{m}} \subseteq \overline{\Omega}_1^{m,\hat{m}} \subseteq \cdots \subseteq \overline{\Omega}_m^{m,\hat{m}} = H^0\big(C, L_W^m \otimes L_r^{\hat{m}}\big). \tag{9}$$

Write $\hat{\beta}_p = \dim \overline{\Omega}_p^{m,\hat{m}}$.

We will get bounds on $\hat{\beta}_p$ which depend on the problem at hand. However, these bounds will always have the same format (described below). The next lemma shows how to estimate the minimal weight $\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda')$ given these bounds on $\hat{\beta}_p$.

Lemma 5.1. In the setup described above, suppose that

$$\hat{\beta}_p \leq (e - \alpha)m + (d - \beta)\hat{m} + \gamma p + \epsilon_p,$$

where $\alpha, \beta, \gamma, \epsilon_p$ are constants. Set

$$\epsilon := \frac{1}{m} \sum_{p=0}^{r_N m - 1} \epsilon_p.$$

Suppose

$$\sum_{j=1}^{n} w_\lambda(w_{i_j})m' = \delta m',$$

where $w_{i_1}, \ldots, w_{i_n}$ are the basis elements of minimal weight satisfying $w_{i_j}(x_j) \neq 0$. Then

$$\begin{aligned}\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') \geq{}& \left(\left(r_N\alpha - r_N^2\frac{\gamma}{2}\right)(e - g + 1) - Re\right)m^2 \\ &+ (r_N\beta(e - g + 1) - Rd)m\hat{m} + (\delta(e - g + 1) - Rn)m' \\ &-\left(\left(r_N(g - 1) - \frac{r_N\gamma}{2} + \epsilon\right)(e - g + 1) + R\right)m.\end{aligned} \tag{10}$$

□

Proof. Suppose we have any monomials $\hat{M}_{i_1}, \ldots, \hat{M}_{i_{P(m,\hat{m})}}$ in $\hat{B}_{m,\hat{m}}$ such that the set $\{\hat{\rho}^C_{m,\hat{m}}(\hat{M}_{i_1}), \ldots, \hat{\rho}^C_{m,\hat{m}}(\hat{M}_{i_{P(m,\hat{m})}})\}$ is a basis of $H^0(C, L_W^m \otimes L_r^{\hat{m}})$. As our filtration is in order of increasing weight, a lower bound for $\sum_{j=1}^{P(m,\hat{m})} w_\lambda(\hat{M}_{i_j})$ is $\sum_{p=1}^{r_N m} p(\hat{\beta}_p - \hat{\beta}_{p-1})$. We calculate

$$\begin{aligned}\sum_{p=1}^{r_N m} p(\hat{\beta}_p - \hat{\beta}_{p-1}) &= r_N m \hat{\beta}_{r_N m} - \sum_{p=0}^{r_N m-1} \hat{\beta}_p \\ &\geq r_N m(em + d\hat{m} - g + 1) - \sum_{p=0}^{r_N m-1} ((e-\alpha)m + (d-\beta)\hat{m} + \gamma p + \epsilon_p) \\ &= \left(r_N\alpha - \frac{r_N^2\gamma}{2}\right)m^2 + r_N\beta m\hat{m} - \left(r_N(g-1) - \frac{r_N\gamma}{2} + \epsilon\right)m,\end{aligned}$$

where $\epsilon := \frac{1}{m}\sum_{p=0}^{r_N m-1}\epsilon_p$. Let λ' be the associated 1-PS of $SL(W)$. Thus, using Lemma 4.5, we calculate

$$\begin{aligned}&\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') \\ &\quad \geq \left(\left(r_N\alpha - r_N^2\frac{\gamma}{2}\right)m^2 + r_N\beta m\hat{m} - \left(r_N(g-1) - \frac{r_N\gamma}{2} + \epsilon\right)m + \delta m'\right)(e-g+1) \\ &\qquad - (m(em + d\hat{m} - g + 1) + nm')\sum_{i=0}^{N} r_i \\ &\quad = \left(\left(r_N\alpha - r_N^2\frac{\gamma}{2}\right)(e-g+1) - Re\right)m^2 + (r_N\beta(e-g+1) - Rd)m\hat{m} \\ &\qquad + (\delta(e-g+1) - Rn)m' - \left(\left(r_N(g-1) - \frac{r_N\gamma}{2} + \epsilon\right)(e-g+1) + R\right)m,\end{aligned}$$

where we have used the bounds $0 \leq \sum_{i=0}^{N} r_i \leq R$ to estimate appropriately, according to the sign of each term. ■

Remark. In general, we shall assume that m is very large, that $\hat{m}$ is proportional to m, and that m' is proportional to m^2.

The following claim is also one which we will refer to frequently in Section 5, and hence we have included it in this reference subsection.

If C is a general curve, we have an inclusion $i : C_{\text{red}} \hookrightarrow C$. The reduced curve C_{red} has normalization $\pi' : \bar{C}_{\text{red}} \to C_{\text{red}}$. Following Gieseker in [9], p. 22, we define the normalization $\pi : \bar{C} \to C$ by letting $\bar{C} := \bar{C}_{\text{red}}$ and $\pi := i \circ \pi'$. Then, whatever the properties of C, the curve $\bar{C}$ is smooth and integral (though possibly disconnected). With these conventions, we may show the following.

Claim 5.2 (cf. [9], p. 52).

(1) Let C be a generically reduced curve over k; we do not assume it has genus g. Let $\pi : \bar{C} \to C$ be the normalization morphism, and let $\mathcal{I}_C$ be the sheaf of nilpotents. Suppose that $C \subset \mathbf{P}(W) \times \mathbf{P}^r$. Define L_{WC} and L_{rC} as in Section 4.4, and let $\bar{L}_{W\bar{C}} := \pi^* L_{WC}$ and $\bar{L}_{r\bar{C}} := \pi^* L_{rC}$. Let

$$\pi_{m,\hat{m}*} : H^0\left(C, L_{WC}^m \otimes L_{rC}^{\hat{m}}\right) \to H^0\left(\bar{C}, \bar{L}_{W\bar{C}}^m \otimes \bar{L}_{r\bar{C}}^{\hat{m}}\right) \tag{11}$$

be the induced morphism. Then

$$\dim \ker \pi_{m,\hat{m}*} = h^0(C, \mathcal{I}_C).$$

(2) Suppose that C is a reduced curve. Let D be an effective divisor on C, and let M be an invertible sheaf on C such that $H^1(C, M) = 0$. Then $h^1(C, M(-D)) \leq \deg D$.

(3) Suppose that C is an integral and smooth curve with genus g_C. Let M be an invertible sheaf on C with $\deg M \geq 2g_C - 1$. Then $H^1(C, M) = 0$. □

5.2 First properties of GIT semistable maps

Proposition 5.3 (cf. [9], 1.0.2). Let M consist of integer triples $(m, \hat{m}, m')$ such that $m, \hat{m} > m_3$ and $m' \geq 1$ with $m > (q_1 - 1)(e - g + 1)$. Let $l \in \mathbf{H}_M(I)$. Suppose that $(C, x_1, \ldots, x_n) \subset \mathbf{P}(W) \times \mathbf{P}^r$ is connected and $SL(W)$-semistable with respect to l. Then $p_W(C)$ is a nondegenerate curve in $\mathbf{P}(W)$, i.e. $p_W(C)$ is not contained in any hyperplane in $\mathbf{P}(W)$. □

Proof. It is enough to prove that the composition

$$H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) \to H^0\left(p_W(C)_{\text{red}}, \mathcal{O}_{p_W(C)_{\text{red}}}(1)\right) \to H^0(C_{\text{red}}, L_{W\text{red}})$$

is injective. So suppose that it has nontrivial kernel W_0. Let λ be the 1-PS of $GL(W)$ which acts with weight 0 on W_0 and with weight 1 on $W_1 = W_2$, and let λ' be the associated 1-PS of $SL(W)$.

We wish to show that $(C, x_1, \ldots, x_n)$ is unstable with respect to any $l \in \mathbf{H}_M(I)$. For this proof we do not follow all parts of the strategy outlined in Section 5.1, as a simpler proof is available. By Lemma 4.4, it is sufficient to show that $\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') > 0$ for all $(m, \hat{m}, m') \in M$, so pick $(m, \hat{m}, m') \in M$.

Let $\hat{B}_{m,\hat{m}}$ be a basis of $H^0(\mathbf{P}(W) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W)}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m}))$ consisting of monomials of bidegree $(m, \hat{m})$. Then the first part of the conclusion of Lemma 4.5 is that there exist monomials $\hat{M}_{i_1}, \ldots, \hat{M}_{i_{P(m,\hat{m})}}$ in $\hat{B}_{m,\hat{m}}$ such that $\{\hat{\rho}^C_{m,\hat{m}}(\hat{M}_{i_1}), \ldots, \hat{\rho}^C_{m,\hat{m}}(\hat{M}_{i_{P(m,\hat{m})}})\}$ is a basis of $H^0(C, L^m_W \otimes L^{\hat{m}}_r)$. For each of the $\hat{M}_{i_j}$, write $\hat{M}_{i_j} = w_0^{\gamma_0} \cdots w_N^{\gamma_N} f_0^{\Gamma_0} \cdots f_r^{\Gamma_r}$.

Recall that, if $\mathcal{I}_C$ denotes the ideal sheaf of nilpotent elements of $\mathcal{O}_C$, then the integer q_1 satisfies $\mathcal{I}_C^{q_1} = 0$. Now suppose that $\sum_{i=0}^{N_0-1} \gamma_i \geq q_1$. It follows that $\hat{\rho}^C_{m,\hat{m}}(\hat{M}_{i_j}) = 0$, and so cannot be in a basis for $H^0(C, L^m_W \otimes L^{\hat{m}}_r)$. Thus $\sum_{i=0}^{N_0-1} \gamma_i \leq q_1 - 1$. The 1-PS λ acts with weight 1 on the factors $w_{N_0}, \ldots, w_N$, and so

$$w_\lambda\big(\hat{M}_{i_j}\big) \geq m - q_1 + 1.$$

Our basis consists of $P(m, \hat{m})$ such monomials, $\hat{M}_{i_1}, \ldots, \hat{M}_{i_{P(m,\hat{m})}}$, so we can estimate their total weight as

$$\sum_{j=1}^{P(m,\hat{m})} w_\lambda\big(\hat{M}_{i_j}\big) \geq P(m, \hat{m})(m - q_1 + 1).$$

We assumed that the n marked points lie on the curve. Hence if $w_{k_l}(x_l) \neq 0$, then $w_\lambda(w_{k_l})$ must be equal to 1, so $\sum_{l=1}^{n} w_\lambda(w_{k_l})m' = nm'$. Finally,

$$\sum_{i=0}^{N} w_\lambda(w_i) = \dim W_1 - \dim W_0 = e - g + 1 - \dim W_0 \leq e - g,$$

because $\dim W_0 \geq 1$.

Combining these estimates with Lemma 4.5, we obtain

$$\begin{aligned}\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') &\geq (P(m, \hat{m})(m - q_1 + 1) + nm')(e - g + 1) - (mP(m, \hat{m}) + nm')(e - g)\\ &\geq P(m, \hat{m})(m - (q_1 - 1)(e - g + 1)).\end{aligned}$$

However, $P(m, \hat{m})$ is positive and by hypothesis $m > (q_1 - 1)(e - g + 1)$; thus $\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') > 0$. Recall that λ' did not depend on the choice of $(m, \hat{m}, m') \in M$. So $\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') > 0$ for all $(m, \hat{m}, m') \in M$. Thus by Lemma 4.4, the curve $(C, x_1, \ldots, x_n)$ is not semistable with respect to any $l \in \mathbf{H}_M(I)$. ∎

Next, we would like to show that no components of a GIT semistable curve collapse under the projection p_W. We must refine the choice of virtual linearization to obtain this result; the proof is spread over Propositions 5.4–5.8.

Let C_i be an irreducible component of C. If the morphism $p_W|_{C_i}$ does not collapse C_i to a point, then it is finite. We shall find a range $M \subset \mathbb{N}^3$, such that if $(C, x_1, \ldots, x_n) \in I^{ss}(l)$ with $l \in \mathbf{H}_M(I)$, and if the morphism $p_W|_{C_i}$ does not collapse C_i to a point, then $p_W|_{C_{i\,\mathrm{red}}}$ is generically 1-1, and C_i is generically reduced.

We use the following notation. Write $C = C' \cup Y$, where C' is the union of all irreducible components of C which collapse under p_W and $Y = \overline{C - C'}$ is the union of all those that do not.

Let $p_W(C)_i$ be the irreducible components of $p_W(C)$, for $i = 1, \ldots, \ell$. We use these to label the components of C' and Y.

(1) Let $C'_{1,1}, \ldots, C'_{1,j'_1}$, up to $C'_{\ell,1}, \ldots, C'_{\ell,j'_\ell}$, be the irreducible components of C', labeled so that $p_W(C'_{i,j'}) \in p_W(C)_i$. If there is a tie (that is, the projection of a component of C' is a point lying on more than one component of $p_W(C)$), index it by the smallest i.

(2) Let $Y_{1,1}, \ldots, Y_{1,j_1}$, up to $Y_{\ell,1}, \ldots, Y_{\ell,j_\ell}$, be the irreducible components of Y so that $p_W(Y_{i,j}) = p_W(C)_i$. Without loss of generality, we may assume that these are ordered in such a way that, if we set

$$b_{i,j} := \deg p_W|_{Y_{i,j}\,\mathrm{red}},$$

then $b_{i,j} \geq b_{i,j+1}$.

Define

$$e_W := \deg_{p_W(C)_{\mathrm{red}}} \mathcal{O}_{\mathbf{P}(W)}(1) \quad e_{W\,i} := \deg_{p_W(C)_{i\,\mathrm{red}}} \mathcal{O}_{\mathbf{P}(W)}(1).$$

Since $p_W(C) \subset \mathbf{P}(W)$, we have $e_{W\,i} \geq 1$ for $i = 1, \ldots, \ell$.

By definition, the degree of L_W on the components $C'_{i,j'}$ is zero, so we define

$$e_{i,j} := \deg_{Y_{i,j\,\mathrm{red}}} L_W$$
$$d_{i,j} := \deg_{Y_{i,j\,\mathrm{red}}} L_r \quad d'_{i,j'} := \deg_{C'_{i,j'}\,\mathrm{red}} L_r.$$

Finally, let $\xi_{i,j}$ be the generic point of $Y_{i,j}$ and ξ_i be the generic point of $p_W(C)_i$. Write

$$k_{i,j} := \mathrm{length}\, \mathcal{O}_{Y_{i,j},\xi_{i,j}} \quad k_i := \mathrm{length}\, \mathcal{O}_{p_W(C)_i,\xi_i}.$$

Then

$$e = \sum k_{i,j} e_{i,j}, \qquad e_{i,j} = b_{i,j} e_{W\,i}, \qquad e_W = \sum k_i e_{W\,i}.$$

Proposition 5.4 ([9], 1.0.3)**.** Let M consist of $(m, \hat{m}, m')$ such that $m, \hat{m} > m_3$ and

$$m > \left(g - \tfrac{1}{2} + e(q_1 + 1) + q_3 + \mu_1 m_2\right)(e - g + 1)$$

with

$$d\frac{\hat{m}}{m} + n\frac{m'}{m^2} < \frac{1}{4}e - \frac{5}{4}g + \frac{3}{4}.$$

Let $l \in \mathbf{H}_M(I)$. Suppose that $(C, x_1, \ldots, x_n)$ is connected and semistable with respect to l. Then, in the notation explained above, the morphism $p_W|_{Y\,\mathrm{red}}$ is generically 1-1, that is, $b_{i,j} = 1$ and $j_i = 1$ for all $i = 1, \ldots, \ell$. Furthermore, Y is generically reduced, i.e. $k_{i,j} = 1$ for all $i = 1, \ldots, \ell$ and $j = 1, \ldots, j_i$. □

Remark. Since $e \geq ag$ and $a \geq 5$, it follows that $e - 5g + 3 > 0$, which means that the condition on $d\frac{\hat{m}}{m} + n\frac{m'}{m^2}$ may be satisfied.

Proof. Suppose not. Then we may assume that at least one of the following is true: $j_1 \geq 2$; or $b_{1,1} \geq 2$; or, for some $1 \leq j \leq j_1$, we have $k_{1,j} \geq 2$. The first condition implies that two irreducible components of Y map to the same irreducible component of $p_W(C)$. The second condition implies that a component of Y is a degree $b_{1,1} \geq 2$ cover of its image. The third condition implies that the subcurve Y is not generically reduced.

Let W_0 be the kernel of the restriction map

$$H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) \to H^0\left(p_W(C)_{1\,\mathrm{red}}, \mathcal{O}_{p_W(C)_{1\,\mathrm{red}}}(1)\right).$$

Step 1. We claim that $W_0 \neq 0$. To see this, suppose $W_0 = 0$. Let D_1 be a divisor on $p_W(C)_{1\,\mathrm{red}}$ corresponding to the invertible sheaf $\mathcal{O}_{p_W(C)_{1\,\mathrm{red}}}(1)$ and having support in the smooth locus of $p_W(C)_{1\,\mathrm{red}}$. We have an exact sequence

$$0 \to \mathcal{O}_{p_W(C)_{1\,\mathrm{red}}} \to \mathcal{O}_{p_W(C)_{1\,\mathrm{red}}}(1) \to \mathcal{O}_{D_1} \to 0.$$

Then the long exact sequence in cohomology implies that

$$h^0\big(p_W(C)_{1\,\mathrm{red}}, \mathcal{O}_{p_W(C)_{1\,\mathrm{red}}}(1)\big) \le h^0\big(p_W(C)_{1\,\mathrm{red}}, \mathcal{O}_{D_1}\big) + h^0\big(p_W(C)_{1\,\mathrm{red}}, \mathcal{O}_{p_W(C)_{1\,\mathrm{red}}}\big).$$

Note that $h^0(p_W(C)_{1\,\mathrm{red}}, \mathcal{O}_{p_W(C)_{1\,\mathrm{red}}}) = 1$, and

$$h^0\big(p_W(C)_{1\,\mathrm{red}}, \mathcal{O}_{D_1}\big) \le \deg D_1 = \deg \mathcal{O}_{p_W(C)_{1\,\mathrm{red}}}(1) = e_{W1}.$$

If $W_0 = 0$, then

$$\begin{aligned} e - g + 1 = h^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) \le h^0\big(p_W(C)_{1\,\mathrm{red}}, \mathcal{O}_{p_W(C)_{1\,\mathrm{red}}}(1)\big) \le e_{W1} + 1 \\ \Rightarrow e - g \le e_{W1} = \frac{e_{1,1}}{b_{1,1}}. \end{aligned} \tag{12}$$

We show that this statement leads to a contradiction. First suppose that $b_{1,1} \ge 2$ or $k_{1,1} \ge 2$. Now we rearrange equation (12) to find

$$\begin{aligned} k_{1,1}b_{1,1}(e-g) \le k_{1,1}e_{1,1} = e - \sum_{(i,j)\neq(1,1)} k_{i,j}e_{i,j} \le e \\ \Rightarrow (k_{1,1}b_{1,1} - 1)e \le k_{1,1}b_{1,1}g. \end{aligned}$$

But our assumptions imply that $\frac{k_{1,1}b_{1,1}-1}{k_{1,1}b_{1,1}} \ge \frac{1}{2}$ and we obtain $\frac{e}{2} \le g$, a contradiction. On the other hand, suppose that $b_{1,1} = k_{1,1} = 1$ but $j_1 \ge 2$. Then

$$e_{W1} = k_{1,1}b_{1,1}e_{W1} = e - \sum_{(i,j)\neq(1,1)} k_{i,j}b_{i,j}e_{Wi} \le e - k_{1,2}b_{1,2}e_{W1} \le e - e_{W1},$$

i.e. $e_{W1} \le \frac{1}{2}e$. Combining this with equation (12), we again obtain the contradiction $\frac{e}{2} \le g$.

Step 2. By Step 1 we have that $W_0 \neq 0$, and in particular that $e_{W1} < (e-g)$, as it is line (12) which leads to the contradiction. Following the strategy prescribed in Section 5.1, let λ be the 1-PS of $GL(W)$ whose weight on W_0 is 0, and whose weight on $W_1 = W_2$ is 1. Let $\overline{\Omega}_p^{m,\hat{m}}$ and $\hat{\beta}_p$ be as defined in Section 5.1.

Pick $(m, \hat{m}, m') \in M$ and suppose $\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') \le 0$. We shall show that this leads to a contradiction.

Let B be the inverse image of $p_W(C)_1$ under p_W, i.e. $B = \bigcup_{j'=1}^{j_1'} C'_{1,j'} \cup \bigcup_{j=1}^{j_1} Y_{1,j}$. Let $\tilde{C}$ be the closure of $C - B$ in C. Since C is connected, there is at least one closed point in

$B \cap \tilde{C}$. Choose one such point P. Let

$$\hat{\rho}_{m,\hat{m}}^{\tilde{C},C} : H^0\big(C, L_W^m \otimes L_r^{\hat{m}}\big) \to H^0\big(\tilde{C}, L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}}\big)$$

be the map induced by restriction.

The following claim is analogous to one of Gieseker and may be proved using a similar argument.

Claim 5.5 (cf. [9], p. 43)**.** $\tilde{C} = \overline{C - B}$ can be given the structure of a closed subscheme of C such that for all $0 \le p \le m - q_1$,

$$\overline{\Omega}_p^{m,\hat{m}} \cap \ker\big\{\hat{\rho}_{m,\hat{m}}^{\tilde{C},C} : H^0\big(C, L_W^m \otimes L_r^{\hat{m}}\big) \to H^0\big(\tilde{C}, L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}}\big)\big\} = 0.$$

□

Let $\mathcal{I}_P$ be the ideal subsheaf of $\mathcal{O}_{\tilde{C}}$ defining the closed point P. We have an exact sequence

$$0 \to \mathcal{I}_P^{m-p} \otimes L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}} \to L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}} \to \mathcal{O}_{\tilde{C}}/\mathcal{I}_P^{m-p} \otimes L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}} \to 0.$$

In cohomology, this induces

$$\begin{aligned} 0 \to\ & H^0\big(\tilde{C}, \mathcal{I}_P^{m-p} \otimes L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}}\big) \to H^0\big(\tilde{C}, L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}}\big) \\ & \to H^0\big(\tilde{C}, \mathcal{O}_{\tilde{C}}/\mathcal{I}_P^{m-p} \otimes L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}}\big) \to H^1\big(\tilde{C}, \mathcal{I}_P^{m-p} \otimes L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}}\big) \to 0. \end{aligned} \tag{13}$$

The following facts are analogous to those stated by Gieseker in ([9], p. 44.)

(1) $h^0\big(\tilde{C}, L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}}\big) = \chi\big(L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}}\big) = \deg_{\tilde{C}} L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}} + \chi(\mathcal{O}_{\tilde{C}})$
$\le \big(e - \sum k_{1,j}e_{1,j}\big)m + \big(d - \sum k_{1,j}d_{1,j} - \sum k_{1,j'}d'_{1,j'}\big)\hat{m} + q_3.$

(2) Since $\mathcal{O}_{\tilde{C}}/\mathcal{I}_P^{m-p} \otimes L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}}$ has support only at the point $P \in \tilde{C}$, we make the estimate $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}/\mathcal{I}_P^{m-p} \otimes L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}}) \ge m - p$.

(3) For $0 \le p \le m_2 - 1$, Proposition 4.6 says that $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}/\mathcal{I}_P^{m-p} \otimes L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}}) \le \mu_1(m-p) + \mu_2$, and so it follows that from the long exact sequence in cohomology, $h^1(\tilde{C}, \mathcal{I}_P^{m-p} \otimes L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}}) \le \mu_1(m-p) + \mu_2$.

(4) For $m_2 \le p < m$, we have $h^1(\tilde{C}, \mathcal{I}_P^{m-p} \otimes L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}}) = 0$ (cf. Proposition 4.6).

(5) $\hat{\rho}_{m,\hat{m}}^{\tilde{C},C}(\overline{\Omega}_p^{m,\hat{m}}) \subset H^0\big(\tilde{C}, \mathcal{I}_P^{m-p} \otimes L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}}\big) \subset H^0\big(\tilde{C}, L_{W\tilde{C}}^m \otimes L_{r\tilde{C}}^{\hat{m}}\big)$.

If $p > m - q_1$, we may make no useful estimate, but if $0 \le p \le m - q_1$ then by Claim 5.5 and fact (5), we have

$$\hat{\beta}_p = \dim \overline{\Omega}_p^{m,\hat{m}} \le h^0(\tilde{C}, \mathcal{I}_P^{m-p} \otimes L_{W\tilde{C}}^{m} \otimes L_{r\tilde{C}}^{\hat{m}}).$$

Now the exact sequence (13) tells us

$$\begin{aligned} h^0(\tilde{C}, \mathcal{I}_P^{m-p} \otimes L_{W\tilde{C}}^{m} \otimes L_{r\tilde{C}}^{\hat{m}}) = h^0(\tilde{C}, L_{W\tilde{C}}^{m} \otimes L_{r\tilde{C}}^{\hat{m}}) - h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}/\mathcal{I}_P^{m-p} \otimes L_{W\tilde{C}}^{m} \otimes L_{r\tilde{C}}^{\hat{m}}) \\ + h^1(\tilde{C}, \mathcal{I}_P^{m-p} \otimes L_{W\tilde{C}}^{m} \otimes L_{r\tilde{C}}^{\hat{m}}). \end{aligned}$$

Thus, using the facts above, we obtain the following estimate:

$$\hat{\beta}_p \le \begin{cases} (e - \sum k_{1,j}e_{1,j})m + (d - \sum k_{1,j}d_{1,j} - \sum k_{1,j'}d'_{1,j'})\hat{m} & \\ \quad + q_3 + p - m + \mu_1(m-p) + \mu_2 & \text{if } 0 \le p \le m_2 - 1 \\ (e - \sum k_{1,j}e_{1,j})m + (d - \sum k_{1,j}d_{1,j} - \sum k_{1,j'}d'_{1,j'})\hat{m} & \\ \quad + q_3 + p - m & \text{if } m_2 \le p \le m - q_1 \\ em + d\hat{m} - g + 1 & \text{if } m - q_1 + 1 \le p \le m. \end{cases} \tag{14}$$

Step 3. We wish to estimate $\sum_{j=1}^{P(m,\hat{m})} w_\lambda(\hat{M}_{i_j})$. Unfortunately, we cannot apply Lemma 5.1 as the estimates in equation (14) do not quite fit the setup there. Instead, we proceed as follows: $\sum_{j=1}^{P(m,\hat{m})} w_\lambda(\hat{M}_{i_j})$ is greater than or equal to $\sum_{p=1}^{m} p(\hat{\beta}_p - \hat{\beta}_{p-1})$, and one may easily calculate

$$\sum_{p=1}^{m} p(\hat{\beta}_p - \hat{\beta}_{p-1}) = m\hat{\beta}_m - \sum_{p=0}^{m-1} \hat{\beta}_p \ge \left(\sum k_{1,j}e_{1,j} + \frac{1}{2}\right) m^2 - S_1 m + c_2, \tag{15}$$

where

$$\begin{aligned} S_1 &= g - \frac{3}{2} + \sum k_{1,j}e_{1,j}(q_1+1) + q_3 + \mu_1 m_2 \\ c_2 &= (q_1 - 1)\left(g + q_3 - \frac{q_1}{2} - 1\right) - \mu_2 m_2 + \mu_1 \frac{m_2(m_2-1)}{2}. \end{aligned}$$

The inequality (15) follows because the term $(\sum k_{1,j}d_{1,j} + \sum k_{1,j'}d_{1,j'})\hat{m}(m - q_1 + 1)$ is positive, since the hypotheses imply that $m > q_1$. Finally, we may estimate $c_2 \ge 0$, since

$q_3 > q_1$ and $\mu_1 > \mu_2$ (see Proposition 4.6), yielding

$$\sum_{j=1}^{P(m,\hat{m})} w_\lambda\big(\hat{M}_{i_j}\big) \geq \left(\sum k_{1,j}e_{1,j} + \frac{1}{2}\right) m^2 - S_1 m. \tag{16}$$

Step 4. We estimate the weight coming from the marked points. We know nothing about which components each marked point lies on, so we can simply state that $\sum_{l=1}^{n} w_\lambda(w_{k_l})m' \geq 0$. Finally, we estimate the sum of the weights

$$\begin{aligned}\sum_{i=0}^{N} w_\lambda(w_i) = \dim W_1 - \dim W_0 &\leq h^0\big(p_W(C_{1,1})_{\text{red}}, \mathcal{O}_{p_W(C_{1,1})_{\text{red}}}(1)\big)\\ &\leq \deg \mathcal{O}_{p_W(C_{1,1})\,\text{red}}(1) + 1 \leq e_{W1} + 1.\end{aligned} \tag{17}$$

Step 5. We combine inequalities (16) and (17) with Lemma 4.5 to obtain a contradiction as follows:

$$\begin{aligned}&\left(\left(\sum k_{1,j}e_{1,j} + \frac{1}{2}\right) m^2 - S_1 m\right)(e-g+1) - (mP(m,\hat{m}) - nm')(e_{W1}+1)\\ &\quad \leq \mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda')\\ &\quad \leq 0\\ &\Longrightarrow \left(\sum k_{1,j}e_{1,j} + \frac{1}{2}\right)(e-g+1)m^2 - \left(e + \frac{1}{m}\right)(e_{W1}+1)m^2 - S_1(e-g+1)m\\ &\quad \leq (e_{W1}+1)\left(d\frac{\hat{m}}{m} + n\frac{m'}{m^2}\right) m^2.\end{aligned}$$

We showed that $e_{W1} < e-g$, and by hypothesis $m > (g - \frac{1}{2} + e(q_1+1) + q_3 + \mu_1 m_2)(e - g + 1) \geq (S_1+1)(e-g+1)$, so we may estimate

$$\frac{(\sum k_{1,j}e_{1,j} + \frac{1}{2})(e-g+1) - e(e_{W1}+1) - 1}{(e_{W1}+1)} \leq d\frac{\hat{m}}{m} + n\frac{m'}{m^2}. \tag{18}$$

Note that since $b_{1,1} \geq 2$ or $k_{1,1} \geq 2$ or $j_1 \geq 2$, we have $\sum k_{1,j}b_{1,j} \geq 2$. Thus

$$\left(e_{W1}\sum k_{1,j}b_{1,j} + \frac{1}{2}\right)(e-g+1) - e(e_{W1}+1) - 1 > 0.$$

Furthermore, the quantity

$$\frac{\left(e_{W1}\sum k_{1,j}b_{1,j}+\frac{1}{2}\right)(e-g+1)-e(e_{W1}+1)-1}{(e_{W1}+1)}$$

is minimized when e_{W1} takes its smallest value, that is, when $e_{W1}=1$. So

$$\begin{aligned}\frac{\left(e_{W1}\sum k_{1,j}b_{1,j}+\frac{1}{2}\right)(e-g+1)-e(e_{W1}+1)-1}{(e_{W1}+1)} &\geq \frac{\frac{5}{2}(e-g+1)-2e-1}{2}\\ &= \frac{1}{4}e-\frac{5}{4}g+\frac{3}{4}.\end{aligned}$$

But by hypothesis, $d\frac{\hat{m}}{m}+n\frac{m'}{m^2}<\frac{1}{4}e-\frac{5}{4}g+\frac{3}{4}$; combining this result with equation (18) gives a contradiction. This implies that $\mu^{L_{m,\hat{m},m'}}((C,x_1,\ldots,x_n),\lambda')>0$, and this holds for all $(m,\hat{m},m')\in M$. It follows by Lemma 4.4 that $(C,x_1,\ldots,x_n)$ is not semistable with respect to l for any $l\in \mathbf{H}_M(I)$. This completes the proof of Proposition 5.4. ■

Next we derive inequality (19), which is similar to Gieseker's so-called Basic Inequality. This turns out to be an extremely useful tool. Later we will show that this inequality holds in more generality than is stated here (see Amplification 5.18). One of the main changes to our potential stability proof compared to Gieseker's is that he derives this inequality much later in his proof. We noticed that the Basic Inequality is something we can prove on components which do not collapse under p_W, and we will use it to prove that there are no components which collapse under p_W.

Notation. Suppose C is a curve which has at least two irreducible components, and suppose it is generically reduced on any components which do not collapse under p_W. Let Y be a union of components which do not collapse under p_W, and let C' be the closure of $C-Y$ in C as constructed above. Let $C'\xrightarrow{\iota_{C'}}C\xrightarrow{\iota_C}\mathbf{P}(W)\times\mathbf{P}^r$ and $Y\xrightarrow{\iota_Y}C$ be the inclusion morphisms. Let

$$\begin{aligned}L_{WY}&:=\iota_Y^*\iota_C^*p_W^*\mathcal{O}_{\mathbf{P}(W)}(1) & L_{WC'}&:=\iota_{C'}^*\iota_C^*p_W^*\mathcal{O}_{\mathbf{P}(W)}(1)\\ L_{rY}&:=\iota_Y^*\iota_C^*p_r^*\mathcal{O}_{\mathbf{P}^r}(1) & L_{rC'}&:=\iota_{C'}^*\iota_C^*p_r^*\mathcal{O}_{\mathbf{P}^r}(1).\end{aligned}$$

Let $\pi:\bar{C}\to C$ be the normalization morphism. Let $\bar{L}_{W\bar{Y}}:=\pi^*L_{WY}$ and similarly define $\bar{L}_{W\bar{C}'}$, $\bar{L}_{r\bar{Y}}$ and $\bar{L}_{r\bar{C}'}$. Normalization induces a homomorphism

$$\pi_{m,\hat{m}*}:H^0\left(C,L_W^m\otimes L_r^{\hat{m}}\right)\to H^0\left(\bar{C},\bar{L}_W^m\otimes\bar{L}_r^{\hat{m}}\right).$$

Define

$$e' := \deg_{\bar{C}'} \bar{L}_{W\bar{C}'} = \deg_{C'} L_{WC'}$$
$$d' := \deg_{\bar{C}'} \bar{L}_{r\bar{C}'} = \deg_{C'} L_{rC'},$$

and let n' be the number of markings on C'. Write $h^0 := h^0(p_W(C'), \mathcal{O}_{p_W(C')}(1))$. Recall that we defined $\bar{g}$ to be

$$\bar{g} := \min\{0, g_{\bar{Y}} \mid \bar{Y} \text{ is the normalization of a complete subcurve } Y \text{ contained in a connected fiber } \mathcal{C}_h \text{ for some } h \in \mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r)\}.$$

Proposition 5.6 ([9], 1.0.7). Let $M \subset \tilde{M}$, where $\tilde{M}$ consists of those $(m, \hat{m}, m')$ such that $m, \hat{m} > m_3$ and

$$m > \left(g - \tfrac{1}{2} + e(q_1 + 1) + q_3 + \mu_1 m_2\right)(e - g + 1)$$

with

$$d\frac{\hat{m}}{m} + n\frac{m'}{m^2} < \frac{1}{4}e - \frac{5}{4}g + \frac{3}{4}.$$

Let $l \in \mathbf{H}_M(I)$. Let $(C, x_1, \ldots, x_n)$ be a connected marked curve which is semistable with respect to l. Suppose C has at least two irreducible components. Let C' and Y be as above; in particular, no component of Y collapses under p_W. The subcurve C' need not be connected. Suppose C' has b connected components. Suppose there exist distinct points $P_1, \ldots, P_k$ on $\bar{Y}$ satisfying

(i) $\pi(P_i) \in Y \cap C'$ for all $1 \le i \le k$;

(ii) for each irreducible component $\bar{Y}_j$ of $\bar{Y}$,

$$\deg_{\bar{Y}_j}(\bar{L}_{W\bar{Y}}(-D)) \ge 0,$$

where $D = P_1 + \cdots + P_k$.

Then there exist $(m, \hat{m}, m') \in M$ such that

$$e' + \frac{k}{2} < \frac{h^0 e + (dh^0 - d'(e-g+1))\frac{\hat{m}}{m} + (nh^0 - n'(e-g+1))\frac{m'}{m^2}}{e-g+1} + \frac{S}{m}, \tag{19}$$

where $S = g + k(2g - \frac{3}{2}) + q_2 - \bar{g} + 1$. □

Proof. We depart from the strategy outlined in Section 5.1 in one important regard: this is not a proof by contradiction in exactly the same way that the other proofs in this section are. However, the following proof will still use some of the notation and constructions described there.

We define the key 1-PS for this case. We shall revisit it later, so we give it the special notation $\lambda_{C'}$. Let

$$W_0 := \ker \left\{ H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) \to H^0\left(p_W(C'), \mathcal{O}_{p_W(C')}(1)\right)\right\}.$$

Let $\lambda_{C'}$ be the 1-PS of $GL(W)$ which acts with weight 0 on W_0 and weight 1 on $W_1 = W_2$. Let $\lambda'_{C'}$ be the associated 1-PS of $SL(W)$.

Claim 5.7. If $(C, x_1, \ldots, x_n)$ satisfies $\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda'_{C'}) \leq 0$ for some $(m, \hat{m}, m') \in M$, then equation (19) is satisfied for this choice of $(m, \hat{m}, m')$. □

Suppose the claim is true. Fix $l \in \mathbf{H}_M(I)$ and suppose that $(C, x_1, \ldots, x_n)$ is semistable with respect to l. If there do no exist $(m, \hat{m}, m') \in M$ satisfying equation (19), then it follows from Claim 5.7 that $\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda'_{C'}) > 0$ for all $(m, \hat{m}, m') \in M$. But then Lemma 4.4 tells us that $(C, x_1, \ldots, x_n)$ is not semistable with respect to l. This contradiction implies the existence of such $(m, \hat{m}, m') \in M$.

It remains to prove Claim 5.7, so assume that $\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda'_{C'}) \leq 0$. We shall derive the fundamental inequality from this, using Lemma 4.5.

Estimate the weights for $\lambda_{C'}$ coming from the marked points. There are n' of these on C', so $\sum_{l=1}^{n} w_{\lambda_{C'}}(w_{k_l})m' \geq n'm'$. Also, estimate the sum of the weights. It is clear from the definition of $\lambda_{C'}$ that $\sum_{i=0}^{N} w_{\lambda_{C'}}(w_i) \leq h^0$.

Now we look at the weight coming from the curve.

Let $\overline{\Omega}_p^{m,\hat{m}}$ and $\hat{\beta}_p$ be as defined in Section 5.1.

For $p = m$, it is clear that $\hat{\beta}_m = h^0(C, L_W^m \otimes L_r^{\hat{m}}) = em + d\hat{m} - g + 1$. We estimate $\hat{\beta}_p$ in the case $p \neq m$. Restriction to Y induces a homomorphism

$$\hat{\rho}^{Y,C}_{m,\hat{m}} : H^0\big(C, L_W^m \otimes L_r^{\hat{m}}\big) \to H^0\big(Y, L_{WY}^m \otimes L_{rY}^{\hat{m}}\big).$$

We restrict this to $\overline{\Omega}_p^{m,\hat{m}}$, where $0 \leq p < m$. Note that if $\hat{M}$ is a monomial in $\overline{\Omega}_p^{m,\hat{m}}$ and $p < m$, then $\hat{M}$ has at least one factor from W_0, and hence by definition $\hat{M}$ vanishes on C'. If such $\hat{M}$ also vanishes on Y, then $\hat{M}$ is zero on C. Hence the restriction $\hat{\rho}^{Y,C}_{m,\hat{m}}|_{\overline{\Omega}_p^{m,\hat{m}}}$ has zero kernel, so is an isomorphism of vector spaces, and thus

$$\dim \hat{\rho}^{Y,C}_{m,\hat{m}}\big(\overline{\Omega}_p^{m,\hat{m}}\big) = \dim \overline{\Omega}_p^{m,\hat{m}} = \hat{\beta}_p.$$

We denote $\hat{\rho}^{Y,C}_{m,\hat{m}}(\overline{\Omega}_p^{m,\hat{m}})$ by $\overline{\Omega}_p^{m,\hat{m}}|_Y$.

The normalization morphism $\pi_Y : \bar{Y} \to Y$ induces a homomorphism

$$\pi_{Y\,m,\hat{m}*} : H^0\big(Y, L_W^m \otimes L_r^{\hat{m}}\big) \to H^0\big(\bar{Y}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}}\big).$$

By definition, the sections in $\pi_{Y\,m,\hat{m}*}(\overline{\Omega}_p^{m,\hat{m}}|_Y)$ vanish to order at least $m - p$ at the points $P_1, \ldots, P_k$. Thus

$$\pi_{Y\,m,\hat{m}*}\big(\overline{\Omega}_p^{m,\hat{m}}\big|_Y\big) \subseteq H^0\big(\bar{Y}, \bar{L}_{W\bar{Y}}^m \otimes \bar{L}_{r\bar{Y}}^{\hat{m}}(-(m-p)D)\big).$$

Then

$$\begin{aligned}\hat{\beta}_p = \dim \big(\overline{\Omega}_p^{m,\hat{m}}\big)_Y &\leq h^0\big(\bar{Y}, \bar{L}_{W\bar{Y}}^m \otimes \bar{L}_{r\bar{Y}}^{\hat{m}}(-(m-p)D)\big) + \dim\ker \pi_{Y\,m,\hat{m}*} \\ &= (e - e')m + (d - d')\hat{m} - k(m-p) - \bar{g} + 1 \\ &\quad + h^1\big(\bar{Y}, \bar{L}_{W\bar{Y}}^m \otimes \bar{L}_{r\bar{Y}}^{\hat{m}}(-(m-p)D)\big) + \dim\ker \pi_{Y\,m,\hat{m}*}.\end{aligned} \tag{20}$$

We apply the estimates of Claim 5.2 to our current situation.

(1) $\dim\ker \pi_{Y\,m,\hat{m}*} < q_2$.

No component of Y collapses under p_W, so by Proposition 5.4 the curve Y is generically reduced. Claim 5.2(1) may be applied to $Y \subset \mathbf{P}(W) \times \mathbf{P}^r$. Let $\mathcal{I}_Y$ denote the ideal sheaf of nilpotents in $\mathcal{O}_Y$. Then $\dim\ker \pi_{Y\,m,\hat{m}*} < h^0(Y, \mathcal{I}_Y)$. In Proposition 4.6, the constant q_2 was defined to have the property $h^0(C, \mathcal{I}_C) < q_2$; hence $h^0(Y, \mathcal{I}_Y) < q_2$ as well.

(2) $h^1(\bar{Y}, \bar{L}_{W\bar{Y}}^m \otimes \bar{L}_{r\bar{Y}}^{\hat{m}}(-(m-p)D)) \leq k(m-p) \leq km$ if $0 \leq p \leq 2g - 2$.

The sheaf $\bar{L}^m_{W\bar{Y}} \otimes \bar{L}^{\hat{m}}_{r\bar{Y}}$ is locally free on $\bar{Y}$, and we have chosen m and $\hat{m}$ so that $H^1(\bar{Y}, \bar{L}^m_{W\bar{Y}} \otimes \bar{L}^{\hat{m}}_{r\bar{Y}}) = 0$. The hypotheses of Claim 5.2(2) hold, and we calculate $\deg(m-p)D = k(m-p)$. We make a coarser estimate than we could as this will be sufficient for our purposes.

(3) $h^1(\bar{Y}, \bar{L}^m_{W\bar{Y}} \otimes \bar{L}^{\hat{m}}_{r\bar{Y}}(-(m-p)D)) = 0$ if $2g-1 \leq p \leq m-1$.

$\bar{Y}$ is reduced, and is a union of disjoint irreducible (and hence integral) components $\bar{Y}_j$ of genus $g_{Y_j} \leq g$. We apply Claim 5.2(3) separately to each component. Our assumption (ii) was that $\deg_{\bar{Y}_j}(\bar{L}_{W\bar{Y}}(-D)) \geq 0$, so

$$\deg_{\bar{Y}_j}(\bar{L}_{W\bar{Y}}) \geq \deg_{\bar{Y}_j} D.$$

If $\deg_{\bar{Y}_j} D \geq 1$, then

$$\begin{aligned}\deg_{\bar{Y}_j}\left(\bar{L}^m_{W\bar{Y}} \otimes \bar{L}^{\hat{m}}_{r\bar{Y}}(-(m-p)D)\right) &\geq m\big(\deg_{\bar{Y}_j} D\big) - (m-p)\big(\deg_{\bar{Y}_j} D\big)\\ &\geq p \geq 2g-1 \geq 2g_{Y_j}-1,\end{aligned}$$

as required. On the other hand, suppose that $\deg_{\bar{Y}_j} D = 0$. We assumed that no component of Y collapses under p_W, and hence for each j, the degree $\deg_{\bar{Y}_j}(\bar{L}_{W\bar{Y}}) \geq 1$. Thus, again

$$\deg_{\bar{Y}_j}\left(\bar{L}^m_{W\bar{Y}} \otimes \bar{L}^{\hat{m}}_{r\bar{Y}}(-(m-p)D)\right) \geq m \geq 2g_{Y_j}-1.$$

Combining this data with our previous formula (20), we have shown

$$\hat{\beta}_p \leq \begin{cases}(e-e'-k)m + (d-d')\hat{m} + kp - \bar{g} + 1 + q_2 + km & 0 \leq p \leq 2g-2\\ (e-e'-k)m + (d-d')\hat{m} + kp - \bar{g} + 1 + q_2, & 2g-1 \leq p \leq m-1.\end{cases}$$

Thus, we may use Lemma 5.1, setting $\alpha = e' + k$, $\beta = d'$, $\gamma = k$, $\delta = n'$, $\epsilon = -\bar{g} + 1 + q_2 + (2g-1)k$, $r_N = 1$ and $R = h^0$. Following Lemma 5.1, we see

$$\begin{aligned}\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') \geq{}& \left(\left(e' + \frac{k}{2}\right)(e-g+1) - h^0 e\right)m^2\\ &+ (d'(e-g+1) - dh^0)m\hat{m} + (n'(e-g+1) - nh^0)m'\\ &- \left(g - \frac{k}{2} - \bar{g} + q_2 + (2g-1)k\right)(e-g+1)m - h^0 m.\end{aligned}$$

Thus, since we assume that $\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda) \le 0$, we may conclude that

$$e' + \frac{k}{2} < \frac{h^0 e + (dh^0 - d'(e-g+1))\frac{\hat{m}}{m} + (nh^0 - n'(e-g+1))\frac{m'}{m^2}}{e-g+1} + \frac{S}{m},$$

where $S = g + k(2g - \frac{3}{2}) + q_2 - \bar{g} + 1$. ■

This fundamental inequality allows us finally to show that no irreducible components of C collapse under projection to $\mathbf{P}(W)$.

Proposition 5.8 ([9], 1.0.3). Let M consist of those $(m, \hat{m}, m')$ such that $m, \hat{m} > m_3$ and

$$m > \max \left\{ \begin{matrix} \left(g - \frac{1}{2} + e(q_1 + 1) + q_3 + \mu_1 m_2\right)(e - g + 1), \\ (6g + 2q_2 - 2\bar{g})(e - g + 1) \end{matrix} \right\}$$

with

$$d\frac{\hat{m}}{m} + n\frac{m'}{m^2} < \frac{1}{4}e - \frac{5}{4}g + \frac{3}{4},$$

while

$$\frac{\hat{m}}{m} > 1 + \frac{\frac{3}{2}g - 1 + d + n\frac{m'}{m^2}}{e - g + 1 - d}.$$

Let $l \in \mathbf{H}_M(I)$. If C is connected and $(C, x_1, \ldots, x_n)$ is semistable with respect to l, then no irreducible components of C collapse under p_W. □

Remark. As the denominator $e - g + 1 - d$ is equal to $(2a-1)(g-1) + an + (ca-1)d$, it is evident that this is positive.

Proof. This is trivial if $d = 0$; assume that $d \ge 1$. Suppose that at least one component of C collapses under p_W. Let C' be the union of all irreducible components of C which collapse under p_W and let $Y := \overline{C - C'}$. Suppose that C' consists of b connected components, namely $C'_1, \ldots, C'_b$. If $d'_i = \deg_{C'_i} L_{r\,C'_i}$ then $d'_i \ge 1$, since $e'_i = 0$, for $i = 1, \ldots, b$. But then $d' = \deg_{C'} L_{r\,C'} = \sum_{i=1}^{b} d'_i \ge b$. Hence

$$1 \le b \le d' \le d.$$

The curve C is connected, so $C' \cap Y \neq \emptyset$. Choose one point $P \in \bar{Y}$ such that $\pi(P) \in C' \cap Y$. We have by definition $\deg_{\bar{Y}_j}(\bar{L}_{W\bar{Y}}) \geq 1$, so $\deg_{\bar{Y}_j}(\bar{L}_{W\bar{Y}}(-P)) \geq 0$ for each irreducible component Y_j of Y. The hypotheses of Proposition 5.6 are satisfied for $k = 1$, and M as in the statement of this proposition. Let $(m, \hat{m}, m') \in M$ be the integers which that corollary provides, satisfying equation (19).

Since C' consists of b connected components, it is collapsed to at most b distinct points under p_W, so we estimate

$$h^0\left(p_W(C'), \mathcal{O}_{p_W(C')}(1)\right) \leq b.$$

Recall that we defined $S = g + k(2g - \frac{3}{2}) + q_2 - \bar{g} + 1$. In the current situation, $k = 1$, so $S < 3g + q_2 - \bar{g}$. The hypotheses on m imply then that $\frac{S}{m}(e - g + 1) < \frac{1}{2}$. Estimate $n' \geq 0$. Then inequality (19) reads

$$\begin{aligned} 0 + \frac{1}{2} \leq e' + \frac{k}{2} &\leq \frac{h^0 e + (dh^0 - d'(e-g+1))\frac{\hat{m}}{m} + (nh^0 - n'(e-g+1))\frac{m'}{m^2}}{e - g + 1} + \frac{S}{m} \\ &\leq \left(1 - \frac{1}{2b}\right) e + \frac{1}{2b} g + n\frac{m'}{m^2} \leq e + \frac{1}{2} g + n\frac{m'}{m^2} \\ \Rightarrow \frac{\hat{m}}{m} &\leq 1 + \frac{\frac{3}{2}g - 1 + d + n\frac{m'}{m^2}}{e - g + 1 - d}. \end{aligned}$$

We have contradicted our hypothesis that $\frac{\hat{m}}{m} > 1 + \frac{\frac{3}{2}g - 1 + d + n\frac{m'}{m^2}}{e - g + 1 - d}$. ■

Remark. We have now described a range M of $(m, \hat{m}, m')$ such that if $l \in \mathbf{H}_M(I)$ and $(C, x_1, \ldots, x_n)$ is semistable with respect to l, then the map $p_W|_C : C \to p_W(C)$ is surjective, finite, and generically 1-1. Further, since no components of C are collapsed under p_W, it follows from Proposition 5.4 that C is generically reduced.

One may check that there exist integers $(m, \hat{m}, m')$ such that all stable maps have a model satisfying inequality (19) of Proposition 5.6. Such a calculation is carried out in ([2], Proposition 5.1.8). It turns out that one may easily show that the inequality is satisfied by any complete subcurve $C' \subset C$, if $\frac{\hat{m}}{m} = \frac{ca}{2a-1}$, and $\frac{m'}{m^2} = \frac{a}{2a-1}$ for $l = 1, \ldots, n$. We will be able to use the theory of variation of GIT to show that in fact the quotient is constant in a small range around this key linearization.

5.3 GIT semistability implies that the only singularities are nodes

The next series of results provides a range M of triples $(m, \hat{m}, m')$ such that if $l \in \mathbf{H}_M(I)$ and if the connected curve $(C, x_1, \ldots, x_n)$ is semistable with respect to l, then any singularities of C are nodes. First we show that C has no cusps by showing that the normalization morphism $\pi : \bar{C} \to C$ is unramified. Singular points are shown to be double points by showing that the inverse image under π of any $P \in C$ contains at most two points. We must also rule out tacnodes; these occur at double points P such that the two tangent lines to C at P coincide.

In the hypotheses of the following lemma, note that $6g + 2q_2 - 2\bar{g} \leq 9g + 3q_2 - 3\bar{g}$ and that $2e - 10g + 6 > e - 9g + 7$, and so in particular the hypotheses of Proposition 5.8 hold.

Proposition 5.9 (cf. [9], 1.0.5)**.** Let a be sufficiently large that $e - 9g + 7 = a(2g - 2 + n + cd) - 9g + 7 > 0$, and let M consist of those $(m, \hat{m}, m')$ such that $m, \hat{m} > m_3$ and

$$m > \max \left\{ \begin{matrix} (g - \frac{1}{2} + e(q_1 + 1) + q_3 + \mu_1 m_2)(e - g + 1), \\ (9g + 3q_2 - 3\bar{g})(e - g + 1) \end{matrix} \right\}$$

with

$$d\frac{\hat{m}}{m} + n\frac{m'}{m^2} < \frac{1}{8}e - \frac{9}{8}g + \frac{7}{8},$$

while

$$\frac{\hat{m}}{m} > 1 + \frac{\frac{3}{2}g - 1 + d + n\frac{m'}{m^2}}{e - g + 1 - d}.$$

Let $l \in \mathbf{H}_M(I)$. If $(C, x_1, \ldots, x_n)$ is connected and semistable with respect to l, then the normalization morphism $\pi : \bar{C} \to C_{\text{red}}$ is unramified. In particular, C possesses no cusps. □

Proof. Suppose π is ramified at $P \in \bar{C}$. Then $p_W \circ \pi : \bar{C} \to p_W(C_{\text{red}})$ is also ramified at P. Recall that by Proposition 5.3, the curve $p_W(C) \subset \mathbf{P}(W)$ is nondegenerate; we can think of $H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1))$ as a subspace of $H^0(p_W(C), L_W)$. Define

$$W_0 = \{s \in H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) | \pi_* p_{W*} s \text{ vanishes to order} \geq 3 \text{ at } P\},$$
$$W_1 = \{s \in H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) | \pi_* p_{W*} s \text{ vanishes to order} \geq 2 \text{ at } P\}.$$

Let λ be the 1-PS of $GL(W)$ whose weights are 0 on W_0, 1 on W_1, and 3 on W_2, and let λ' be the associated 1-PS of $SL(W)$. Pick some $(m, \hat{m}, m') \in M$.

As discussed in Section 5.1, to make an estimate for $\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda')$, we need to find an estimate for $\hat{\beta}_i := \dim \overline{\Omega}_i^{m,\hat{m}}$ for $0 \le i \le 3m$.

We use the homomorphism $\pi_{m,\hat{m}*}$ induced by the normalization morphism (see equation (11)). We show that

$$\pi_{m,\hat{m}*}\big(\overline{\Omega}_i^{m,\hat{m}}\big) \subseteq H^0\big(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}((-3m+i)P)\big), \tag{21}$$

for $0 \le i \le 3m$. When $i = 0$, this follows from the definitions. For $1 \le i \le 3m$, it is enough to show that monomial $\hat{M} \in (\overline{\Omega}_i^{m,\hat{m}} - \overline{\Omega}_{i-1}^{m,\hat{m}})$ vanishes at P to order at least $3m - i$. Suppose that such $\hat{M}$ has i_0 factors from W_0, i_1 factors from W_1, and i_2 factors from W_3. Then $i_0 + i_1 + i_2 = m$ and $i_1 + 3i_2 = i$. By definition, $\hat{M}$ vanishes at P to order at least $3i_0 + 2i_1$. But

$$3i_0 + 2i_1 = 3(i_0 + i_1 + i_2) - (i_1 + 3i_2) = 3m - i,$$

so the monomial vanishes as required, and hence equation (21) is satisfied.

By equation (21) and Riemann–Roch,

$$\begin{aligned}
\hat{\beta}_i := \dim \overline{\Omega}_i^{m,\hat{m}} &\le h^0\big(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(3m-i)P)\big) + \dim\ker \pi_{m,\hat{m}*} \\
&\le em + d\hat{m} - 3m + i - \bar{g} + 1 \\
&\quad + h^1\big(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(3m-i)P)\big) + \dim\ker \pi_{m,\hat{m}*}.
\end{aligned}$$

We may use Claim 5.2 in a straightforward way to show that $\dim\ker \pi_{m,\hat{m}*} < q_2$ and that $h^1(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(3m-i)P)) \le 3m - i \le 3m$ if $0 \le i \le 2g-2$. More care is needed to show that the h^1 term vanishes for higher values of i.

Let $\bar{C}_i$ be an irreducible component of $\bar{C}$. Suppose $\bar{C}_i$ does not contain $P \in \bar{C}$. We have shown (Proposition 5.8) that $\deg_{\bar{C}_i} \bar{L}_W = \deg_{C_i} L_W \ge 1$. Thus

$$\deg_{\bar{C}_i}\big(\bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(3m-i)P)\big) = \deg_{\bar{C}_i}\big(\bar{L}_W^m \otimes \bar{L}_r^{\hat{m}}\big) \ge m \ge 2g_{C_i} - 1.$$

On the other hand, suppose that $\bar{C}_i$ is the component of $\bar{C}$ on which P lies. The morphism $\bar{C}_i \to p_W(C_{i\,\mathrm{red}})$ is ramified at P, so $p_W(C_{i\,\mathrm{red}})$ is singular and integral in $\mathbf{P}(W)$. Were $p_W(C_{i\,\mathrm{red}})$ an integral curve of degree 1 or 2 in $\mathbf{P}(W)$, it would be either a line or a conic, and hence nonsingular. We conclude that $\deg_{\bar{C}_i} \bar{L}_W = \deg_{C_{i\,\mathrm{red}}} L_W \geq 3$. Then

$$\deg_{\bar{C}_i}\left(\bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(3m-i)P)\right) \geq 3m - 3m + i = i.$$

Thus, Claim 5.2(3) shows that $h^1(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(3m-i)P)) = 0$ if $2g-1 \leq i \leq 3m-1$.

Combining these inequalities, we have

$$\hat{\beta}_i \leq \begin{cases} (e-3)m + d\hat{m} + i - \bar{g} + q_2 + 1 + 3m, & 0 \leq i \leq 2g-2 \\ (e-3)m + d\hat{m} + i - \bar{g} + q_2 + 1, & 2g-1 \leq i \leq 3m-1. \end{cases}$$

Thus, in the language of Lemma 5.1, we shall set $\alpha = 3$, $\beta = 0$, $\gamma = 1$, and $\epsilon = -3\bar{g} + 3q_2 + 6g$. We may estimate the minimum weight of the action of λ on the marked points x_i as zero, so we set $\delta = 0$. We know that $r_N = 3$. It remains to find an upper bound for $\sum w_\lambda(w_i)$.

Recall that we are regarding W as a subspace of $H^0(p_W(C), L_W)$. Note that the image of W_0 under π_* is contained in $H^0(\bar{C}, \bar{L}_W(-3P))$, and the image of W_1 under π_* is contained in $H^0(\bar{C}, \bar{L}_W(-2P))$. We have two exact sequences

$$\begin{array}{l} 0 \to \bar{L}_W(-P) \to \bar{L}_W \to k(P) \to 0 \\ 0 \to \bar{L}_W(-3P) \to \bar{L}_W(-2P) \to k(P) \to 0, \end{array}$$

which give rise to long exact sequences in cohomology

$$\begin{array}{l} 0 \to H^0(\bar{C}, \bar{L}_W(-P)) \to H^0(\bar{C}, \bar{L}_W) \to H^0(\bar{C}, k(P)) \to \cdots \\ 0 \to H^0(\bar{C}, \bar{L}_W(-3P)) \to H^0(\bar{C}, \bar{L}_W(-2P)) \to H^0(\bar{C}, k(P)) \to \cdots. \end{array}$$

The second long exact sequence implies that $\dim W_1/W_0 \leq 1$. Now recall that $\bar{L}_W := \pi^*(L_W)$ and π is ramified at P. The ramification index must be at least 2, so we have $H^0(\bar{C}, \bar{L}_W(-P)) = H^0(\bar{C}, \bar{L}_W(-2P))$. Then the first long exact sequence implies that $\dim W_2/W_1 \leq 1$. We conclude that $\sum_{i=1}^{N+1} w_\lambda(w_i) \leq 1 + 3 = 4 =: R$.

We may now estimate the λ'-weight for $(C, x_1, \ldots, x_n)$, using Lemma 5.1,

$$\begin{aligned} \mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') &\geq \left(\frac{9}{2}(e-g+1) - 4e\right) m^2 - 4dm\hat{m} - 4nm' \\ &\quad - \left(\left(9g - 3\bar{g} + 3q_2 - \frac{9}{2}\right)(e-g+1) + 4\right) m \\ &\geq \left(\frac{1}{2}e - \frac{9}{2}(g-1) - 4d\frac{\hat{m}}{m} - 4n\frac{m'}{m^2}\right) m^2 \\ &\quad - (9g - 3\bar{g} + 3q_2)(e-g+1)m. \end{aligned}$$

We assumed that $(9g - 3\bar{g} + 3q_2)(e - g + 1) < m$, so we have shown that

$$\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') \geq \left(\frac{1}{2}e - \frac{9}{2}g + \frac{7}{2} - 4d\frac{\hat{m}}{m} - 4n\frac{m'}{m^2}\right) m^2.$$

This is clearly positive, as we assumed that

$$\frac{1}{8}e - \frac{9}{8}g + \frac{7}{8} > d\frac{\hat{m}}{m} + n\frac{m'}{m^2},$$

so m^2 has a positive coefficient.

Thus $\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') > 0$. This is true for all $(m, \hat{m}, m') \in M$, and therefore by Lemma 4.4, the n-pointed curve $(C, x_1, \ldots, x_n)$ is not semistable with respect to l for any $l \in \mathbf{H}_M(I)$. ■

Remark. Note that the value $e - 9g + 7$ is positive for any (g, n, d) as long as $a \geq 10$, but smaller values of a will suffice in many cases; for example, if $g \geq 3$ then $a \geq 5$ is sufficient.

Proposition 5.10 (cf. [9], 1.0.4)**.** Let a be sufficiently large that $e - 9g + 7 > 0$, and let M consist of those $(m, \hat{m}, m')$ such that $m, \hat{m} > m_3$ and

$$m > \max \left\{ \begin{matrix} (g - \frac{1}{2} + e(q_1 + 1) + q_3 + \mu_1 m_2)(e - g + 1), \\ (9g + 3q_2 - 3\bar{g})(e - g + 1) \end{matrix} \right\}$$

with

$$d\frac{\hat{m}}{m} + n\frac{m'}{m^2} < \frac{1}{8}e - \frac{9}{8}g + \frac{7}{8},$$

while

$$\frac{\hat{m}}{m} > 1 + \frac{\frac{3}{2}g - 1 + d + n\frac{m'}{m^2}}{e - g + 1 - d}.$$

Let $l \in \mathbf{H}_M(I)$. If $(C, x_1, \ldots, x_n)$ is connected, and semistable with respect to l, then all singular points of C_{red} are double points. □

Proof. Suppose there exists a point $P \in C$ with multiplicity ≥ 3 on C_{red}. Let $ev : H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) \to k(P)$ be the evaluation map. Let $W_0 = \ker ev$. We have $N_0 :=$ dim

$W_0 = N$. We take $W_1 = W_2$ and let λ be the 1-PS of $GL(W)$ which acts with weight 0 on W_0 and weight 1 on W_1. Let λ' be the associated 1-PS of $SL(W)$ and pick $(m, \hat{m}, m') \in M$.

We follow the strategy of Section 5.1. We need to find an upper bound for $\hat{\beta}_p := \dim \overline{\Omega}_p^{m,\hat{m}}$.

Define a divisor D on $\bar{C}$ as follows: Let $\pi : \bar{C} \to C$ be the normalization morphism. The hypotheses of Proposition 5.9 are satisfied, so π is unramified; as P has multiplicity at least 3, there must be at least three distinct points in the preimage $\pi^{-1}(P)$. Let $D = Q_1 + Q_2 + Q_3$ be three such points. Note that if any two of these points lie on the same component $\bar{C}_1 \subset \bar{C}$, then the corresponding component $C_1 \subset C$ must have $\deg_{C_1} L_W \geq 3$, by the same argument as in the proof of Proposition 5.9.

The normalization morphism induces a homomorphism $\pi_{m,\hat{m}*}$ (see equation (11)). Note that $\pi_{m,\hat{m}*}(\overline{\Omega}_p^{m,\hat{m}}) \subseteq H^0(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(m-p)D))$. We have

$$\begin{aligned}
\hat{\beta}_p := \dim \overline{\Omega}_p^{m,\hat{m}} &\leq h^0\big(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(m-p)D)\big) + \dim \ker \pi_{m,\hat{m}*} \\
&\leq em + d\hat{m} - 3(m-p) - \bar{g} + 1 \\
&\quad + h^1\big(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(m-p)D)\big) + \dim \ker \pi_{m,\hat{m}*}.
\end{aligned}$$

We may use Claim 5.2(1) and (2) to make the estimates that $h^0(C, \mathcal{I}_C) < q_2$ and that $h^1(\bar{C}, \bar{L}_W^m(-(m-p)D) \otimes \bar{L}_r^{\hat{m}}) \leq 3(m-p) \leq 3m$ if $0 \leq p \leq 2g-2$. To show, as one would wish, that $h^1(\bar{C}, \bar{L}_W^m(-(m-p)D) \otimes \bar{L}_r^{\hat{m}}) = 0$ if $p \geq 2g-1$, we may verify that the degree of L_W on any component C_1 meeting P implies that the hypothesis of Claim 5.2(3) is satisfied.

Thus

$$\hat{\beta}_p \leq \begin{cases} (e-3)m + d\hat{m} + 3p - \bar{g} + q_2 + 1 + 3m, & 0 \leq p \leq 2g-2 \\ (e-3)m + d\hat{m} + 3p - \bar{g} + q_2 + 1, & 2g-1 \leq p \leq m. \end{cases}$$

We may apply Lemma 5.1, setting $\alpha = 3$, $\beta = 0$, $\gamma = 3$, and $\epsilon = -\bar{g} + q_2 + 6g - 2$. We know that $r_N = 1$ and may estimate the weight of the action of λ on the marked points $x_1, \ldots, x_n$ as greater than or equal to zero, so we set $\delta = 0$. Finally, note that $\sum_{i=0}^N w_\lambda(w_i) = 1 =: R$. Now, substituting in these values,

$$\begin{aligned}
\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') &\geq \left(\frac{3}{2}(e-g+1) - e\right) m^2 - dm\hat{m} - nm' \\
&\quad - \left(\left(7g - \bar{g} + q_2 - \frac{9}{2}\right)(e-g+1) + 1\right) m \\
&\geq \left(\frac{1}{2}e - \frac{3}{2}(g+1) - d\frac{\hat{m}}{m} - n\frac{m'}{m^2}\right) m^2 - (7g - \bar{g} + q_2)(e-g+1)m.
\end{aligned}$$

Our assumptions imply that $(7g - \bar{g} + q_2)(e - g + 1) < m$. We have shown that

$$\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') \geq \left(\frac{1}{2}e - \frac{3}{2}g + \frac{1}{2} - d\frac{\hat{m}}{m} - n\frac{m'}{m^2}\right) m^2.$$

However, we assumed that

$$d\frac{\hat{m}}{m} + n\frac{m'}{m^2} < \frac{1}{8}e - \frac{9}{8}g + \frac{7}{8},$$

and $\frac{1}{8}e - \frac{9}{8}g + \frac{7}{8} < \frac{1}{2}e - \frac{3}{2}g + \frac{1}{2}$, since $e - g \geq 4$, so the coefficient of m^2 is positive.

Thus $\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') > 0$. This holds for all $(m, \hat{m}, m') \in M$, so by Lemma 4.4, we see that $(C, x_1, \ldots, x_n)$ is not semistable with respect to l for any $l \in \mathbf{H}_M(I)$. ■

The remaining case we must rule out is that C possesses a tacnode. The analogous proposition in [9] is 1.0.6, but the proof there contains at least two errors (one should use Ω_i^m accounting rather than the Tata notes' $W_i^{m-r} W_j^r$ when the filtration has more than two stages, and tacnodes need not be separating). These may be avoided if we simply follow ([11], 4.53) instead.

Proposition 5.11 (cf. [11], 4.53)**.** Let a be sufficiently large that $e - 9g + 7 > 0$, and let M consist of those $(m, \hat{m}, m')$ such that $m, \hat{m} > m_3$ and

$$m > \max \left\{ \begin{matrix} (g - \frac{1}{2} + e(q_1 + 1) + q_3 + \mu_1 m_2)(e - g + 1), \\ (10g + 3q_2 - 3\bar{g})(e - g + 1) \end{matrix} \right\}$$

with

$$d\frac{\hat{m}}{m} + n\frac{m'}{m^2} < \frac{1}{8}e - \frac{9}{8}g + \frac{7}{8},$$

while

$$\frac{\hat{m}}{m} > 1 + \frac{\frac{3}{2}g - 1 + d + n\frac{m'}{m^2}}{e - g + 1 - d}.$$

Let $l \in \mathbf{H}_M(I)$. If C is connected and $(C, x_1, \ldots, x_n)$ is semistable with respect to l, then C_{red} does not have a tacnode. □

Proof. Suppose that C_{red} has a tacnode at P. Let $\pi : \bar{C} \to C$ be the normalization. There exist two distinct points, $Q_1, Q_2 \in \bar{C}$, such that $\pi(Q_1) = \pi(Q_2) = P$. Moreover, the two tangent lines to C at P are coincident. Define the divisor $D := Q_1 + Q_2$ on $\bar{C}$.

We consider $H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1))$ as a subspace of $H^0(p_w(C), L_W)$. Thus we may define subspaces

$$W_0 := \{s \in H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) | \pi_* p_{W*} x \text{ vanishes to order} \geq 2 \text{ at } Q_1 \text{ and } Q_2\},$$
$$W_1 := \{s \in H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) | \pi_* p_{W*} x \text{ vanishes to order} \geq 1 \text{ at } Q_1 \text{ and } Q_2\}.$$

Let λ be the 1-PS of $GL(W)$ which acts with weight 0 on W_0, 1 on W_1, and 2 on W_2. Let λ' be the associated 1-PS of $SL(W)$ and fix $(m, \hat{m}, m') \in M$.

Following the strategy of Section 5.1, we wish to estimate $\hat{\beta}_i = \dim \overline{\Omega}_i^{m,\hat{m}}$.

As in Proposition 5.6, we use the homomorphism $\pi_{m,\hat{m}*}$ induced by the normalization morphism (see equation (11)). Similarly to as in Proposition 5.9, we show that

$$\pi_{m,\hat{m}*}\big(\overline{\Omega}_i^{m,\hat{m}}\big) \subseteq H^0\big(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(2m-i)D)\big), \tag{22}$$

for $0 \leq i \leq 2m$. When $i = 0$, this follows from the definitions. For $1 \leq i \leq 2m$ it is enough to show that for any monomial $\hat{M} \in \overline{\Omega}_i^{m,\hat{m}} - \overline{\Omega}_{i-1}^{m,\hat{m}}$, we have

$$\pi_{m,\hat{m}*}(\hat{M}) \in H^0\big(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(2m-i)D)\big).$$

Suppose that such $\hat{M}$ has j_k factors from W_k, for $k = 0, 1, 2$. Then $j_0 + j_1 + j_2 = m$ and $j_1 + 2j_2 = i$. By definition, $\pi_{m,\hat{m}*}(\hat{M})$ vanishes at both Q_1 and Q_2 to order at least $2j_0 + j_1$. But

$$2j_0 + j_1 = 2(j_0 + j_1 + j_2) - (j_1 + 2j_2) = 2m - i,$$

giving the required vanishing of $\pi_{m,\hat{m}*}(\hat{M})$.

We shall also prove at this stage that $\sum_{i=1}^{N+1} w_\lambda(w_i) \leq 3$. We have two exact sequences

$$\begin{array}{ccccccccc} 0 \to & \bar{L}_W(-Q_1) & \to & \bar{L}_W & \to & k(Q_1) & \to & 0 \\ 0 \to & \bar{L}_W(-2Q_1) & \to & \bar{L}_W(-Q_1) & \to & k(Q_1) & \to & 0, \end{array}$$

which give rise to long exact sequences in cohomology

$$\begin{array}{lllll} 0 \to & H^0(\bar{C}, \bar{L}_W(-Q_1)) & \to & H^0(\bar{C}, \bar{L}_W) & \to H^0(\bar{C}, k(P)) \\ 0 \to & H^0(\bar{C}, \bar{L}_W(-2Q_1)) & \to & H^0(\bar{C}, \bar{L}_W(-Q_1)) & \to H^0(\bar{C}, k(P)). \end{array}$$

If we let $\bar{W}_2$ be the image of $W_2 = H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1))$ in $H^0(\bar{C}, \bar{L}_W)$, we may intersect each of the spaces in the sequences with $\bar{W}_2$. Note that the image of W_1 in $H^0(\bar{C}, \bar{L}_W)$ is precisely $\bar{W}_2 \cap H^0(\bar{C}, \bar{L}_W(-Q_1))$; if a section of $\bar{L}_W$ which vanishes at Q_1 is the pullback a section of C, then it automatically vanishes at Q_2 as well. Similarly, the image of W_0 in $H^0(\bar{C}, \bar{L}_W)$ is $\bar{W}_2 \cap H^0(\bar{C}, \bar{L}_W(-2Q_1))$. We have obtained

$$\begin{array}{l} 0 \to \bar{W}_1 \to \bar{W}_2 \to \bar{W}_2 \cap H^0(\bar{C}, k(P)) \\ 0 \to \bar{W}_0 \to \bar{W}_1 \to \bar{W}_2 \cap H^0(\bar{C}, k(P)). \end{array}$$

It follows that $\dim W_2/W_1 \leq 1$ and $\dim W_1/W_0 \leq 1$. Thus $\sum_{i=1}^{N+1} w_\lambda(w_i) \leq 1 + 2 = 3 =: R$.

The weight coming from the marked points will always be estimated as zero. In order to estimate $\hat{\beta}_p$, there are unfortunately various cases to consider, which shall give rise to different inequalities.

(1) There is one irreducible component C_1 of C passing through P.
(2) There are two irreducible components C_1 and C_2 of C passing through P, and $\deg_{C_{i\,\mathrm{red}}} L_W \otimes L_r \geq 2$ for $i = 1, 2$.
(3) There are two irreducible components C_1 and C_2 of C passing through P, and $\deg_{C_{1\,\mathrm{red}}} L_W \otimes L_r = 1$, while $\deg_{C_{2\,\mathrm{red}}} L_W \otimes L_r \geq 2$.

There cannot be two degree 1 curves meeting at a tacnode, so these are the only cases. Note that in Case 1, since C_1 is an irreducible curve with a tacnode, $\deg_{C_{1\,\mathrm{red}}} L_W \otimes L_r \geq 4$. For Case 3, we know by Proposition 5.8 that $\deg_{C_{1\,\mathrm{red}}} L_W \geq 1$, and hence we see that $\deg_{C_{1\,\mathrm{red}}} L_r = 0$.

Cases 1 and 2. We estimate $\hat{\beta}_p$. By equation (22) and Riemann–Roch,

$$\begin{aligned} \hat{\beta}_i := \dim \overline{\Omega}_i^{m,\hat{m}} &\leq h^0\big(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(2m-i)D)\big) + \dim\ker \pi_{m,\hat{m}*} \\ &\leq em + d\hat{m} - 2(2m-i) - \bar{g} + 1 \\ &\quad + h^1\big(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(2m-i)D)\big) + \dim\ker \pi_{m,\hat{m}*}. \end{aligned}$$

Claim 5.2 allows us to estimate, as usual, the upper bounds $\dim\ker \pi_{m,\hat{m}*} < q_2$ and $h^1\left(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(2m-i)D)\right) \leq 4m - 2i \leq 4m$ if $0 \leq i \leq 2g-2$. To show that the h^1

term vanishes for larger values of i, recall that our assumptions imply that $\hat{m} > m$. Thus, for any component C_i of the curve,

$$\deg_{\bar{C}_i} \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \geq \deg_{\bar{C}_i} \bar{L}_W^m \otimes \bar{L}_r^m = m \cdot \deg_{\bar{C}_i} \bar{L}_W \otimes \bar{L}_r.$$

If we are in Case 1, then we conclude that $\deg_{\bar{C}_1} \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(2m-i)D) \geq 4m - 2 \cdot 2m + i \geq 2g-2$ if $2g-2 \leq i \leq 2m-1$. The other components of the curve do not meet D, and so one sees from Claim 5.2(3) that h^1 is zero there. Case 2 follows similarly.

Combining these inequalities, we have

$$\hat{\beta}_i \leq \begin{cases} (e-4)m + d\hat{m} + 2i - \bar{g} + 1 + q_2 + 4m, & 0 \leq i \leq 2g-2 \\ (e-4)m + d\hat{m} + 2i - \bar{g} + 1 + q_2, & 2g-1 \leq i \leq 3m-1. \end{cases}$$

In the language of Lemma 5.1, we set $\alpha = 4$, $\beta = 0$, $\gamma = 2$, $\delta = 0$, $\epsilon = 8g - 3\bar{g} + 3q_2 - 1$, $r_N = 2$, and $R = 3$. Now

$$\begin{aligned}\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') &\geq (4(e-g+1) - 3e)m^2 - 3dm\hat{m} - 3nm' \\ &\quad - ((10g - 3\bar{g} + 3q_2 - 5)(e-g+1) + 3)m \\ &\geq \left(e - 4(g-1) - 3d\frac{\hat{m}}{m} - 3n\frac{m'}{m^2}\right)m^2 - (10g - 3\bar{g} + 3q_2)(e-g+1)m.\end{aligned}$$

This is clearly positive for large m. In particular, as we set $m > (10g - 3\bar{g} + 3q_2)(e-g+1)$, we must infer that

$$\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') \geq \left(e - 4g + 3 - 3d\frac{\hat{m}}{m} - 3n\frac{m'}{m^2}\right)m^2.$$

Then, since

$$d\frac{\hat{m}}{m} + n\frac{m'}{m^2} < \frac{1}{8}e - \frac{9}{8}g + \frac{7}{8} < \frac{1}{3}e - \frac{4}{3}g + 1$$

(where the latter inequality may be seen to hold, since $e - g \geq 4$), we conclude that $\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') > 0$.

Case 3. Now that $\deg_{\bar{C}_1} \bar{L}_W \otimes \bar{L}_r = 1$, we need a new way to estimate $\hat{\beta}_i := \dim \overline{\Omega}_p^{m,\hat{m}}$. However, we do know that the genus of C_1 is zero. Let $Y := \overline{C - C_1}$. Noting in line (22) that $\bar{C}_1$ and $\bar{Y}$ are disjoint, we may write

$$\begin{aligned}\pi_{m,\hat{m}*}(\overline{\Omega}_p^{m,\hat{m}}) &\subset H^0\big(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(2m-i)D)\big)\\ &= H^0\big(\bar{C}_1, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}}(-(2m-i)Q_1)\big) \oplus H^0\big(\bar{Y}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}}(-(2m-i)Q_2)\big).\end{aligned}$$

Thus

$$\begin{aligned}\hat{\beta}_i := \dim \overline{\Omega}_i^{m,\hat{m}} \leq h_i^0 + (e-1)m + d\hat{m} - (2m-i) - g_{\bar{C}} + 1\\ + h^1\big(\bar{Y}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{Y}}(-(2m-i)Q_2)\big) + \dim\ker \pi_{m,\hat{m}*},\end{aligned}$$

where we write h_i^0 for $h^0(\bar{C}_1, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}_1}(-(2m-i)Q_1))$.

As usual, Claim 5.2 allows us to estimate the upper bounds $\dim\ker \pi_{m,\hat{m}*} < q_2$ and $h^1(\bar{Y}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{Y}}(-(2m-i)Q_2)) \leq 2m-i \leq 2m$ if $0 \leq i \leq 2g-2$. The assumptions for Case 3 tell us directly that we may apply Claim 5.2 (3) and obtain, as we desire, $h^1(\bar{Y}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{Y}}(-(2m-i)Q_2)) = 0$ if $2g-1 \leq i \leq 2m-1$.

Combining these inequalities, we have

$$\hat{\beta}_i \leq \begin{cases}(e-3)m + d\hat{m} + i - \bar{g} + 1 + q_2 + 2m + h_i^0, & 0 \leq i \leq 2g-2\\ (e-3)m + d\hat{m} + i - \bar{g} + 1 + q_2 + h_i^0, & 2g-1 \leq i \leq 2m-1.\end{cases} \tag{23}$$

To calculate the ϵ term in the language of Lemma 5.1, we must calculate $\sum_{i=0}^{2m-1} h_i^0$. We recall that $\bar{C}_1 \cong \mathbf{P}^1$, that $\deg_{\bar{C}_1} \bar{L}_W = 1$, and that $\deg_{\bar{C}_1} \bar{L}_r = 0$. Thus $\bar{L}_{W\bar{C}_1} = \mathcal{O}_{\mathbf{P}^1}(1)$ and $\bar{L}_{r\bar{C}_1} = \mathcal{O}_{\mathbf{P}^1}$, so

$$h_i^0 = h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m) \otimes \mathcal{O}_{\mathbf{P}^1}(-(2m-i)Q_1)) = \begin{cases}0 & i \leq m-1\\ -m+i+1 & i \geq m.\end{cases}$$

Thus, in particular,

$$\sum_{i=0}^{2m-1} h_i^0 = \sum_{j=1}^{m} j = \frac{1}{2}m^2 + \frac{1}{2}m.$$

Hence, we calculate ϵ to be $= 4g - 2\bar{g} + 2q_2 + \frac{1}{2} + \frac{1}{2}m$. For the rest of the dictionary for Lemma 5.1, we set $\alpha = 4$, $\beta = 0$, $\gamma = 1$, $\delta = 0$, $r_N = 2$ and $R = 3$. Then

$$\begin{aligned}\mu^{L_{m,\hat{m},m'}}((C,x_1,\dots,x_n),\lambda') &\geq (4(e-g+1)-3e)m^2 - 3dm\hat{m} - 3nm' \\ &\quad - \left(\left(6g-2\bar{g}+2q_2-\frac{5}{2}+\frac{1}{2}m\right)(e-g+1)+3\right)m \\ &\geq \left(\frac{1}{2}e-\frac{7}{2}(g-1)-3d\frac{\hat{m}}{m}-3n\frac{m'}{m^2}\right)m^2-(6g-2\bar{g}+2q_2)(e-g+1)m.\end{aligned}$$

Our estimates for m imply that $m > (e-g+1)(6g-2\bar{g}+2q_2)$, so we have shown that

$$\mu^{L_{m,\hat{m},m'}}((C,x_1,\dots,x_n),\lambda') \geq \left(\frac{1}{2}e-\frac{7}{2}g+\frac{5}{2}-3d\frac{\hat{m}}{m}-3n\frac{m'}{m^2}\right)m^2.$$

This is clearly positive, as

$$d\frac{\hat{m}}{m}+n\frac{m'}{m^2} < \frac{1}{8}e-\frac{9}{8}g+\frac{7}{8} < \frac{1}{6}e-\frac{7}{6}g+\frac{5}{6},$$

where the second inequality holds, since $e-g \geq 4$.

Thus $\mu^{L_{m,\hat{m},m'}}((C,x_1,\dots,x_n),\lambda) > 0$ for all triples $(m,\hat{m},m') \in M$, and hence by Lemma 4.4, we see that $(C,x_1,\dots,x_n)$ is unstable with respect to l for any $l \in \mathbf{H}_M(I)$. ■

5.4 Marked points are nonsingular and distinct

We now turn to the marked points, which we would like to be nonsingular and distinct. This is ensured in the following two propositions.

Proposition 5.12. Let a be sufficiently large that $e-9g+7>0$, and let M consist of those $(m,\hat{m},m')$ such that $m,\hat{m} > m_3$ and

$$m > \max\left\{\begin{matrix}\left(g-\frac{1}{2}+e(q_1+1)+q_3+\mu_1 m_2\right)(e-g+1), \\ (10g+3q_2-3\bar{g})(e-g+1)\end{matrix}\right\}$$

with

$$d\frac{\hat{m}}{m}+n\frac{m'}{m^2} < \frac{1}{8}e-\frac{9}{8}g+\frac{7}{8},$$

while

$$\frac{\hat{m}}{m} > 1+\frac{\frac{3}{2}g-1+d+n\frac{m'}{m^2}}{e-g+1-d} \qquad \frac{m'}{m^2} > \frac{g+d\frac{\hat{m}}{m}}{e-g+1-n}.$$

Let $l \in \mathbf{H}_M(I)$. If $(C, x_1, \ldots, x_n)$ is connected, and semistable with respect to l, then all the marked points lie on the nonsingular locus of C. □

Remark. As $e - g + 1 - n = (2a - 1)(g - 1) + (a - 1)n + cad$, it is evident that this is positive.

Proof. Suppose there exists a point $P \in C$, which is singular and also the location of a marked point. By Proposition 5.10, this point is a double point. Let $ev : H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) \to k(P)$ be the evaluation map. Let $W_0 = \ker ev$. We have $N_0 := \dim W_0 = N - 1$. Let λ be the 1-PS of $GL(W)$ which acts with weight 0 on W_0 and with weight 1 on W_1. Let λ' be the associated 1-PS of $SL(W)$ and fix $(m, \hat{m}, m') \in M$.

We have assumed that at least one marked point, say, x_i lies at P. If $w \in W_0$ then $w(x_i) = 0$, so w_{k_i} must be w_{e-g}, whose λ-weight is 1. Hence $\sum_{l=1}^{n} w_\lambda(w_{k_l})m' \geq m'$. Note also that $\sum_{i=0}^{N} w_\lambda(w_i) = 1 =: R$. As usual, construct a filtration of $H^0(C, L_W^m \otimes L_r^{\hat{m}})$ of increasing weight as in equation (9). We need to find an upper bound for $\hat{\beta}_p = \dim \overline{\Omega}_p^{m,\hat{m}}$.

Let $\pi : \bar{C} \to C$ be the normalization morphism, which is unramified as the hypotheses of Proposition 5.9 are satisfied. There are two distinct points in $\pi^{-1}(P)$, by Proposition 5.10. Let the divisor $D := Q_1 + Q_2$ on $\bar{C}$ consist of these points. Should Q_1 and Q_2 lie on the same component $\bar{C}_1$ of $\bar{C}$, we see as in the proof of Proposition 5.9 that $\deg_{\bar{C}_1} \bar{L}_W \geq 3$.

The normalization morphism induces a homomorphism $\pi_{m,\hat{m}*}$ (see equation (11)) with $\pi_{m,\hat{m}*}(\overline{\Omega}_p^{m,\hat{m}}) \subseteq H^0(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(m-p)D))$. We have

$$\begin{aligned}\hat{\beta}_p := \dim \overline{\Omega}_p^{m,\hat{m}} &\leq h^0\big(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(m-p)D)\big) + \dim \ker \pi_{m,\hat{m}*} \\ &= em + d\hat{m} - 2(m-p) - g_{\bar{C}} + 1 + h^1\big(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}} \otimes \mathcal{O}_{\bar{C}}(-(m-p)D)\big) + \dim \ker \pi_{m,\hat{m}*}.\end{aligned}$$

We may use Claim 5.2 as usual to establish the estimates that $\dim \ker \pi_{m,\hat{m}*} < q_2$, that $h^1(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}}(-(m-p)D)) \leq 2(m-p) \leq 2m$ if $0 \leq p \leq 2g-2$, and that, as one would wish, $h^1(\bar{C}, \bar{L}_W^m \otimes \bar{L}_r^{\hat{m}}(-(m-p)D)) = 0$ if $p \geq 2g-1$.

We can now estimate $\hat{\beta}_p$,

$$\hat{\beta}_p \leq \begin{cases} (e-2)m + d\hat{m} + 2p - \bar{g} + q_2 + 1 + 2m, & 0 \leq p \leq 2g-2 \\ (e-2)m + d\hat{m} + 2p - \bar{g} + q_2 + 1, & 2g-1 \leq p \leq m. \end{cases}$$

We may apply Lemma 5.1, setting $\alpha = 2$, $\beta = 0$, $\gamma = 2$, $\delta = 1$, $\epsilon = -\bar{g} + q_2 + 4g - 1$, $r_N = 1$, and $R = 1$; thus we estimate

$$\begin{aligned}\mu^{L_{m,\hat{m},m'}}((C, x_1, \dots, x_n), \lambda') &\geq ((e - g + 1) - e)m^2 - dm\hat{m} + ((e - g + 1) - n)m' \\ &\quad - ((5g - \bar{g} + q_2 - 3)(e - g + 1) + 1)m \\ &\geq \left(-g + 1 - d\frac{\hat{m}}{m} + (e - g + 1 - n)\frac{m'}{m^2}\right) m^2 \\ &\quad - (5g - \bar{g} + q_2)(e - g + 1)m.\end{aligned}$$

Our assumptions imply that $m > (5g - \bar{g} + q_2)(e - g + 1)$, and so we have shown that

$$\mu^{L_{m,\hat{m},m'}}((C, x_1, \dots, x_n), \lambda') \geq \left(-g - d\frac{\hat{m}}{m} + (e - g + 1 - n)\frac{m'}{m^2}\right) m^2.$$

This, however is positive, as we assumed that

$$\frac{m'}{m^2} > \frac{g + d\frac{\hat{m}}{m}}{e - g + 1 - n}.$$

Thus $\mu^{L_{m,\hat{m},m'}}((C, x_1, \dots, x_n), \lambda') > 0$. This is true for any $(m, \hat{m}, m') \in M$, so it follows by Lemma 4.4 that $(C, x_1, \dots, x_n)$ is not semistable with respect to l, for any $l \in \mathbf{H}_M(I)$. ■

Proposition 5.13. Let a be sufficiently large that $e - 9g + 7 > 0$, and let M consist of those $(m, \hat{m}, m')$ such that $m, \hat{m} > m_3$ and

$$m > \max \left\{ \begin{matrix} (g - \frac{1}{2} + e(q_1 + 1) + q_3 + \mu_1 m_2)(e - g + 1), \\ (10g + 3q_2 - 3\bar{g})(e - g + 1) \end{matrix} \right\}$$

with

$$d\frac{\hat{m}}{m} + n\frac{m'}{m^2} < \frac{1}{8}e - \frac{9}{8}g + \frac{7}{8},$$

while

$$\frac{\hat{m}}{m} > 1 + \frac{\frac{3}{2}g - 1 + d + n\frac{m'}{m^2}}{e - g + 1 - d} \qquad \frac{m'}{m^2} > \frac{1}{4} + \frac{g + \frac{n}{4} + d\frac{\hat{m}}{m}}{2(e - g + 1) - n}.$$

Let $l \in \mathbf{H}_M(I)$. If $(C, x_1, \dots, x_n)$ is connected, and semistable with respect to l, then all the marked points are distinct. □

Remark. The denominator $2(e-g+1)-n$ is easily checked to be positive, as it is equal to $(4a-1)(g-1)+(2a-1)n+cad$.

Proof. Suppose two marked points, x_i and x_j meet at $P \in C$. The hypotheses of the previous proposition hold, and so P is a nonsingular point. Let $ev : H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) \to k(P)$ be the evaluation map. Let $W_0 = \ker ev$; thus $N_0 := \dim W_0 = N$. Let λ be the 1-PS of $GL(W)$ which acts with weight 0 on W_0 and with weight 1 on W_1. Let λ' be the associated 1-PS of $SL(W)$. Fix $(m, \hat{m}, m') \in M$.

As we assume that x_i and x_j lie at P, it follows that $\sum_{l=1}^{n} w_\lambda(w_{k_l})m' \geq 2m'$. Again, note that $\sum_{i=0}^{N} w_\lambda(w_i) = 1 =: R$. Construct a filtration of $H^0(C, L_W^m \otimes L_r^{\hat{m}})$ of increasing weight as in equation (9). We need to find an upper bound for $\hat{\beta}_p := \dim \overline{\Omega}_p^{m,\hat{m}}$.

This time, we do not need to use the normalization to estimate $\hat{\beta}_p$; by Proposition 5.12, we know that C is smooth at P. It is clear that the space of monomials $\overline{\Omega}_p^{m,\hat{m}} \subseteq H^0(C, L_W^m \otimes L_r^{\hat{m}} \otimes \mathcal{O}_C(-(m-p)P))$. We have

$$\begin{aligned}\hat{\beta}_p := \dim \overline{\Omega}_p^{m,\hat{m}} &\leq h^0\big(C, L_W^m \otimes L_r^{\hat{m}} \otimes \mathcal{O}_C(-(m-p)P)\big)\\ &= em + d\hat{m} - (m-p) - g + 1 + h^1\big(C, L_W^m \otimes L_r^{\hat{m}} \otimes \mathcal{O}_C(-(m-p)P)\big).\end{aligned}$$

We use Claim 5.2 to estimate that $h^1(C, L_W^m \otimes L_r^{\hat{m}}(-(m-p)P)) \leq (m-p) \leq m$ and that $h^1(C, L_W^m \otimes L_r^{\hat{m}}(-(m-p)P)) = 0$ if $p \geq 2g-1$.

These give us upper bounds for $\hat{\beta}_p$,

$$\hat{\beta}_p \leq \begin{cases} (e-1)m + d\hat{m} + p - g + 1 + m, & 0 \leq p \leq 2g-2 \\ (e-1)m + d\hat{m} + p - g + 1, & 2g-1 \leq p \leq m-1. \end{cases}$$

We may apply Lemma 5.1, setting $\alpha = 1$, $\beta = 0$, $\gamma = 1$, $\delta = 2$, $\epsilon = g-1$, $r_N = 1$, and $R = 1$. Thus we estimate

$$\begin{aligned}\mu^{L_{m,\hat{m},m'}}((C, x_1, \dots, x_n), \lambda') &\geq \left(\frac{1}{2}(e-g+1) - e\right)m^2 - dm\hat{m}\\ &\quad + (2(e-g+1)-n)m' - \left(\left(2g - \frac{5}{2}\right)(e-g+1) + 1\right)m\\ &\geq \left(-\frac{1}{2}e - \frac{1}{2}g - \frac{1}{2} - d\frac{\hat{m}}{m} + (2(e-g+1)-n)\frac{m'}{m^2}\right)m^2,\end{aligned}$$

where we have used the fact that our assumptions imply $m > (2g - \frac{5}{2})(e-g+1) - (g-1)$.

However, this must be positive, as we assumed that

$$\frac{m'}{m^2} > \frac{1}{4} + \frac{g + \frac{n}{4} + d\frac{\hat{m}}{m}}{2(e-g+1)-n} = \frac{\frac{1}{2}e + \frac{1}{2}g + \frac{1}{2} + d\frac{\hat{m}}{m}}{2(e-g+1)-n}.$$

Thus $\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda') > 0$, and therefore by Lemma 4.4, we know that $(C, x_1, \ldots, x_n)$ is not semistable with respect to l for any $l \in \mathbf{H}_M(I)$. ■

5.5 GIT semistable curves are reduced and "potentially stable"

The next three results show that, if $(C, x_1, \ldots, x_n)$ is semistable with respect to some l in our range $\in \mathbf{H}_M(I)$ of virtual linearizations, then the curve C is reduced. We begin with a generalized Clifford's theorem.

Lemma 5.14 (cf. [9], p. 18)**.** Let C be a reduced curve with only nodes, and let L be a line bundle generated by global sections which is not trivial on any irreducible component of C. If $H^1(C, L) \neq 0$, then there is a connected subcurve $C' \subset C$ such that

$$h^0(C', L) \leq \frac{\deg_{C'}(L)}{2} + 1. \tag{24}$$

Furthermore, $C' \ncong \mathbf{P}^1$. □

Proof. Gieseker proves nearly all of this. It remains only to show that C' may be taken to be connected and $C' \ncong \mathbf{P}^1$. Firstly, if equation (24) is satisfied by $C' \subset C$, then it is clear that equation (24) must be satisfied by some connected component of C'. So assume that C' is connected and suppose that $C' \cong \mathbf{P}^1$. Now, every line bundle on $\mathbf{P}^1$ is isomorphic to $\mathcal{O}_{\mathbf{P}^1}(m)$ for some $m \in \mathbb{Z}$. By hypothesis, L is generated by global sections and is nontrivial on C'; this implies that $m > 0$. However, combining this with equation (24) implies that $m + 1 = h^0(C', L) \leq \frac{m}{2} + 1$ which implies that $m \leq 0$, a contradiction. ■

Lemma 5.15 ([9], p. 79)**.** Let a be sufficiently large that $e - 9g + 7 > 0$, and let M consist of those $(m, \hat{m}, m')$ such that $m, \hat{m} > m_3$ and

$$m > \max\left\{\begin{matrix}\left(g - \frac{1}{2} + e(q_1+1) + q_3 + \mu_1 m_2\right)(e-g+1), \\ (10g + 3q_2 - 3\bar{g})(e-g+1)\end{matrix}\right\}$$

with

$$d\frac{\hat{m}}{m}+n\frac{m'}{m^2}<\frac{1}{8}e-\frac{9}{8}g+\frac{7}{8},$$

while

$$\frac{\hat{m}}{m}>1+\frac{\frac{3}{2}g-1+d+n\frac{m'}{m^2}}{e-g+1-d}\qquad \frac{m'}{m^2}>\frac{1}{4}+\frac{g+\frac{n}{4}+d\frac{\hat{m}}{m}}{2(e-g+1)-n}.$$

Let $l\in \mathbf{H}_M(I)$. If $(C,x_1,\dots,x_n)$ is connected and semistable with respect to l, then $H^1(C_{\rm red},L_{W{\rm red}})=0$. □

Proof. Since $C_{\rm red}$ is nodal, it has a dualizing sheaf ω. Suppose $H^1(C_{\rm red},L_{W{\rm red}})\neq 0$. Then by duality,

$$H^0\big(C_{\rm red},\omega\otimes L_{W{\rm red}}^{-1}\big)\cong H^1(C_{\rm red},L_{W{\rm red}})\neq 0.$$

By Proposition 5.8, the line bundle $L_{W{\rm red}}$ is not trivial on any component of $C_{\rm red}$. Then by Lemma 5.14, there is a connected subcurve $C'\ncong \mathbf{P}^1$ of $C_{\rm red}$ for which $e'>1$ and $h^0(C',L_{WC'})\leq \frac{e'}{2}+1$.

Let $Y:=\overline{C-C'}_{\rm red}$ and pick a point P on the normalization $\bar{Y}$, so that $\pi(P)\in C'\cap Y$. By Proposition 5.8, we know that $\deg_{\bar{Y}_j}\bar{L}_W(-P)\geq 0$ for every component Y_j of Y. We may apply Proposition 5.6, setting $k=1$ and $b=1$. There exists $(m,\hat{m},m')\in M$ satisfying inequality (19). Estimate $d'\geq 0$, and $n'\geq 0$. Recall that in this case, $S=3g+q_2-\bar{g}'+\frac{1}{2}$, and so the hypotheses on m certainly imply that $\frac{S}{m}(e-g+1)\leq\frac{1}{2}$. We obtain

$$\begin{aligned} e'+\frac{1}{2}=e'+\frac{k}{2}&\leq\frac{(\frac{e'}{2}+1)e+d(\frac{e'}{2}+1)\frac{\hat{m}}{m}+n(\frac{e'}{2}+1)\frac{m'}{m^2}}{e-g+1}+\frac{1}{2(e-g+1)}\\ &\Rightarrow 0<-\left(e'+\frac{1}{2}\right)(e-g+1)+\left(\frac{1}{2}e'+1\right)\left(e+d\frac{\hat{m}}{m}+n\frac{m'}{m^2}\right)+\frac{1}{2}.\end{aligned}$$

Use the bound $d\frac{\hat{m}}{m}+n\frac{m'}{m^2}<\frac{1}{8}e-\frac{9}{8}g+\frac{7}{8}$ to obtain

$$\begin{aligned} 0&<-\left(e'+\frac{1}{2}\right)(e-g+1)+\left(\frac{1}{2}e'+1\right)\left(\frac{9}{8}e-\frac{9}{8}g+\frac{7}{8}\right)+\frac{1}{2}\\ &=-\left(\frac{7}{16}e'-\frac{5}{8}\right)(e-g)-\frac{9}{16}e'+\frac{7}{8}.\end{aligned}$$

Since $e' > 1$, we may substitute in $e' \geq 2$

$$0 < -\frac{1}{4}(e-g) - \frac{1}{4},$$

a contradiction. Thus $H^1(C_{\text{red}}, L_{W\text{red}}) = 0$. ■

We may now, finally, show that our semistable curves are reduced.

Proposition 5.16 (cf. [9], 1.0.8)**.** Let a be sufficiently large that $e - 9g + 7 > 0$, and let M consist of those $(m, \hat{m}, m')$ such that $m, \hat{m} > m_3$ and

$$m > \max \begin{Bmatrix} \left(g - \frac{1}{2} + e(q_1+1) + q_3 + \mu_1 m_2\right)(e-g+1), \\ (10g + 3q_2 - 3\bar{g})(e-g+1), \end{Bmatrix}$$

with

$$d\frac{\hat{m}}{m} + n\frac{m'}{m^2} < \frac{1}{8}e - \frac{9}{8}g + \frac{7}{8},$$

while

$$\frac{\hat{m}}{m} > 1 + \frac{\frac{3}{2}g - 1 + d + n\frac{m'}{m^2}}{e-g+1-d} \qquad \frac{m'}{m^2} > \frac{1}{4} + \frac{g + \frac{n}{4} + d\frac{\hat{m}}{m}}{2(e-g+1)-n}.$$

Let $l \in \mathbf{H}_M(I)$. If $(C, x_1, \ldots, x_n)$ is connected and semistable with respect to l, then C is reduced. □

Proof. Let $\iota : C_{\text{red}} \to C$ be the canonical inclusion. The exact sequence of sheaves on C

$$0 \to \mathcal{I}_C \otimes L_W \to L_W \to \iota_* L_{W\text{red}} \to 0$$

gives rise to a long exact sequence in cohomology

$$\cdots \to H^1(C, \mathcal{I}_C \otimes L_W) \to H^1(C, L_W) \to H^1(C, \iota_* L_{W\text{red}}) \to 0.$$

Since C is generically reduced, $\mathcal{I}_C$ has finite support, hence $H^1(C, \mathcal{I}_C \otimes L_W) = 0$. Lemma 5.15 tells us us that $H^1(C, \iota_* L_{W\text{red}}) = H^1(C_{\text{red}}, L_{W\text{red}}) = 0$. The exact sequence implies

$H^1(C, L_W) = 0$ as well. Next, the map

$$H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) \to H^0\big(p_W(C)_{\mathrm{red}}, \mathcal{O}_{p_W(C)_{\mathrm{red}}}(1)\big) \to H^0(C_{\mathrm{red}}, L_{W\mathrm{red}})$$

is injective by Proposition 5.3. Then

$$\begin{aligned} e - g + 1 = h^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) &\le h^0(C_{\mathrm{red}}, L_{W\mathrm{red}}) \\ &= h^0(C, L_W) - h^0(C, \mathcal{I}_C \otimes L_W) = e - g + 1 - h^0(C, \mathcal{I}_C \otimes L_W). \end{aligned}$$

Therefore $h^0(C, \mathcal{I}_C \otimes L_W) = 0$. Since $\mathcal{I}_C \otimes L_W$ has finite support, $\mathcal{I}_C = 0$, so C is reduced. ∎

Next, we improve on Proposition 5.6. If C is connected and $(C, x_1, \ldots, x_n)$ is semistable with respect to some $l \in \mathbf{H}_M(I)$, and if C' is a connected subcurve of C, then we know that our "fundamental inequality" is satisfied without needing to verify condition (ii). This inequality is then an extra property of semistable curves, which we will use in Theorem 5.21 to show that $\overline{J}^{ss}(l) \subseteq J$ for the right range of virtual linearizations.

We repeat the definition of $\lambda'_{C'}$: Let

$$W_0 := \ker\left\{H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) \to H^0\big(p_W(C'), \mathcal{O}_{p_W(C')}(1)\big)\right\}.$$

Choose a basis $w_0, \ldots, w_N$ of $H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1))$ relative to the filtration $0 \subset W_0 \subset W_1 = H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1))$. Let $\lambda_{C'}$ be the 1-PS of $GL(W)$ whose action is given by

$$\begin{aligned} \lambda_{C'}(t) w_i &= w_i, t \in \mathbb{C}^*, 0 \le i \le N_0 - 1 \\ \lambda_{C'}(t) w_i &= t w_i, t \in \mathbb{C}^*, N_0 \le i \le N, \end{aligned}$$

and let $\lambda'_{C'}$ be the associated 1-PS of $SL(W)$. It is more convenient to prove that the inequality holds for linearizations $L_{m,\hat{m},m'}$ before inferring the result in general.

Lemma 5.17 (cf. [9], p. 83 and Proposition 5.6 above)**.** Let a be sufficiently large that $e - 9g + 7 > 0$, and suppose that $m, \hat{m} > m_3$ and

$$m > \max\left\{\begin{array}{c}\big(g - \frac{1}{2} + e(q_1 + 1) + q_3 + \mu_1 m_2\big)(e - g + 1), \\ (10g + 3q_2 - 3\bar{g})(e - g + 1),\end{array}\right\}$$

with

$$d\frac{\hat{m}}{m} + n\frac{m'}{m^2} < \frac{1}{8}e - \frac{9}{8}g + \frac{7}{8},$$

while

$$\frac{\hat{m}}{m} > 1 + \frac{\frac{3}{2}g - 1 + d + n\frac{m'}{m^2}}{e - g + 1 - d} \qquad \frac{m'}{m^2} > \frac{1}{4} + \frac{g + \frac{n}{4} + d\frac{\hat{m}}{m}}{2(e - g + 1) - n}.$$

Let $(C, x_1, \dots, x_n) \subset \mathbf{P}(W) \times \mathbf{P}^r$ be a connected curve whose only singularities are nodes, and such that no irreducible component of C collapses under projection to $\mathbf{P}(W)$. Suppose C has at least two irreducible components. Let $C' \neq C$ be a reduced, connected, complete subcurve of C, and let Y be the closure of $C - C'$ in C with the reduced structure. Suppose there exist points $P_1, \dots, P_k$ on $\bar{Y}$, the normalization of Y, satisfying $\pi(P_i) \in Y \cap C'$ for all $1 \leq i \leq k$. Write $h^0(p_W(C'), \mathcal{O}_{p_W(C')}(1)) =: h^0$.

Finally, suppose that

$$\mu^{L_{m,\hat{m},m'}}((C, x_1, \dots, x_n), \lambda'_{C'}) \leq 0.$$

Then

$$e' + \frac{k}{2} < \frac{h^0 e + (dh^0 - d'(e - g + 1))\frac{\hat{m}}{m} + (nh^0 - n'(e - g + 1))\frac{m'}{m^2}}{e - g + 1} + \frac{S}{m}, \tag{25}$$

where $S = g + k(2g - \frac{3}{2}) + q_2 - \bar{g} + 1$. □

Proof. Arguing similarly to ([9], pp. 83–85), we prove the result by contradiction. We first assume that $k = \#(Y \cap C')$ and then show that this implies the general case.

Let C' be a connected subcurve of C, and let $P_1, \dots, P_k$ be all the points on $\bar{Y}$ satisfying $\pi(P_i) \in Y \cap C'$. We assume that equation (25) is not satisfied for C', and further that C' is maximal with this property. Namely, if C'' is complete and connected, and $C' \subsetneq C'' \subset C$, then equation (25) does hold for C''. Since equation (25) does not hold for C',

$$\left(e' + \frac{k}{2}\right)(e - g + 1) \geq (e' - g' + 1)e + (d(e' - g' + 1) - d'(e - g + 1))\frac{\hat{m}}{m}$$
$$+ (n(e' - g' + 1) - n'(e - g + 1))\frac{m'}{m^2} + \frac{S'}{m}(e - g + 1). \tag{26}$$

As all other hypotheses of Proposition 5.6 have been met, we must conclude that condition (ii) there fails. Thus there is some irreducible component $\bar{Y}_j$ of $\bar{Y}$, the normalization of Y, such that

$$\deg_{\bar{Y}_j}(\bar{L}_{W\bar{Y}}(-(P_1+\cdots+P_k)))<0.$$

Let Y_j be the corresponding irreducible component of Y. By assumption, Y_j does not collapse under projection to $\mathbf{P}(W)$, and so $\deg_{\bar{Y}_j}(\bar{L}_{W\bar{Y}})=\deg_{Y_j}(L_W)>0$. Putting this together,

$$0<\deg_{\bar{Y}_j}(\bar{L}_{W\bar{Y}})<\#(\bar{Y}_j\cap\{P_1,\ldots,P_k\})=\#(Y_j\cap C')=: i_{Y_j,C'}.$$

Thus $i_{Y_j,C'}\geq 2$. We define D to be the connected subcurve $Y_j\cup C'$. By the maximality assumption on C', it follows that equation (25) does hold for D. We define constants e_D, k_D, h^0_D, d_D, n_D, and S_D in analogy with the constants e', k, h^0, d', n', and S' for C'. Similarly, we define constants pertaining to Y_j. Then

$$e_D+\frac{k_D}{2}<\frac{h^0_D e+\left(dh^0_D-d_D(e-g+1)\right)\frac{\hat{m}}{m}+\left(nh^0_D-n_D(e-g+1)\right)\frac{m'}{m^2}}{e-g+1}+\frac{S_D}{m}. \tag{27}$$

Observe that $e_D=e'+e_{Y_j}$, $d_D=d'+d_{Y_j}$, and $n_D=n'+n_{Y_j}$. The curve C is nodal, so we conclude that

(i) $k_D=\#((Y_j\cup C')\cap(\overline{Y-Y_j}))=\#(C'\cap Y)+\#(Y_j\cap\overline{Y-Y_j})-\#(C'\cap Y_j)$; if we set $i_{Y_j,Y}:=\#(Y_j\cap Y)$, then $k_D=k+i_{Y_j,Y}-i_{Y_j,C'}$;

(ii) $g_D=g'+g_{Y_j}+i_{Y_j,C'}-1$;

(iii) $h^0_D=e_D-g_D+1=e'+e_{Y_j}-g'-g_{Y_j}-i_{Y_j,C'}+2$.

Note in particular that, since $i_{Y_j,C'}\geq 2$ and since C' and Y_j are connected, (ii) implies that $g_D\geq 1$. But $g_D\leq g$, so if $g=0$ then we already have the required contradiction. We henceforth assume that $g\geq 1$. Equation (27) may be rearranged to form

$$\begin{aligned}\left(e'+e_{Y_j}+\frac{k+i_{Y_j,Y}-i_{Y_j,C'}}{2}\right)(e-g+1)&<\left(e'+e_{Y_j}-g'-g_{Y_j}-i_{Y_j,C'}+2\right)e\\&+\left(d\left(e'+e_{Y_j}-g'-g_{Y_j}-i_{Y_j,C'}+2\right)-\left(d'+d_{Y_j}\right)(e-g+1)\right)\frac{\hat{m}}{m}\\&+\left(n\left(e'+e_{Y_j}-g'-g_{Y_j}-i_{Y_j,C'}+2\right)-\left(n'+n_{Y_j}\right)(e-g+1)\right)\frac{m'}{m^2}+\frac{S_D}{m}(e-g+1).\end{aligned}$$

We subtract our assumption, line (26),

$$\begin{aligned}\left(e_{Y_j} + \frac{i_{Y_j,Y} - i_{Y_j,C'}}{2}\right)(e-g+1) < {} & \left(e_{Y_j} - g_{Y_j} - i_{Y_j,C'} + 1\right)e \\ & + \left(d\left(e_{Y_j} - g_{Y_j} - i_{Y_j,C'} + 1\right) - d_{Y_j}(e-g+1)\right)\frac{\hat{m}}{m} \\ & + \left(n\left(e_{Y_j} - g_{Y_j} - i_{Y_j,C'} + 1\right) - n_{Y_j}(e-g+1)\right)\frac{m'}{m^2} \\ & + \frac{\left(i_{Y_j,Y} - i_{Y_j,C'}\right)\left(2g - \frac{1}{2}\right)}{m}(e-g+1). \end{aligned} \tag{28}$$

We rearrange, and use the inequality $e_{Y_j} \leq i_{Y_j,C'} - 1$.

$$\begin{aligned} & \left(\frac{i_{Y_j,Y} + i_{Y_j,C'}}{2} + g_{Y_j} - 1 + d_{Y_j}\frac{\hat{m}}{m} + n_{Y_j}\frac{m'}{m^2} + \frac{\left(i_{Y_j,C'} - i_{Y_j,Y}\right)\left(2g - \frac{1}{2}\right)}{m}\right)e \\ & \quad < \left(\frac{i_{Y_j,Y} + i_{Y_j,C'}}{2} - 1 + d_{Y_j}\frac{\hat{m}}{m} + n_{Y_j}\frac{m'}{m^2} + \frac{\left(i_{Y_j,C'} - i_{Y_j,Y}\right)\left(2g - \frac{1}{2}\right)}{m}\right) \\ & \quad \times (g-1) - g_{Y_j}\left(d\frac{\hat{m}}{m} + n\frac{m'}{m^2}\right), \end{aligned} \tag{29}$$

so that finally we may estimate

$$\begin{aligned} & \left(\frac{i_{Y_j,Y} + i_{Y_j,C'}}{2} + g_{Y_j} - 1 + d_{Y_j}\frac{\hat{m}}{m} + n_{Y_j}\frac{m'}{m^2} + \frac{\left(i_{Y_j,C'} - i_{Y_j,Y}\right)\left(2g - \frac{1}{2}\right)}{m}\right)(e-g+1) \\ & \quad < \; -g_{Y_j}\left(d\frac{\hat{m}}{m} + n\frac{m'}{m^2}\right) \leq 0. \end{aligned} \tag{30}$$

Recall that $e - g + 1 = \dim W > 0$, that $i_{Y_j,C'} \geq 2$, and that $g \geq 1$. Thus the left-hand side of equation (30) is strictly positive. This is a contradiction. No such C' exists, i.e. all subcurves C' of C satisfy inequality (25), provided that $k = \#(C' \cap Y)$.

Finally, suppose that we choose any k points $P_1 \ldots, P_k$ on $\bar{Y}$ such that $\pi(Y_i) \in (C' \cap Y)$. Then $k \leq \#(C' \cap Y) := k'$. We proved that equation (25) is true for k', and though we must take a little care with the dependence of S on k, it follows that equation (25) is true for k. ■

Now we may extend this to general $l \in \mathbf{H}_M(I)$, to provide the promised extension of the fundamental basic inequality.

Amplification 5.18. Let a be sufficiently large that $e - 9g + 7 > 0$, and let $M \subset \tilde{M}$, where $\tilde{M}$ consists of those $(m, \hat{m}, m')$ such that $m, \hat{m} > m_3$ and

$$m > \max \left\{ \begin{array}{c} (g - \frac{1}{2} + e(q_1 + 1) + q_3 + \mu_1 m_2)(e - g + 1), \\ (10g + 3q_2 - 3\bar{g})(e - g + 1), \end{array} \right\}$$

with

$$d\frac{\hat{m}}{m} + n\frac{m'}{m^2} < \frac{1}{8}e - \frac{9}{8}g + \frac{7}{8},$$

while

$$\frac{\hat{m}}{m} > 1 + \frac{\frac{3}{2}g - 1 + d + n\frac{m'}{m^2}}{e - g + 1 - d} \qquad \frac{m'}{m^2} > \frac{1}{4} + \frac{g + \frac{n}{4} + d\frac{\hat{m}}{m}}{2(e - g + 1) - n}.$$

Let $l \in \mathbf{H}_M(I)$. Let $(C, x_1, \ldots, x_n)$ be semistable with respect to l, where C is a connected curve. Suppose C has at least two irreducible components. Let $C' \neq C$ be a reduced, complete subcurve of C and let $Y := \overline{C - C'}$. The subcurves C' and Y need not be connected; suppose C has b connected components. Suppose there exist points $P_1, \ldots, P_k$ on $\bar{Y}$, the normalization of Y, satisfying $\pi(P_i) \in Y \cap C'$ for all $1 \leq i \leq k$. Write $h^0(p_W(C'), \mathcal{O}_{p_W(C')}(1)) =: h^0$. Then there exists a triple $(m, \hat{m}, m') \in M$ such that

$$e' + \frac{k}{2} < \frac{h^0 e + (dh^0 - d'(e - g + 1))\frac{\hat{m}}{m} + (nh^0 - n'(e - g + 1))\frac{m'}{m^2}}{e - g + 1} + \frac{bS}{m}, \tag{31}$$

where $S = g + k(2g - \frac{3}{2}) + q_2 - \bar{g} + 1$. □

Proof. First assume that C' is connected, and suppose that inequality (31) fails for all $(m, \hat{m}, m') \in M$. It must follow that $(C, x_1, \ldots, x_n)$ does not satisfy the hypotheses of Lemma 5.17. However, as $(C, x_1, \ldots, x_n)$ is semistable with respect to l, all the other hypotheses of that lemma are verified, so we must conclude that $\mu^{L_{m,\hat{m},m'}}((C, x_1, \ldots, x_n), \lambda'_{C'}) > 0$ for all $(m, \hat{m}, m') \in M$. It follows by Lemma 4.4 that $(C, x_1, \ldots, x_n)$ is unstable with respect to l. The contradiction implies that there do indeed exist some $(m, \hat{m}, m') \in M$ such that equation (31) is satisfied.

Now, let $C'_1, \ldots, C'_b$ be the connected components of C'. We may prove a version of equation (31) for each C_i, for $i = 1, \ldots, b$. When we sum these inequalities over i, it follows that

$$e' + \frac{k}{2} < \frac{h^0 e + (dh^0 - d'(e-g+1))\frac{\hat{m}}{m} + (nh^0 - n'(e-g+1))\frac{m'}{m^2}}{e-g+1} + \frac{bS}{m}.$$

■

We summarize the results of Sections 5.2 to 5.5. Recall again that since e is defined to be $a(2g-2+n+cd)$, and since $2g-2+n+cd$ is always at least 1, the denominators $e-g+1-d$ and $2(e-g+1)-n$ are both positive.

Theorem 5.19. Let a be sufficiently large that $e - 9g + 7 > 0$, and let $M \subset \tilde{M}$, where $\tilde{M}$ consists of those $(m, \hat{m}, m')$ such that $m, \hat{m} > m_3$ and

$$m > \max\left\{\begin{matrix}\left(g - \frac{1}{2} + e(q_1+1) + q_3 + \mu_1 m_2\right)(e-g+1), \\ (10g + 3q_2 - 3\bar{g})(e-g+1),\end{matrix}\right\}$$

with

$$d\frac{\hat{m}}{m} + n\frac{m'}{m^2} < \frac{1}{8}e - \frac{9}{8}g + \frac{7}{8},$$

while

$$\frac{\hat{m}}{m} > 1 + \frac{\frac{3}{2}g - 1 + d + n\frac{m'}{m^2}}{e-g+1-d} \qquad \frac{m'}{m^2} > \frac{1}{4} + \frac{g + \frac{n}{4} + d\frac{\hat{m}}{m}}{2(e-g+1)-n}.$$

Let $l \in \mathbf{H}_M(I)$ and let $(C, x_1, \ldots, x_n)$ be a connected curve, semistable with respect to l. Then $(C, x_1, \ldots, x_n)$ satisfies that

(i) $(C, x_1, \ldots, x_n)$ is a reduced, connected, nodal curve, and the marked points are distinct and nonsingular;

(ii) the map $C \to \mathbf{P}(W)$ collapses no component of C, and induces an injective map

$$H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) \to H^0(C, L_W);$$

(iii) $h^1(C, L_W) = 0$;

(iv) any complete subcurve $C' \subset C$ with $C' \neq C$ satisfies the inequality

$$e' + \frac{k}{2} < \frac{h^0 e + (dh^0 - d'(e-g+1))\frac{\hat{m}}{m} + (nh^0 - n'(e-g+1))\frac{m'}{m^2}}{e-g+1} + \frac{bS}{m}$$

of Amplification 5.18, where C' consists of b connected components and $S = g + k(2g - \frac{3}{2}) + q_2 - \bar{g} + 1$. □

Definition 5.20. If $(C, x_1, \ldots, x_n) \subset \mathbf{P}(W) \times \mathbf{P}^r$ satisfies conditions (i)–(iv) of Theorem 5.19, then the corresponding map $(C, x_1, \ldots, x_n) \xrightarrow{p_r} \mathbf{P}^r$ is referred to as a *potentially stable map*. □

Remark. Gieseker defines his "potentially stable curves" (which have no marked points) using the analogous statements, and the additional condition if the curve is not a moduli stable curve, then destabilizing components must have two nodes and be embedded as lines. A similar condition can be given here, and shown to be a corollary of the fundamental inequality (for a restricted range M).

Namely, for a certain M, we can show that if $l \in \mathbf{H}_M(I)$ and $(C, x_1, \ldots, x_n)$ is semistable with respect to l, and if C' is a rational component of C which is collapsed under projection to $\mathbf{P}^r$, then C' has at least two special points; if it has precisely two, then it is embedded in $\mathbf{P}(W)$ as a line. The proof of this follows ([9], Proposition 1.0.9) and full details may be seen in ([2], Corollary 5.5.1), but it has been omitted here for brevity, as it is not needed to prove Theorem 6.1.

5.6 GIT semistable maps represented in $\overline{J}$ are moduli stable

In the previous sections, we have been studying I^{ss}. In this section, we focus on $\overline{J}^{ss}$. Recall the definitions of I and J, given in Sections 3.1 and 3.2: the scheme I is the Hilbert scheme of n-pointed curves in $\mathbf{P}(W) \times \mathbf{P}^r$, and $J \subset I$ is the locally closed subscheme such that for each $(h, x_1, \ldots, x_n) \in J$,

(i) $(\mathcal{C}_h, x_1, \ldots, x_n)$ is prestable, i.e. $\mathcal{C}_h$ is projective, connected, reduced and nodal, and the marked points are distinct and nonsingular;

(ii) the projection map $\mathcal{C}_h \to \mathbf{P}(W)$ is a nondegenerate embedding;

(iii) the invertible sheaves $(\mathcal{O}_{\mathbf{P}(W)}(1) \otimes \mathcal{O}_{\mathbf{P}^r}(1))|_{\mathcal{C}_h}$ and $(\omega_{\mathcal{C}_h}^{\otimes a}(ax_1 + \cdots + ax_n) \otimes \mathcal{O}_{\mathbf{P}^r}(ca+1))|_{\mathcal{C}_h}$ are isomorphic, where c is a positive integer; see Section 2.4.

Moreover, recall from the discussion at the end of Section 5.1 that we had set out to find a linearization such that $\overline{J}^{ss}(L) \subseteq J$. This, together with nonemptiness of $\overline{J}^{ss}(L)$, is sufficient to show that $\overline{J}//_L SL(W) \cong \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ (Theorem 3.6).

In this section, we find a range M of $(m, \hat{m}, m')$ such that $\overline{J}^{ss}(l) \subseteq J$ when $l \in \mathbf{H}_M(I)$. This range is much narrower than those we have considered so far.

Here is the result we have been seeking.

Theorem 5.21 (cf. [11], 4.55). Let M consist of those $(m, \hat{m}, m')$ such that $m, \hat{m} > m_3$ and

$$m > \max \left\{ \begin{matrix} \left(g - \frac{1}{2} + e(q_1 + 1) + q_3 + \mu_1 m_2\right)(e - g + 1), \\ (10g + 3q_2 - 3\bar{g})(e - g + 1) \\ (6g + 2q_2 - 2\bar{g} - 1)(2a - 1) \end{matrix} \right\}$$

with

$$\frac{\hat{m}}{m} = \frac{ca}{2a - 1} + \delta \tag{32}$$

$$\frac{m'}{m^2} = \frac{a}{2a - 1} + \eta, \tag{33}$$

where

$$|n\eta| + |d\delta| \leq \frac{1}{4a - 2} - \frac{3g + q_2 - \bar{g} - \frac{1}{2}}{m}. \tag{34}$$

In addition, ensure that a is sufficiently large that

$$d\frac{\hat{m}}{m} + n\frac{m'}{m^2} < \frac{1}{8}e - \frac{9}{8}g + \frac{7}{8}. \tag{35}$$

Let $l \in \mathbf{H}_M(\overline{J})$. Then $\overline{J}^{ss}(l)$ is contained in J. □

Remark. The final assumption on the magnitude of m ensures that the right-hand side of equation (34) is positive, and so equation (34) may be satisfied. There may seem to be many competing bounds on the ratios $\frac{\hat{m}}{m}$ and $\frac{m'}{m^2}$, and on a. However, one may show that equation (35) is implied by equations (32), (33), and (34) for all g, n, and d as long as $a \geq 10$ (cf. [2], proof of Theorem 5.21.1). Smaller values of a are possible for most g, n, and d. Once a large enough a has been chosen, it is always possible to satisfy the rest of the inequalities; the simplest way is to set δ and η to zero and pick large $m, \hat{m}, m'$ with the desired ratios.

Proof. The range M here is contained within the range for Theorem 5.19. We may thus apply Theorem 5.19: if $(h, x_1, \dots, x_n) \in I^{ss}(l')$ for some $l' \in \mathbf{H}_M(I)$, then $(\mathcal{C}_h, x_1, \dots, x_n)$ is nodal and reduced, and one can find $(m, \hat{m}, m') \in M$ satisfying inequality (31). However, this theorem in fact deals with $\mathbf{H}_M(\overline{J})$ and not $\mathbf{H}_M(I)$; on the other hand, any $l \in \mathbf{H}_M(\overline{J})$ may be regarded as the restriction of some $l' \in \mathbf{H}_M(I)$ to $\overline{J}$, and then $\overline{J}^{ss}(l) = \overline{J} \cap I^{ss}(l')$. Thus, we may use all our previous results.

Suppose we can show that $J \cap \overline{J}^{ss}(l)$ is closed in $\overline{J}^{ss}(l)$. Then if $x \in \overline{J}^{ss}(l) - J \cap \overline{J}^{ss}(l)$, there must be an open neighborhood of x in $\overline{J}^{ss}(l) - J \cap \overline{J}^{ss}(l)$, but this is a contradiction as x is in $\overline{J}$, so x is a limit point of J. It follows that $J \cap \overline{J}^{ss}(l) = \overline{J}^{ss}(l)$, i.e. $\overline{J}^{ss}(l) \subseteq J$.

We shall proceed by using the valuative criterion of properness to show that the inclusion $J \cap \overline{J}^{ss}(l) \hookrightarrow \overline{J}^{ss}(l)$ is proper, whence $J \cap \overline{J}^{ss}(l)$ is closed in $\overline{J}^{ss}(l)$, as required. Let R be a discrete valuation ring, with generic point ξ and closed point 0. Let $\alpha : \operatorname{Spec} R \to \overline{J}^{ss}(l)$ be a morphism such that $\alpha(\xi) \subset J \cap \overline{J}^{ss}(l)$. Then we will show that $\alpha(0) \in J \cap \overline{J}^{ss}(l)$.

Define a family $\mathcal{D}$ of n-pointed curves in $\mathbf{P}(W) \times \mathbf{P}^r$ by the following pullback diagram:

$$\begin{array}{ccc} \mathcal{D} & \to & \tilde{\mathcal{C}}|_{\overline{J}^{ss}(l)} \\ \downarrow\uparrow \sigma_i & & \downarrow\uparrow \sigma_i \\ \operatorname{Spec} R & \xrightarrow{\alpha} & \overline{J}^{ss}(l), \end{array}$$

where $\sigma_1, \dots, \sigma_n : \operatorname{Spec} R \to \mathcal{D}$ are the sections giving the marked points. The images of the σ_i in $\mathcal{D}$ are divisors, denoted $\sigma_i(\operatorname{Spec} R)$. By definition of J, we have

$$(\mathcal{O}_{\mathbf{P}(W)}(1) \otimes \mathcal{O}_{\mathbf{P}^r}(1))|_{\mathcal{D}_\xi} \cong \omega_{\mathcal{D}_\xi}^{\otimes a}(a\sigma_1(\xi) + \cdots + a\sigma_n(\xi)) \otimes \mathcal{O}_{\mathbf{P}^r}(ca + 1)|_{\mathcal{D}_\xi}.$$

We will write $(\mathcal{D}_0, \sigma_1(0), \dots, \sigma_n(0)) =: (C, x_1, \dots, x_n)$, and show that its representative in the universal family I is in fact in J. The curve C is connected, as a limit of connected curves. We assumed that $\alpha(0) \in \overline{J}^{ss}(l) = \overline{J} \cap I^{ss}(l')$, where $l' \in \mathbf{H}_M(I)$, and so $(C, x_1, \dots, x_n)$ satisfies conditions (i) above, and the curve $p_W(C) \subset \mathbf{P}(W)$ is nondegenerate. We will show that the line bundles in condition (iii) are isomorphic. It follows from this that

$$\mathcal{O}_{\mathbf{P}(W)}(1)|_C \cong \omega_C^a(ax_1 + \cdots + ax_n) \otimes \mathcal{O}_{\mathbf{P}^r}(ca)|_C, \tag{36}$$

and so this line bundle has positive degree on every component of C. However, we know that $\mathcal{L} = \omega_C(x_1 + \cdots + x_n) \otimes \mathcal{O}_{\mathbf{P}^r}(c)$ has positive degree on each component of C if and only if $p_r : C \to \mathbf{P}^r$ is a stable map, and that when this true then $\mathcal{L}^a$ is very ample. Hence $\mathcal{L}^a$ embeds C in $\mathbf{P}(W)$, i.e. $p_W : C \cong p_W(C)$, and we have verified condition (ii) in the definition of J. Thus, checking condition (iii) is sufficient to show that $(C, x_1, \dots, x_n)$ is represented in J.

Decompose $C = \bigcup C_i$ into its irreducible components. Then we can write

$$(\mathcal{O}_{\mathbf{P}W}(1) \otimes \mathcal{O}_{\mathbf{P}^r}(1))|_{\mathcal{D}} \cong \omega^{\otimes a}_{\mathcal{D}/\text{Spec } R}(a\sigma_1(\text{Spec } R) + \cdots + a\sigma_n(\text{Spec } R)) \\ \otimes \mathcal{O}_{\mathbf{P}^r}(ca+1)|_{\mathcal{D}} \otimes \mathcal{O}_{\mathcal{D}}\left(\sum a_i C_i\right),$$

where the a_i are integers. As $\mathcal{O}_{\mathcal{D}}$-modules, $\mathcal{O}_{\mathcal{D}}(C) \cong \mathcal{O}_{\mathcal{D}}$, so we can normalize the integers a_i so that they are all non-negative and at least one of them is zero. Separate C into two subcurves $Y := \bigcup_{a_i=0} C_i$ and $C' := \bigcup_{a_i>0} C_i$. Since at least one of the a_i is zero, we have $Y \neq \emptyset$ and $C' \neq C$. Suppose for a contradiction that $C' \neq \emptyset$ (hence $Y \neq C$). Let $k = \#(Y \cap C')$ and let b be the number of connected components of C'. Since C is connected, we must have $k \geq b$. We will obtain our contradiction by showing that $\frac{k}{b} < 1$.

Any local equation for the divisor $\mathcal{O}_{\mathcal{D}}(\sum a_i C_i)$ must vanish identically on every component of C' and on no component of Y. Such an equation is zero therefore at each of the k nodes in $Y \cap C'$. Thus we obtain the inequality

$$k \leq \deg_Y\left(\mathcal{O}_{\mathcal{D}}\left(-\sum a_i C_i\right)\right) = \deg_Y\left(\mathcal{O}_{\mathbf{P}(W)}(1) \otimes \omega^{\otimes -a}_{\mathcal{D}_0}(-a\sigma_1(0) - \cdots - a\sigma_n(0)) \otimes \mathcal{O}_{\mathbf{P}^r}(-ca)\right) \\ = e_Y - a(2g_Y - 2 + n_Y + k) - cad_Y.$$

Substituting $e' = e - e_Y$, $d' = d - d_Y$, $g' = g - g_Y - k + 1$, and $e = a(2g - 2 + n + cd)$, this is equivalent to

$$e' - a(2g' - 2 + n' + cd') \leq (a-1)k. \tag{37}$$

The hypotheses of Amplification 5.18 are satisfied for C' and $k = \#(Y \cap C')$, with M as in the statement of this theorem, and $l' \in \mathbf{H}_M(I)$: there exist $(m, \hat{m}, m') \in M$ satisfying equation (25). Write $\frac{\hat{m}}{m} = \frac{ca}{2a-1} + \delta$ and $\frac{m'}{m^2} = \frac{a}{2a-1} + \eta$, assuming that δ and η satisfy the hypotheses above.

$$e' + \frac{k}{2} < \frac{(e' - g' + 1)e + \frac{\hat{m}}{m}((e' - g' + 1)d - (e - g + 1)d')}{e - g + 1} \\ + \frac{\frac{m'}{m^2}((e' - g' + 1)n - (e - g + 1)n')}{e - g + 1} + \frac{bS}{m}$$

$$\begin{aligned}\Leftrightarrow \frac{k}{2}(e-g+1) &< e'(g-1)-e(g'-1)+\frac{a}{2a-1}((e'-g'+1)(n+cd)-(e-g+1)(n'+cd'))\\ &\quad +\eta((e'-g'+1)n-(e-g+1)n')+\delta((e'-g'+1)d-(e-g+1)d')\\ &\quad +\frac{bS}{m}(e-g+1). \qquad (38)\end{aligned}$$

The final terms of the right-hand side are already in the form we want them, so for brevity we shall only work on the first line.

$$\begin{aligned}&e'(g-1)-e(g'-1)+\frac{a}{2a-1}((e'-g'+1)(n+cd)-(e-g+1)(n'+cd'))\\ &\quad = e'\left((g-1)+\frac{a}{2a-1}(n+cd)\right)-(g'-1)\left(e+\frac{a}{2a-1}(n+cd)\right)\\ &\qquad -\frac{a}{2a-1}(e-g+1)(n'+cd')\\ &\quad =\frac{e'}{2a-1}((2a-1)(g-1)+a(n+cd))-(g'-1)(e-g+1)\\ &\qquad +\frac{1}{2a-1}((2a-1)(g-1)+a(n+cd))-\frac{a}{2a-1}(e-g+1)(n'+cd')\\ &\quad =\frac{e'}{2a-1}(e-g+1)-(g'-1)(e-g+1)\left(1+\frac{1}{2a-1}\right)-\frac{a}{2a-1}(e-g+1)(n'+cd'),\end{aligned}$$

where for the last equality we have recalled that $(2a-1)(g-1)+a(n+cd)=e-g+1$. We substitute this in line (38) and then multiply by $\frac{2a-1}{e-g+1}$ to obtain

$$\begin{aligned}(2a-1)\frac{k}{2} &< e'-a(2g'-2+n'+cd')+(2a-1)n\eta\left(\frac{e'-g'+1}{e-g+1}-\frac{n'}{n}\right)\\ &\quad +(2a-1)d\delta\left(\frac{e'-g'+1}{e-g+1}-\frac{d'}{d}\right)+(2a-1)\frac{bS}{m}.\end{aligned}$$

Now use equation (37) to see

$$\begin{aligned}\frac{k}{2} &< (2a-1)n\eta\left(\frac{e'-g'+1}{e-g+1}-\frac{n'}{n}\right)+(2a-1)d\delta\left(\frac{e'-g'+1}{e-g+1}-\frac{d'}{d}\right)+(2a-1)\frac{bS}{m}\\ \Rightarrow \frac{k}{2b} &< (2a-1)n\eta\left(\frac{e'-g'+1}{e-g+1}-\frac{n'}{n}\right)+(2a-1)d\delta\left(\frac{e'-g'+1}{e-g+1}-\frac{d'}{d}\right)+(2a-1)\frac{S}{m}. \qquad (39)\end{aligned}$$

We must take care as S varies with k; explicitly, $S = g + k(2g - \frac{3}{2}) + q_2 - \bar{g} + 1$. Thus the inequality we wish to contradict becomes

$$\frac{k}{b}\left(\frac{1}{4a-2} - \frac{\left(2g - \frac{3}{2}\right)}{m}\right) < n\eta\left(\frac{e' - g' + 1}{e - g + 1} - \frac{n'}{n}\right) + d\delta\left(\frac{e' - g' + 1}{e - g + 1} - \frac{d'}{d}\right) + \frac{g + q_2 - \bar{g} + 1}{m}. \tag{40}$$

It is time to use our bounds for η and δ. Note that

$$-1 \leq \frac{e' - g' + 1}{e - g + 1} - \frac{n'}{n} \leq 1 \qquad -1 \leq \frac{e' - g' + 1}{e - g + 1} - \frac{d'}{d} \leq 1.$$

We assumed that $|n\eta| + |d\delta| \leq \frac{1}{4a-2} - \frac{3g + q_2 - \bar{g} - \frac{1}{2}}{m}$. It follows that

$$n\eta\left(\frac{e' - g' + 1}{e - g + 1} - \frac{n'}{n}\right) + d\delta\left(\frac{e' - g' + 1}{e - g + 1} - \frac{d'}{d}\right) \leq \frac{1}{4a-2} - \frac{3g + q_2 - \bar{g} - \frac{1}{2}}{m}.$$

Hence line (40) says

$$\frac{k}{b}\left(\frac{1}{4a-2} - \frac{\left(2g - \frac{3}{2}\right)}{m}\right) < \frac{1}{4a-2} - \frac{2g - \frac{3}{2}}{m}.$$

By hypothesis, $m > (6g + 2q_2 - 2\bar{g} + 1)(2a - 1) > (2g - \frac{3}{2})(4a - 2)$, and so we know that $\frac{1}{4a-2} - \frac{2g - \frac{3}{2}}{m} > 0$. Thus we have proved that $\frac{k}{b} < 1$, a contradiction.

The contradiction implies that we cannot decompose C into two strictly smaller sub-curves C' and Y as described. Thus all the coefficients a_i must be zero, and we have an isomorphism

$$(\mathcal{O}_{\mathbf{P}(W)}(1) \otimes \mathcal{O}_{\mathbf{P}^r}(1))|_{\mathcal{D}} \cong \omega_{\mathcal{D}/\text{Spec } R}^{\otimes a}(a\sigma_1(\text{Spec } R) + \cdots + a\sigma_n(\text{Spec } R)) \otimes \mathcal{O}_{\mathbf{P}^r}(ca + 1)|_{\mathcal{D}}.$$

In particular, $(\mathcal{D}_0, \sigma_1(0), \ldots, \sigma_n(0))$ satisfies condition (iii) of Definition 3.2. We conclude as described that it is represented in J, and so $\alpha(0) \in J \cap \overline{J}^{ss}$. Hence $J \cap \overline{J}^{ss}(l)$ is closed in $\overline{J}^{ss}(l)$, which completes the proof. ■

Remark. A slightly larger range of values for $\frac{m'}{m^2}$ and $\frac{\hat{m}}{m}$ is possible; note that in fact $\frac{e'-g'+1}{e-g+1} - \frac{n'}{n} > -1$, enabling us to drop our lower bound to below $\frac{1}{4a-1}$. It is not clear whether the upper bound can be improved.

Let us review what we know, given this result. It is time to apply the theory of variation of GIT, to show that the semistable set $\overline{J}^{ss}(l)$ is the same for all $l \in \mathbf{H}_M(\overline{J})$, where

M is as in the statement of Theorem 5.21. Recall the definitions from Section 2.3. In particular, we will make use of Proposition 2.12.

Corollary 5.22. Let M be as given in the statement of Theorem 5.21. Let $l \in \mathbf{H}_M(I)$.

(i) If $l \in \mathbf{H}_M(\overline{J})$, then $\overline{J}^{ss}(l) = \overline{J}^{s}(l) \subseteq J$.

(ii) If $(m, \hat{m}, m') \in M$ and if $\overline{J}^{ss}(L_{m,\hat{m},m'}) \neq \emptyset$ then, when we work over $\mathbb{C}$,

$$\overline{J}//_{L_{m,\hat{m},m'}} SL(W) \cong \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d).$$

(iii) The semistable set $\overline{J}^{ss}(l)$ is the same for all $l \in \mathbf{H}_M(\overline{J})$. □

Proof. Parts (i) and (ii) follow from Proposition 3.5, Theorem 3.6, and Theorem 5.21.

Part (iii): The region $\mathbf{H}_M(\overline{J})$ is by definition convex, and lies in the ample cone $\mathbf{A}^G(X)$. Part (i) has shown us that if $l \in \mathbf{H}_M(\overline{J})$, then $\overline{J}^{ss}(l) = \overline{J}^{s}(l)$. Now the result follows from Proposition 2.12. ■

We have completed the first part of the proof. By Corollary 5.22, it only remains to show that $\overline{J}^{ss}(l) \neq \emptyset$ for at least one $l \in \mathbf{H}_M(\overline{J})$.

6 The Construction Finished

6.1 Statement of theorems

We are now in a position to state the main theorem of this paper: for a specified range M of values $(m, \hat{m}, m')$, the GIT quotient $\overline{J}//_{L_{m,\hat{m},m'}} SL(W)$ is isomorphic to $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$. First we recall the notation from Section 3.1. The vector space W is of dimension $e - g + 1$, where $e = a(2g - 2 + n + cd)$, the integer c being sufficiently large that this is positive. We embed the domains of stable maps into $\mathbf{P}(W)$. We denote by I the Hilbert scheme of n-pointed curves in $\mathbf{P}(W) \times \mathbf{P}^r$ of bidegree (e, d). The subspace $J \subset I$ corresponds to a-canonically embedded curves, such that the projection to $\mathbf{P}^r$ is a moduli stable map; this is laid out precisely in Definition 3.2.

The constants m_1, m_2, m_3, q_1, q_2, q_3, μ_1, and μ_2 are all defined in Section 4.4. In particular, we recall m_3 and q_2: if $m, \hat{m} \geq m_3$, then the morphism from I to projective space, defined by $(h, x_1, \ldots, x_n) \mapsto \hat{H}_{m,\hat{m},m'}(h, x_1, \ldots, x_n)$, is a closed immersion. The constant q_2 is chosen so that $h^0(C, \mathcal{I}_C) \leq q_2$, for any curve $C \subset \mathbf{P}(W) \times \mathbf{P}^r$. We also defined $\bar{g}$,

$$\bar{g} := \min\{0, g_{\bar{Y}} \mid \bar{Y} \text{ is the normalization of a complete subcurve } Y \text{ contained in a connected fiber } C_h \text{ for some } h \in \mathrm{Hilb}(\mathbf{P}(W) \times \mathbf{P}^r)\}.$$

Recall that $\bar{g}$ is bounded below by $-(e+d)+1$.

In the statement of the theorem, note that the conditions on m explicitly ensure that $\frac{1}{4a-2} - \frac{3g+q_2-\bar{g}-\frac{1}{2}}{m} > 0$, and hence that the condition on η and δ is satisfied when η and δ are sufficiently small. Further remarks on the bounds for $m, \hat{m}, m'$, and a are given after the statement of Theorem 5.21, which concerns the same range of linearizations.

Theorem 6.1. Fix integers g, n, and $d \geq 0$ such that there exist smooth stable n-pointed maps of genus g and degree d. Suppose $m, \hat{m} > m_3$ and

$$m > \max \left\{ \begin{matrix} \left(g - \frac{1}{2} + e(q_1+1) + q_3 + \mu_1 m_2\right)(e-g+1), \\ (10g + 3q_2 - 3\bar{g})(e-g+1) \\ (6g + 2q_2 - 2\bar{g} - 1)(2a-1) \end{matrix} \right\}$$

with

$$\frac{\hat{m}}{m} = \frac{ca}{2a-1} + \delta, \qquad \frac{m'}{m^2} = \frac{a}{2a-1} + \eta,$$

where

$$|n\eta| + |d\delta| \leq \frac{1}{4a-2} - \frac{3g+q_2-\bar{g}-\frac{1}{2}}{m},$$

and in addition, let a be sufficiently large that

$$d\frac{\hat{m}}{m} + n\frac{m'}{m^2} < \frac{1}{8}e - \frac{9}{8}g + \frac{7}{8}.$$

Then over $\mathbb{C}$,

$$\overline{J}//_{L_{m,\hat{m},m'}} SL(W) = \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d).$$

□

Corollary 6.2 (cf. [8], Lemma 8). Let $m, \hat{m}, m'$ satisfy the conditions of Theorem 6.1. Let $X \overset{\iota}{\hookrightarrow} \mathbf{P}^r$ be a projective variety. Let $\beta \in H_2(X)^+$ be the homology class of some stable map. If $\beta = 0$, suppose that $2g - 2 + n \geq 1$. Write $d := \iota_*(\beta) \in H_2(\mathbf{P}^r)^+$. Then there exists a closed subscheme $J_{X,\beta}$ of J such that over $\mathbb{C}$,

$$\overline{J}_{X,\beta}//_{L_{m,\hat{m},m'}|_{\overline{J}_{X,\beta}}} SL(W) \cong \overline{\mathcal{M}}_{g,n}(X, \beta),$$

where $\overline{J}_{X,\beta}$ is the closure of $J_{X,\beta}$ in $\overline{J}$. □

Theorem 6.1 and Corollary 6.2 will have been proved when we know that $\overline{J}^{ss}(l) = \overline{J}^{s}(l) = J$ for $l \in \mathbf{H}_M(\overline{J})$. By Corollary 5.22, it only remains to show nonemptiness of $\overline{J}^{ss}(l)$ for one such l. This nonemptiness is proved by induction on n, the number of marked points. The base case, $n = 0$, closely follows Gieseker's method, ([9], Theorem 1.0.0), and was given in Swinarski's thesis [28].

The inductive step is different. We take a stable map $f : (C_0, x_1, \ldots, x_n) \to \mathbf{P}^r$, and remove one of the marked points, attaching a new genus 1 component to C_0 in its place. We extend f over the new curve by defining it to contract the new component to a point. The result is a Deligne–Mumford stable map of genus $g+1$, with $n-1$ marked points. This is inductively known to have a GIT semistable model, and semistability of a model for $f : (C_0, x_1, \ldots, x_n) \to \mathbf{P}^r$ follows.

In fact, the inductive step shows directly that $\overline{J}^{ss}(l) = J$ for all $l \in \mathbf{H}_M(\overline{J})$. It follows from Proposition 5.21 that $\overline{J}^{ss}(l) = \overline{J}^{s}(l) = J$ for such l. Hence we know that $\overline{J}//_{L_{m,\hat{m},m'}} SL(W) \cong \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ for $(m, \hat{m}, m') \in M$, without needing to appeal to independent constructions. Of course, such constructions are still needed for the base case. However, when $r = d = 0$, the base case is $\overline{\mathcal{M}}_g$, constructed by Gieseker over Spec $\mathbb{Z}$. The theory we are using is valid over any field, as we have extended the results we need from variation of GIT. Thus $\overline{\mathcal{M}}_{g,n}$ is constructed over Spec k for any field k. As we shall show, this is sufficient to show that $\overline{\mathcal{M}}_{g,n}$ is in fact constructed over Spec $\mathbb{Z}$.

As the constant $\hat{m}$ is irrelevant in the case $r = d = 0$, we set it to zero and suppress it in the notation $L_{m,\hat{m},m'}$.

Theorem 6.3. Let g and $n \geq 0$ be such that $2g - 2 + n > 0$. Set $e = a(2g - 2 + n)$. Suppose $m > m_3$ and

$$m > \max \left\{ \begin{matrix} \left(g - \frac{1}{2} + e(q_1 + 1) + q_3 + \mu_1 m_2\right)(e - g + 1), \\ (10g + 3q_2 - 3\bar{g})(e - g + 1) \\ (6g + 2q_2 - 2\bar{g} - 1)(2a - 1) \end{matrix} \right\}$$

with

$$\frac{m'}{m^2} = \frac{a}{2a - 1} + \eta,$$

where

$$|n\eta| \leq \frac{1}{4a - 2} - \frac{3g + q_2 - \bar{g} - 1}{m},$$

and in addition, ensure a is sufficiently large that

$$n\frac{m'}{m^2} < \frac{1}{8}e - \frac{9}{8}g + \frac{7}{8}.$$

Then, as schemes over Spec $\mathbb{Z}$,

$$\overline{J}//_{L_{m,m'}} SL(W) = \overline{\mathcal{M}}_{g,n}. \qquad \square$$

6.2 The base case: no marked points

Before we can state the theorem that maps from smooth domain curves are semistable, there is a little more notation to mention. The constant ϵ is found by Gieseker in the following lemma, which is based on ([18], Theorem 4.1). Note that the hypothesis printed in [9] is that $e \geq 20(g-1)$, but careful examination of the proof and [18] shows that $e \geq 2g+1$ suffices.

Lemma 6.4 ([9], Lemma 0.2.4)**.** Fix two integers $g \geq 2$, $e \geq 2g+1$ and write $N = e - g$. Then there exists $\epsilon > 0$ such that for all integers $r_0 \leq \cdots \leq r_N$ (not all zero) with $\sum r_i = 0$, and for all integers $0 = e_0 \leq \cdots \leq e_N = e$ satisfying

(i) if $e_j > 2g-2$, then $e_j \geq j+g$;
(ii) if $e_j \leq 2g-2$, then $e_j \geq 2j$;

there exists a sequence of integers $0 = i_1 \leq \cdots \leq i_k = N$ verifying the following inequality:

$$\sum_{t=1}^{k-1} \left(r_{i_{t+1}} - r_{i_t}\right)\left(e_{i_{t+1}} + e_{i_t}\right) > 2r_N e + 2\epsilon(r_n - r_0).$$

$\square$

Now the statement of the theorem is as follows.

Theorem 6.5 (cf. [9], 1.0.0)**.** For all $K > 0$ there exist integers p, b satisfying $m = (p+1)b > K$, such that for any $\hat{m}_1 > 2g-1$ satisfying $\hat{m} := b\hat{m}_1 > m_3$, if $C \subset \mathbf{P}(W) \times \mathbf{P}^r \to \mathbf{P}^r$ is a stable map, if C is nonsingular, if the map

$$H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) \xrightarrow{\rho} H^0\big(p_W(C), \mathcal{O}_{p_W(C)}(1)\big)$$

is an isomorphism, and if L_W is very ample (so that $C \cong p_W(C)$), then $C \in I^{ss}(L_{m,\hat{m}})$. $\square$

Remark. The values that m and $\hat{m}$ must take will be made clear in the course of the proof.

Proof. Let $C \subset \mathbf{P}(W) \times \mathbf{P}^r \to \mathbf{P}^r$ be such a map. Let λ be a 1-PS of $SL(W)$. There exist a basis $\{w_0, \ldots, w_N\}$ of $H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1))$ and integers $r_0 \leq \cdots \leq r_N$ such that $\sum r_i = 0$ and the action of λ is given by $\lambda(t) w_i = t^{r_i} w_i$. By our hypotheses, the map $p_{W*}\rho : H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1)) \to H^0(C, L_W)$ is injective. Write $w_i' := p_{W*}\rho(w_i)$. Let E_j be the invertible subsheaf of L_W generated by $w_0', \ldots, w_j'$ for $0 \leq j \leq N = e - g$, and write $e_j = \deg E_j$. Note that $E_N = L_W$, since L_W is very ample, hence generated by global sections, $h^0(C, L_W) = e - g + 1$ and $w_0', \ldots, w_N'$ are linearly independent. The integers $e_0, \ldots, e_N = e$ satisfy the following two properties:

(i) If $e_j > 2g - 2$, then $e_j \geq j + g$.
(ii) If $e_j \leq 2g - 2$, then $e_j \geq 2j$.

To see this, note that since by definition E_j is generated by $j + 1$ linearly independent sections, we have $h^0(C, E_j) \geq j + 1$. If $e_j = \deg E_j > 2g - 2$ then $H^1(C, E_j) = 0$, so by Riemann–Roch, $e_j = h^0 - h^1 + g - 1 \geq j + g$. If $e_j \leq 2g - 2$ then $H^0(C, \omega_C \otimes E_j^{-1}) \neq 0$, so by Clifford's theorem, $j + 1 \leq h^0 \leq \frac{e_j}{2} + 1$.

The hypotheses of Lemma 6.4 are satisfied with these r_i and e_j, so there exist integers $0 = i_1, \ldots, i_k = N$ such that

$$\sum_{t=1}^{k-1} \left(r_{i_{t+1}} - r_{i_t}\right)\left(e_{i_{t+1}} + e_{i_t}\right) > 2 r_N e + 2\epsilon (r_N - r_0).$$

Suppose p and b are positive integers, and set $m = (p + 1)b$; assume that $m > m_3$. Recall that $H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}((p + 1)b))$ has a basis consisting of monomials of degree $(p + 1)b$ in $w_0, \ldots, w_N$. For all $1 \leq t \leq k$, let

$$V_{i_t} \subset H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(1))$$

be the subspace spanned by $\{w_0, \ldots, w_{i_t}\}$. Let $\hat{m}_1$ be another positive integer such that $\hat{m} := b\hat{m}_1 > m_3$, so that

$$\hat{\rho}^C_{(p+1)b, \hat{m}} : H^0(\mathbf{P}(W) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W)}((p + 1)b) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})) \to H^0\left(C, L_W^{(p+1)b} \otimes L_r^{\hat{m}}\right)$$

is surjective. For all triples (t_1, t_2, s) with $1 \leq t_1 < t_2 \leq k$ and $0 \leq s \leq p$, let

$$V_{i_{t_1}}^{p-s} V_{i_{t_2}}^{s} V_N \subset H^0(\mathbf{P}(W) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W)}(p + 1) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m}_1))$$

be the subspace spanned by monomials of bidegree $(p+1, \hat{m}_1)$, where the degree $p+1$ part has the following form:

$$x_{j_1} \cdots x_{j_{p-s}} y_{j_1} \cdots y_{j_s} z_j, \text{ where } x_{j_k} \in V_{i_{t_1}},\ y_{j_k} \in V_{i_{t_2}},\ z_j \in V_N = W.$$

We write $(V_{i_{t_1}}^{p-s} V_{i_{t_2}}^{s} V_N)^b$ for $\mathrm{Sym}^b(V_{i_{t_1}}^{p-s} V_{i_{t_2}}^{s} V_N)$, which consists of monomials of bidegree $((p+1)b, b\hat{m}_1)$ in $H^0(\mathbf{P}(W) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W)}((p+1)b) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m}))$. Now set

$$\big(\overline{V_{i_{t_1}}^{p-s} V_{i_{t_2}}^{s} V_N}\big)^b := \hat{\rho}^C_{(p+1)b,\hat{m}}\big(\big(V_{i_{t_1}}^{p-s} V_{i_{t_2}}^{s} V_N\big)^b\big).$$

We have obtained a filtration of $H^0(C, L_W^{(p+1)b} \otimes L_r^{\hat{m}})$,

$$\begin{array}{llll} 0 \subset \big(\overline{V_{i_1}^{p} V_{i_2}^{0} V_N}\big)^b & \subset \big(\overline{V_{i_1}^{p-1} V_{i_2}^{1} V_N}\big)^b & \subset \cdots \subset \big(\overline{V_{i_1}^{1} V_{i_2}^{p-1} V_N}\big)^b \\ \subset \big(\overline{V_{i_2}^{p} V_{i_3}^{0} V_N}\big)^b & \subset \big(\overline{V_{i_2}^{p-1} V_{i_3}^{1} V_N}\big)^b & \subset \cdots \subset \big(\overline{V_{i_2}^{1} V_{i_3}^{p-1} V_N}\big)^b \\ \vdots\ \vdots & \vdots\ \vdots & \vdots\ \ \vdots\ \ \vdots\ \vdots \\ \subset \big(\overline{V_{i_{k-1}}^{p} V_{i_k}^{0} V_N}\big)^b & \subset \big(\overline{V_{i_{k-1}}^{p-1} V_{i_k}^{1} V_N}\big)^b & \subset \cdots \subset \big(\overline{V_{i_{k-1}}^{1} V_{i_k}^{p-1} V_N}\big)^b \\ \subset \big(\overline{V_{i_{k-1}}^{0} V_{i_k}^{p} V_N}\big)^b & = H^0\big(C, L_W^{(p+1)b} \otimes L_r^{\hat{m}}\big). & \end{array} \tag{41}$$

Claim 6.6 ([9], p. 29)**.** There exists an integer b' which is independent of C, t_1, and t_2 such that if $b \geq b'$, then

$$\big(\overline{V_{i_{t_1}}^{p-s} V_{i_{t_2}}^{s} V_N}\big)^b = H^0\big(C, \big(E_{i_{t_1}}^{p-s} \otimes E_{i_{t_2}}^{s} \otimes L_W\big)^b \otimes L_r^{\hat{m}}\big).$$

□

Proof of Claim 6.6. By hypothesis, L_W is very ample. Note that $\deg L_r = d > 0$, since C is nonsingular, hence irreducible. Thus if $\hat{m}_1 > 2g+1$, then $L_r^{\hat{m}_1}$ is very ample.

By definition of the sheaves E_j, it follows that the linear system $(V_{i_{t_1}}^{p-s} V_{i_{t_2}}^{s} V_N)^b$ restricted to C generates

$$\big(E_{i_{t_1}}^{p-s} \otimes E_{i_{t_2}}^{s} \otimes L_W\big)^b \otimes L_r^{\hat{m}}$$

and

$$\hat{\rho}^C_{(p+1)b,\hat{m}}\big(\big(V_{i_{t_1}}^{p-s} V_{i_{t_2}}^{s} V_N\big)^b\big) \subseteq H^0\big(C, \big(E_{i_{t_1}}^{p-s} \otimes E_{i_{t_2}}^{s} \otimes L_W\big)^b \otimes L_r^{\hat{m}}\big). \tag{42}$$

Recall that $\hat{m} = b\hat{m}_1$ and suppose $\hat{m}_1 > 2g+1$, so that $L_r^{\hat{m}_1}$ is very ample, hence generated by global sections. $E_{i_{t_1}}$ and $E_{i_{t_2}}$ are generated by global sections, so it follows that $\hat{\rho}^C_{p+1,\hat{m}_1}(V_{i_{t_1}}^{p-s} V_{i_{t_2}}^s V_N)$ is a very ample base point free linear system on C.

Let $\psi = \psi_{p+1,\hat{m}_1}$ be the projective embedding corresponding to the linear system $\hat{\rho}^C_{p+1,\hat{m}_1}(V_{i_{t_1}}^{p-s} V_{i_{t_2}}^s V_N)$. Let $\mathcal{I}_{C/\mathbf{P}}$ be the ideal sheaf defining C as a closed subscheme of $\mathbf{P} := \mathbf{P}(\hat{\rho}^C_{p+1,\hat{m}_1}(V_{i_{t_1}}^{p-s} V_{i_{t_2}}^s V_N))$. There is an exact sequence of sheaves on $\mathbf{P}$ as follows:

$$0 \to \mathcal{I}_{C/\mathbf{P}} \to \mathcal{O}_{\mathbf{P}} \to \psi_* \mathcal{O}_C \to 0.$$

Tensoring by the very ample sheaf $\mathcal{O}_{\mathbf{P}}(b)$, we obtain

$$0 \to \mathcal{I}_{C/\mathbf{P}}(b) \to \mathcal{O}_{\mathbf{P}}(b) \to (\psi_* \mathcal{O}_C)(b) \to 0. \tag{43}$$

Write

$$\mathcal{F} := E_{i_{t_1}}^{p-s} \otimes E_{i_{t_2}}^s \otimes L_W \otimes L_r^{\hat{m}_1} \cong \psi^* \mathcal{O}_{\mathbf{P}}(1).$$

We have

$$\begin{aligned}(\psi_* \mathcal{O}_C)(b) &:= \psi_* \mathcal{O}_C \otimes_{\mathcal{O}_{\mathbf{P}}} \mathcal{O}_{\mathbf{P}}(b) \\ &\cong \psi_*(\mathcal{O}_C \otimes_{\mathcal{O}_C} \psi^* \mathcal{O}_{\mathbf{P}}(1)^b) \\ &\cong \psi_*(\mathcal{F}^b) \text{ since } \psi^* \mathcal{O}_{\mathbf{P}}(1) \cong \mathcal{F}.\end{aligned}$$

Now the exact sequence (43) reads

$$0 \to \mathcal{I}_{C/\mathbf{P}}(b) \to \mathcal{O}_{\mathbf{P}}(b) \to \psi_*(\mathcal{F}^b) \to 0.$$

In the corresponding long exact sequence in cohomology, we have

$$\cdots \to H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(b)) \to H^0(\mathbf{P}, \psi_* \mathcal{F}^b) \to H^1(\mathbf{P}, \mathcal{I}_{C/\mathbf{P}}(b)) \to \cdots. \tag{44}$$

The so-called "Uniform m Lemma" (cf. [11], Lemma 1.11) ensures that there is an integer $b' > 0$ depending on the Hilbert polynomial P, but not on the curve C such that

$H^1(\mathbf{P}, \mathcal{I}_{C/\mathbf{P}}(b)) = 0$ if $b > b'$. Then for such b, the exact sequence (44) implies that the map

$$H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(b)) \to H^0(C, \mathcal{F}^b)$$

is surjective. Recall that $\mathbf{P} := \mathbf{P}(\hat{\rho}^C_{p+1,\hat{m}_1}(V^{p-s}_{i_{t_1}} V^s_{i_{t_2}} V_N))$. Then

$$H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(b)) \cong \mathrm{Sym}^b\big(\hat{\rho}^C_{p+1,\hat{m}_1}\big(V^{p-s}_{i_{t_1}} V^s_{i_{t_2}} V_N\big)\big).$$

Also there is a surjection

$$\mathrm{Sym}^b\big(V^{p-s}_{i_{t_1}} V^s_{i_{t_2}} V_N\big) \to \mathrm{Sym}^b\big(\hat{\rho}^C_{p+1,\hat{m}_1}\big(V^{p-s}_{i_{t_1}} V^s_{i_{t_2}} V_N\big)\big),$$

so putting this all together we have a surjection

$$\mathrm{Sym}^b\big(V^{p-s}_{i_{t_1}} V^s_{i_{t_2}} V_N\big) \to H^0(C, \mathcal{F}^b),$$

that is,

$$\hat{\rho}^C_{(p+1)b,\hat{m}} : \big(V^{p-s}_{i_{t_1}} V^s_{i_{t_2}} V_N\big)^b \to H^0\big(C, \big(E^{p-s}_{i_{t_1}} \otimes E^s_{i_{t_2}} \otimes L_W\big)^b \otimes L^{\hat{m}}_r\big) \tag{45}$$

is surjective. It follows from lines (42) and (45) that

$$\big(\overline{V^{p-s}_{t_1} V^s_{t_2} V_N}\big)^b = H^0\big(C, \big(E^{p-s}_{i_{t_1}} \otimes E^s_{i_{t_2}} \otimes L_W\big)^b \otimes L^{\hat{m}}_r\big),$$

completing the proof of Claim 6.6. ■

Proof of Theorem 6.5 continued. Take $b \geq 2g+1$ so that we have the vanishing $H^1(C, (E^{p-s}_{i_{t_1}} \otimes E^s_{i_{t_2}} \otimes L_W)^b \otimes L^{\hat{m}}_r) = 0$. We use Riemann–Roch to calculate

$$\begin{aligned}\dim\big(\overline{V^{p-s}_{i_t} V^s_{i_{t+1}} V}\big)^b &= h^0\big(C, \big(E^{p-s}_{i_t} \otimes E^s_{i_{t+1}} \otimes L_W\big)^b \otimes L^{\hat{m}}_r\big)\\ &= b((p-s)e_{i_t} + se_{i_{t+1}} + e) + d\hat{m} - g + 1.\end{aligned}$$

We assume for the rest of the proof that p, b, and $\hat{m}_1$ are sufficiently large that

$$\begin{aligned} p &> \max\left\{e+g, \frac{\frac{3}{2}e+1}{\epsilon}\right\}, \\ b &> \max\{p, (2g+1)b'\}, \\ m &:= (p+1)b > \max\{m_3, K\}, \\ \hat{m}_1 &> 2g+1, \\ \hat{m} &:= b\hat{m}_1 > m_3. \end{aligned}$$

Choose a basis $\hat{B}_{(p+1)b,\hat{m}}$ of $H^0(\mathbf{P}(W)\times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W)}((p+1)b)\otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m}))$ of monomials $\hat{M}_i$ of bidegree $((p+1)b, \hat{m})$. Pick monomials $\hat{M}_1, \ldots, \hat{M}_{P((p+1)b,\hat{m})}$ in $\hat{B}_{(p+1)b,\hat{m}}$ such that $\hat{\rho}^C_{(p+1)b,\hat{m}}(\hat{M}_1), \ldots, \hat{\rho}^C_{(p+1)b,\hat{m}}(\hat{M}_{P((p+1)b,\hat{m})})$ is a basis of $H^0(C, L_W^{(p+1)b}\otimes L_r^{\hat{m}})$, which respects the filtration (41). Observe that if a monomial $\hat{M} \in (\bar{V}_{i_1}^{p-s}\bar{V}_{i_2}^{s}\bar{V})^b - (\bar{V}_{i_1}^{p-s+1}\bar{V}_{i_2}^{s-1}\bar{V})^b$, then $\hat{M}$ has λ-weight $w_\lambda(\hat{M}) \leq n((p-s)r_{i_t} + sr_{i_{t+1}} + r_N)$. Moreover, as our basis respects the filtration (41), we may count how many such $\hat{M}$ there are.

We now estimate the total λ-weight of $\hat{M}_1, \ldots, \hat{M}_{P((p+1)b,\hat{m})}$, which gives an upper bound for $\mu^{L_{m,\hat{m}}}(C, \lambda)$ as follows:

$$\begin{aligned} \mu^{L_{m,\hat{m}}}(C,\lambda) \leq & \sum_{i=1}^{P(m)+d\hat{m}} w_\lambda(\hat{M}_i) \leq b\big(pr_{i_1} + r_N\big)\dim\left(\overline{V_{i_1}^{p}V_{i_2}^{0}V}\right)^b \\ & + \sum_{\substack{0\leq s\leq p \\ 1\leq t\leq k-1}} b\big((p-s)r_{i_t} + sr_{i_{t+1}} + r_N\big)\left(\dim\left(\overline{V_{i_t}^{p-s}V_{i_{t+1}}^{s}V}\right)^b - \dim\left(\overline{V_{i_t}^{p-s+1}V_{i_t}^{s-1}V}\right)^b\right). \end{aligned} \tag{46}$$

The first term on the right-hand side of equation (46) is

$$b\big(pr_{i_1} + r_N\big)\big(b\big(pe_{i_1} + e_{i_k}\big) + d\hat{m} - g + 1\big) = b(pr_0 + r_N)d\hat{m} + b(pr_0 + r_N)(b(pe_0 + e) - g + 1).$$

The factor $\dim(\overline{V_{i_t}^{p-s}V_{i_{t+1}}^{s}V})^b - \dim(\overline{V_{i_t}^{p-s+1}V_{i_t}^{s-1}V})^b$ of the summand is

$$\begin{aligned} &\big(b\big((p-s)e_{i_t} + se_{i_{t+1}} + e\big) + d\hat{m} - g + 1\big) \\ &\quad - \big(b\big((p-s+1)e_{i_t} + (s-1)e_{i_{t+1}} + e\big) + d\hat{m} - g + 1\big) = b\big(e_{i_{t+1}} - e_{i_t}\big). \end{aligned}$$

Note that nearly all of the terms having $d\hat{m}$ as a factor have "telescoped." We have

$$\mu^{L_{m,\hat{m}}}(C,\lambda) \le db(pr_0 + r_N)\hat{m} + b(pr_0 + r_N)(b(pe_0 + e) - g + 1) + \sum_{\substack{0 \le s \le p \\ 1 \le t \le k-1}} b\big((p-s)r_{i_t} + sr_{i_{t+1}} + r_N\big)\big(b\big(e_{i_{t+1}} - e_{i_t}\big)\big).$$

The sum of the second two terms is *exactly* the expression Gieseker obtains at the bottom of page 30 in [9]. Following Gieseker's calculations up to page 34, we see that

$$\mu^{L_{m,\hat{m}}}(C,\lambda) < db\big(pr_{i_1} + r_N\big)\hat{m} + b^2 p(r_N - r_0)\left(-\epsilon p + \frac{3e}{2} + \frac{e+g-1}{p}\right) < db\big(pr_{i_1} + r_N\big)\hat{m},$$

where the last inequality follows because $p > \max\{e+g, \frac{\frac{3}{2}e+1}{\epsilon}\}$. Next, we may estimate $r_N = \sum_{i=0}^{N-1} -r_i \le -Nr_0$, and we know that $r_0 < 0$, so we have shown that

$$\mu^{L_{m,\hat{m}}}(C,\lambda) \le dbr_0(p-N)d\hat{m} < 0,$$

as, by hypothesis, $p > e - g = N$.

Nowhere in the proof have we placed any conditions on the 1-PS λ, so the result is true for every 1-PS of $SL(W)$. Thus C is $SL(W)$-stable with respect to $L_{m,\hat{m}}$. ■

Note in particular that the hypotheses are satisfied by all smooth maps represented in J, by definition of $\mathcal{L}$ and a (cf. Section 3.1). We may state the base cases of our induction.

Proposition 6.7. Fix $n = 0$. Let M consist of those $(m, \hat{m}, m')$ such that $m' \ge 1$, and $m, \hat{m} > m_3$ with

$$m > \max\left\{\begin{array}{c} \big(g - \frac{1}{2} + e(q_1+1) + q_3 + \mu_1 m_2\big)(e - g + 1), \\ (10g + 3q_2 - 3\bar{g})(e - g + 1) \\ (6g + 2q_2 - 2\bar{g} - 1)(2a - 1) \end{array}\right\},$$

while

$$\frac{\hat{m}}{m} = \frac{ca}{2a-1} + \delta,$$

where

$$|d\delta| \leq \frac{1}{4a-2} - \frac{3g + q_2 - \bar{g} - \frac{1}{2}}{m},$$

and in addition a is sufficiently large that

$$d\frac{\hat{m}}{m} < \frac{1}{8}e - \frac{9}{8}g + \frac{7}{8}.$$

Let $l \in \mathbf{H}_M(\overline{J})$. Then, as schemes over $\mathbb{C}$,

$$\overline{J}^{ss}(l) = \overline{J}^{s}(l) = J.$$

□

Proof. By Theorem 6.5, there exists $(m, \hat{m}, m') \in M$ such that $I^{ss}(L_{m,\hat{m},m'})$ is nonempty. In particular, any smooth curve in J satisfies the hypotheses of Theorem 6.5, and so $\overline{J}^{ss}(L_{m,\hat{m},m'})$ is nonempty. By Theorem 5.21, we know that $\overline{J}^{ss}(L_{m,\hat{m},m'}) \subseteq J$. Then by Theorem 3.6(ii), we see that $\overline{J}^{ss}(L_{m,\hat{m},m'}) = \overline{J}^{s}(L_{m,\hat{m},m'}) = J$. The result follows for all $l \in \mathbf{H}_M(\overline{J})$ by Corollary 5.22. ■

The base case for stable curves is proved over Spec $\mathbb{Z}$.

Proposition 6.8. Fix $n = 0$, and $r = d = 0$. Let M consist of those $(m, m') \in \mathbb{N}^2$ such that $m' \geq 1$, and $m > m_3$ with

$$m > \max\left\{\begin{matrix}(g - \frac{1}{2} + e(q_1 + 1) + q_3 + \mu_1 m_2)(e - g + 1), \\ (10g + 3q_2 - 3\bar{g})(e - g + 1)\end{matrix}\right\}.$$

Let $l \in \mathbf{H}_M(\overline{J})$. Then, as schemes over Spec $\mathbb{Z}$,

$$\overline{J}^{ss}(l) = \overline{J}^{s}(l) = J.$$

□

Proof. By Theorem 6.5, there exists $(m, m') \in M$ such that $I^{ss}(L_{m,\hat{m}})$ is nonempty. In particular, any smooth curve in J satisfies the hypotheses of Theorem 6.5, and so $\overline{J}^{ss}(L_{m,m'})$ is nonempty. However, by ([9], Theorem 2.0.2), the GIT quotient $\overline{J}//_{L_{m,m'}} SL(W)$ is isomorphic to the moduli space of stable curves, $\overline{\mathcal{M}}_g$. In particular, Gieseker proves here that every stable curve is represented in $\overline{J}//_{L_{m,m'}} SL(W)$. Then by Corollary 5.22(iii), we see that $\overline{J}^{ss}(l) = J$ for every $l \in \mathbf{H}_M(\overline{J})$. ■

6.3 The general case

We suppose that $n > 0$ and fix M as in the statement of Theorem 6.1, then use induction to show that $\overline{J}^{ss}_{g,n,d}(l) = \overline{J}^{s}_{g,n,d}(l) = J$ for all $l \in \mathbf{H}_M(\overline{J})$. By Corollary 3.5, it follows that $\overline{J}//_{L_{m,\hat{m},m'}} SL(W) \cong \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$ for any $(m, \hat{m}, m') \in M$. As our inductive work uses various spaces of maps with differing genera and numbers of marked points, we shall be more precise about using the subscripts g, n, d in this section. Note that, indeed m_1, m_2, m_3, q_1, q_2, q_3, μ_1, and μ_2 all depend on the genus. We may specify in addition that these are increasing as functions of the genus.

The inductive hypothesis is given in the following proposition. We fix g, n, and d such that $n \geq 1$ and smooth stable n-pointed maps of genus g and degree d do exist. Fix an integer $c \geq 2$ (the case $(g, n, d) = (0, 0, 1)$ does not arise here, so this will suffice according to the remark in Section 2.4). We suppose that the theorem has been proved to have stable maps of degree d, from curves of genus $g + 1$ with $n - 1$ marked points. As the base case, where the map has no marked points, works for any genus, this is a valid inductive hypothesis to make. Since our assumptions imply $2g - 2 + n + cd > 0$, it follows that $2(g + 1) - 2 + (n - 1) + cd > 0$, so smooth stable maps of this type do exist.

Proposition 6.9. Let g, n, and d be such that $2g - 2 + n + cd > 0$ and assume in addition that $n \geq 1$. Let $M_{g,n,d}$ consist of those $(m, \hat{m}, m')$ such that $m, \hat{m} > m_3$ and

$$m > \max \left\{ \begin{array}{c} \left(g - \frac{1}{2} + e_{g,n,d}(q_1 + 1) + q_3 + \mu_1 m_2\right)(e_{g,n,d} - g + 1), \\ (10g + 3q_2 - 3\bar{g})(e_{g,n,d} - g + 1) \\ (6g + 2q_2 - 2\bar{g} - 1)(2a - 1) \end{array} \right\}, \tag{47}$$

where m_1, m_2, m_3, q_1, q_2, q_3, μ_1, and μ_2, are those defined in Section 4.4, taken to be functions of the genus g; in addition,

$$\frac{\hat{m}}{m} = \frac{ca}{2a - 1} + \delta \qquad \frac{m'}{m^2} = \frac{a}{2a - 1} + \eta,$$

where

$$|n\eta| + |d\delta| \leq \frac{1}{4a - 2} - \frac{3g + q_2 - \bar{g} - \frac{1}{2}}{m},$$

and ensure in addition that a is sufficiently large that

$$d\frac{\hat{m}}{m} + n\frac{m'}{m^2} < \frac{1}{8}e - \frac{9}{8}g + \frac{7}{8}.$$

Work over Spec k. Assume that

$$\overline{J}^{ss}_{g+1,n-1,d}(l') = \overline{J}^{s}_{g+1,n-1,d}(l') = J_{g+1,n-1,d}$$

for all $l' \in \mathbf{H}_{M_{g+1,n-1,d}}(\overline{J}_{g+1,n-1,d})$. Then for all $l \in \mathbf{H}_{M_{g,n,d}}(\overline{J}_{g,n,d})$,

$$\overline{J}^{ss}_{g,n,d}(l) = \overline{J}^{s}_{g,n,d}(l) = J_{g,n,d}$$

as schemes over an arbitrary field k. □

Proof. Note that if $n \geq 1$, then

$$e_{g+1,n-1,d} - (g+1) + 1 = (2a-1)g + a(n-1+cd) > (2a-1)(g-1) + a(n+cd) = e - g + 1.$$

We conclude, by the definition and specifications given above, that $M_{g+1,n-1,d} \subseteq M_{g,n,d}$. Fix specific integers $(m, \hat{m}, m') \in M_{g+1,n-1,d}$, satisfying

$$\frac{\hat{m}}{m} = \frac{ca}{2a-1} \qquad \frac{m'}{m^2} = \frac{a}{2a-1},$$

and also such that $\frac{m}{n}(1 - S_{14})$ is an integer, where $S_{14} := \frac{g(a-1)}{(2a-1)g + a(n-1+cd)}$. Our inductive hypothesis implies in particular that

$$\overline{J}^{ss}_{g+1,n-1,d}(L_{m,\hat{m},m'}) = J_{g+1,n-1,d}.$$

We shall now find m'' such that $(m, \hat{m}, m'') \in M_{g,n,d}$, with $\overline{J}^{ss}_{g,n,d}(L_{m,\hat{m},m''}) = J_{g,n,d}$.

Fix some $(h, x_1, \ldots, x_n) \in J_{g,n,d}$. Write $C_0 := \mathcal{C}_h$, so that $(h, x_1, \ldots, x_n)$ models a stable map $p_r : (C_0, x_1, \ldots, x_n) \to \mathbf{P}^r$ in $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$. Also fix an elliptic curve $(C_1, y) \subset \mathbf{P}(W_{1,1,0})$ represented in $J_{1,1,0}$.

Let $ev : H^0(\mathbf{P}(W_{g,n,d}), \mathcal{O}_{\mathbf{P}(W_{g,n,d})}(1)) \to k$ be the evaluation map at the closed point $p_{W_{g,n,d}}(x_n) \in \mathbf{P}(W_{g,n,d})$, and let $V_{g,n,d}$ be its kernel so that $V_{g,n,d}$ is the codimension 1 subspace of $W_{g,n,d}$ consisting of sections vanishing at $p_{W_{g,n,d}}(x_n)$. Similarly, let $V_{1,1,0}$ be the codimension 1 subspace of $W_{1,1,0}$ corresponding to sections vanishing at y.

Now note that

$$\dim V_{g,n,d} + \dim V_{1,1,0} + 1 = a(2g-2+n+cd) - g + a \cdot 1 - 1 + 1 = \dim W_{g+1,n-1,d}.$$

Hence, if we let U be a dimension 1 vector space over k, we may pick an isomorphism

$$W_{g+1,n-1,d} \cong V_{g,n,d} \oplus U \oplus V_{1,1,0}.$$

We further choose isomorphisms $W_{g,n,d} \cong V_{g,n,d} \oplus U$ and $W_{1,1,0} \cong U \oplus V_{1,1,0}$ which fix $V_{g,n,d}$ and $V_{1,1,0}$, respectively. Thus we regard $W_{g,n,d}$ and $W_{1,1,0}$ as subspaces of $W_{g+1,n-1,d}$. The most important of these identifications we shall write as

$$W_{g+1,n-1,d} = W_{g,n,d} \oplus V_{1,1,0}.$$

We project $W_{g+1,n-1,d} \to W_{g,n,d}$ along $V_{1,1,0}$ and induce an embedding

$$\mathbf{P}(W_{g,n,d}) \hookrightarrow \mathbf{P}(W_{g+1,n-1,d});$$

similarly $\mathbf{P}(W_{1,1,0}) \hookrightarrow \mathbf{P}(W_{g+1,n-1,d})$. Then we induce closed immersions $p_{W_{g,n,d}}(C_0) \hookrightarrow \mathbf{P}(W_{g+1,n-1,d})$ and $C_1 \hookrightarrow \mathbf{P}(W_{g+1,n-1,d})$; we shall consider the curves as embedded in this space.

If $s \in V_{1,1,0} \subset W_{g+1,n-1,d}$ is regarded as a section of $\mathcal{O}_{\mathbf{P}(W_{g+1,n-1,d})}(1)$, then $s(x) = 0$ for any x in $\mathbf{P}(W_{g,n,d})$, and in particular $s(x) = 0$ for any x in $p_W(C_0)$. In other words, $\rho^{p_W(C_0)}(V_{1,1,0}) = \{0\}$, where we write $\rho^{p_W(C_0)}$ for restriction of sections to $p_W(C_0)$. Similarly, $\rho^{C_1}(V_{g,n,d}) = \{0\}$.

The images of $\mathbf{P}(W_{g,n,d})$ and $\mathbf{P}(W_{1,1,0})$ meet only at one point, $\mathbf{P}(U) \in \mathbf{P}(W_{g+1,n-1,d})$. We shall denote this point by P. If the curves C_0 and C_1 meet, it could only be at this point. Consider $p_W(x_n)$; by the definitions, we know $s(p_W(x_n)) = 0$ for all $s \in V_{1,1,0}$ and for all $s \in V_{g,n,d}$. We conclude that $p_W(x_n) \in \mathbf{P}(U)$. Similarly, $y \in \mathbf{P}(U)$. Thus, after the curves have been embedded in $\mathbf{P}(W_{g+1,n-1,d})$, the points $p_W(x_n)$ and y coincide at P. We define

$$(C, x_1, \ldots, x_{n-1}) := (C_0 \cup C_1, x_1, \ldots, x_{n-1}).$$

As the curves C_0 and C_1 are smooth at x_n and y, respectively, and they lie in two linear subspaces meeting transversally at P, the singular point of C at P is a node.

The map $p_r : (C_0, x_1, \ldots, x_n) \to \mathbf{P}^r$ may be extended over C if we define it to contract C_1 to the point $p_r(x_n)$. Thus we have the graph of a prestable map,

$$(C, x_1, \ldots, x_{n-1}) \subset \mathbf{P}(W_{g+1,n-1,d}) \times \mathbf{P}^r.$$

We wish to show that $(C, x_1, \ldots, x_{n-1})$ is represented by a point in $J_{g+1,n-1,d}$, so we check conditions (i)–(iii) of Definition 3.2. Clearly (i) is satisfied: C is projective, connected, reduced, and nodal, and the $n-1$ marked points are distinct and nonsingular. By construction, $C \to \mathbf{P}(W_{g+1,n-1,d})$ is a nondegenerate embedding, so (ii) is satisfied. We must check (iii), the isomorphism of line bundles.

In general, if C is a nodal curve and $C' \subset C$ is a complete subcurve, meeting the rest of C in only one node at Q, then

$$\omega_C|_{C'} = \omega_{C'}(Q).$$

Thus in our situation, as $(C_0, x_1, \ldots, x_n)$ is represented in $J_{g,n,d}$,

$$\begin{aligned} \left(\mathcal{O}_{\mathbf{P}(W_{g+1,n-1,d})}(1) \otimes \mathcal{O}_{\mathbf{P}^r}(1)\right)\Big|_{C_0} &= \left(\mathcal{O}_{\mathbf{P}(W_{g,n,d})}(1) \otimes \mathcal{O}_{\mathbf{P}^r}(1)\right)\Big|_{C_0} \\ &\cong \omega_{C_0}^{\otimes a}(ax_1 + \cdots + ax_n) \otimes \mathcal{O}_{\mathbf{P}^r}(ca+1)|_{C_0} \\ &= \left(\omega_C^{\otimes a}(ax_1 + \cdots + ax_{n-1}) \otimes \mathcal{O}_{\mathbf{P}^r}(ca+1)|_C\right)\Big|_{C_0}. \end{aligned} \tag{48}$$

To analyze C_1, we also observe that $\mathcal{O}_{\mathbf{P}^r}(1)|_{C_1}$ is trivial, as f contracts C_1 to a point.

$$\begin{aligned} \left(\mathcal{O}_{\mathbf{P}(W_{g+1,n-1,d})}(1) \otimes \mathcal{O}_{\mathbf{P}^r}(1)\right)\Big|_{C_1} &= \mathcal{O}_{\mathbf{P}(W_{1,1,0})}(1)|_{C_1} \\ &\cong \omega_{C_1}^{\otimes a}(ay) \\ &= \left(\omega_C^{\otimes a}(ax_1 + \cdots + ax_{n-1}) \otimes \mathcal{O}_{\mathbf{P}^r}(ca+1)|_C\right)\Big|_{C_1}. \end{aligned} \tag{49}$$

The curve C was defined as $C_0 \cup C_1$. We have an induced isomorphism of line bundles

$$\left(\mathcal{O}_{\mathbf{P}(W_{g+1,n-1,d})}(1) \otimes \mathcal{O}_{\mathbf{P}^r}(1)\right)\Big|_{C\setminus\{P\}} \cong \left(\omega_C^{\otimes a}(ax_1 + \cdots + ax_{n-1}) \otimes \mathcal{O}_{\mathbf{P}^r}(ca+1)|_C\right)\Big|_{C\setminus\{P\}}$$

found by excluding P from the two isomorphisms above. To extend this over P, we simply need to insist that the isomorphisms over C_0 and C_1 are consistent at P. When we restrict to the fiber over P, the two isomorphisms (48) and (49) are scalar multiples of one another, so we obtain consistency at P by multiplying the isomorphism (49) by a suitable nonzero scalar, once the isomorphism (48) is given.

Thus, $(C, x_1, \ldots, x_{n-1})$ is indeed represented in $J_{g+1,n-1,d}$, as required. Now we use our inductive hypothesis: we know that $\overline{J}^{ss}_{g+1,n-1,d}(L_{m,\hat{m},m'}) = J_{g+1,n-1,d}$, so in particular $(C, x_1, \ldots, x_{n-1})$ is GIT semistable with respect to $L_{m,\hat{m},m'}$.

For the following analysis, we must clarify the notation for our standard line bundles. For $i = 1, 2$, let

$$C_i \xrightarrow{\iota_{C_i,C}} C \xrightarrow{\iota_C} \mathbf{P}(W_{g+1,n-1,d}) \times \mathbf{P}^r$$

be the inclusion morphisms. Note that the composition

$$C_0 \xrightarrow{\iota_{C_0,\mathbf{P}(W_{g,n,d})}} \mathbf{P}(W_{g,n,d}) \times \mathbf{P}^r \xrightarrow{\iota_{\mathbf{P}(W_{g,n,d})}} \mathbf{P}(W_{g+1,n-1,d}) \times \mathbf{P}^r$$

is equal to $\iota_{C_0,C} \circ \iota_C$. Thus

$$\iota^*_{C_0,\mathbf{P}(W_{g,n,d})} p^*_{W_{g,n,d}} \mathcal{O}_{\mathbf{P}(W_{g,n,d})}(1) = \iota^*_{C_0,C} \iota^*_C p^*_{W_{g+1,n-1,d}} \mathcal{O}_{\mathbf{P}(W_{g+1,n-1,d})}(1).$$

Therefore we may denote this line bundle by L_{WC_0}, as we need not specify which "W" space we have used. L_{WC_1} is defined similarly. We let

$$L_{rC_0} := \iota^*_{C_0,C} \iota^*_C p^*_r \mathcal{O}_{\mathbf{P}^r}(1) \quad L_{rC_1} := \iota^*_{C_1,C} \iota^*_C p^*_r \mathcal{O}_{\mathbf{P}^r}(1),$$

though L_{rC_1} is in fact trivial, since C_1 is collapsed by projection to $\mathbf{P}(r)$.

For $i = 0, 1$, define restriction maps to our subcurves

$$\hat{\rho}^{C_i}_{m,\hat{m}} : H^0(\mathbf{P}(W_{g+1,n-1,d}) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W_{g+1,n-1,d})}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})) \to H^0\big(C_i, L^m_{WC_i} \otimes L^{\hat{m}}_{rC_i}\big).$$

We show that these are surjective for m and $\hat{m}$ sufficiently large. The conditions on m and $\hat{m}$ imply $m, \hat{m} > m_3$, so the restriction map

$$\hat{\rho}^{C_0,\mathbf{P}(W_{g,n,d})}_{m,\hat{m}} : H^0\big(\mathbf{P}(W_{g,n,d}) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W_{g,n,d})}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})\big) \to H^0\big(C_0, L^m_{WC_0} \otimes L^{\hat{m}}_{rC_0}\big)$$

is surjective by Grothendieck's uniform m lemma (cf. Proposition 4.6(i)). The restriction to a linear subspace

$$\hat{\rho}^{\mathbf{P}(W_{g,n,d})}_{m,\hat{m}} : H^0\big(\mathbf{P}(W_{g+1,n-1,d}) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W_{g+1,n-1,d})}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})\big) \\ \to H^0\big(\mathbf{P}(W_{g,n,d}) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W_{g,n,d})}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})\big)$$

is surjective, hence by composition $\hat{\rho}^{C_0}_{m,\hat{m}} = \hat{\rho}^{C_0,\mathbf{P}(W_{g,n,d})}_{m,\hat{m}} \circ \hat{\rho}^{\mathbf{P}(W_{g,n,d})}_{m,\hat{m}}$ is surjective. Surjectivity of $\hat{\rho}^{C_1}_{m,\hat{m}}$ is shown similarly. Moreover, for $i = 0, 1$, the restriction $\hat{\rho}^{C_i}_{m,\hat{m}}$ factors through maps which restrict sections of C to those on one of the subcurves

$$\hat{\rho}^{C_i,C}_{m,\hat{m}} : H^0\big(C, L^m_{WC} \otimes L^{\hat{m}}_{rC}\big) \to H^0\big(C_i, L^m_{WC_i} \otimes L^{\hat{m}}_{rC_i}\big),$$

and thus the maps $\hat{\rho}^{C_0,C}_{m,\hat{m}}$ and $\hat{\rho}^{C_1,C}_{m,\hat{m}}$ are also surjective.

To relate GIT semistability of $(C, x_1, \dots, x_{n-1})$ to that of $(C_0, x_1, \dots, x_n)$, we shall display the vector space $H^0(C, L^m_{WC} \otimes L^{\hat{m}}_{rC})$ as the direct sum of two subspaces. Recall that we wrote

$$W_{g+1,n-1,d} = W_{g,n,d} \oplus V_{1,1,0}.$$

Fix a basis $w_0, \dots w_{N_{g+1,n-1,d}}$ for $W_{g+1,n-1,d}$ respecting this decomposition. Let $\hat{B}_{m,\hat{m}}$ be a basis of $H^0(\mathbf{P}(W_{g+1,n-1,d}) \otimes \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W_{g+1,n-1,d})}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m}))$ of monomials of bidegree $(m, \hat{m})$, where the degree m part is a monomial in $w_0, \dots w_{N_{g+1,n-1,d}}$.

Let $\Omega^{m,\hat{m}}_+$ be the subspace of $H^0(\mathbf{P}(W_{g+1,n-1,d}) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W_{g+1,n-1,d})}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m}))$ spanned by all monomials in $\hat{B}_{m,\hat{m}}$ which have at least one factor from $V_{1,0,0}$. Namely, $\Omega^{m,\hat{m}}_+$ is spanned by monomials of bidegree $(m, \hat{m})$, where the degree m part has form $w_{i_1} w_{i_2} \cdots w_{i_m}$, with $w_{i_1} \in V_{1,1,0}$. The remaining factors may come from either $W_{g,n,d}$ or $V_{1,0,0}$.

Similarly, let $\Omega^{m,\hat{m}}_0 \subset H^0(\mathbf{P}(W_{g+1,n-1,d}) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W_{g+1,n-1,d})}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m}))$ be the subspace spanned by monomials in $\hat{B}_{m,\hat{m}}$ which have no factors from $V_{1,0,0}$. In other words, all of the factors in the degree m part come from $W_{g,n,d}$.

By inspecting the basis $\hat{B}_{m,\hat{m}}$, we see that as vector spaces,

$$H^0\big(\mathbf{P}(W_{g+1,n-1,d}) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W_{g+1,n-1,d})}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})\big) = \Omega^{m,\hat{m}}_+ \oplus \Omega^{m,\hat{m}}_0. \tag{50}$$

We wish to show that this decomposition restricts to the curve C. Set

$$\overline{\Omega}^{m,\hat{m}}_+ := \hat{\rho}^C_{m,\hat{m}}\big(\Omega^{m,\hat{m}}_+\big) \subset H^0\big(C, L^m_{WC} \otimes L^{\hat{m}}_{rC}\big) \\ \overline{\Omega}^{m,\hat{m}}_0 := \hat{\rho}^C_{m,\hat{m}}\big(\Omega^{m,\hat{m}}_0\big) \subset H^0\big(C, L^m_{WC} \otimes L^{\hat{m}}_{rC}\big).$$

We first make some technical observations on where such sections vanish.

Claim 6.10. Let $\overline{\Omega}_+^{m,\hat{m}}$ and $\overline{\Omega}_0^{m,\hat{m}}$ be defined as above.

(i) If $0 \neq s \in \overline{\Omega}_+^{m,\hat{m}}$, then $\hat{\rho}_{m,\hat{m}}^{C_0,C}(s) = 0$ and $\hat{\rho}_{m,\hat{m}}^{C_1,C}(s) \neq 0$.

(ii) The image $\hat{\rho}_{m,\hat{m}}^{C_1,C}(\overline{\Omega}_0^{m,\hat{m}})$ is 1-dimensional, and if $s \in \overline{\Omega}_0^{m,\hat{m}}$ with $s(P) = 0$, then $\hat{\rho}_{m,\hat{m}}^{C_1,C}(s) = 0$. □

Proof of Claim 6.10. (i) Recall that we observed that, for any $w \in V_{1,1,0}$, the restriction $\rho^{C_0}(w) = 0$. The space $\Omega_+^{m,\hat{m}}$ is spanned by monomials containing a factor from $V_{0,1,1}$, so it follows that if $s \in \overline{\Omega}_+^{m,\hat{m}}$, then $\hat{\rho}_{m,\hat{m}}^{C_0,C}(s) = 0$. However, if $s \in \overline{\Omega}_+^{m,\hat{m}} \subset H^0(C, L_{WC}^m \otimes L_{rC}^{\hat{m}})$ and $s \neq 0$, then s must be nonzero on one of C_0 and C_1, whence $\hat{\rho}_{m,\hat{m}}^{C_1,C}(s) \neq 0$.

(ii) Recall that we wrote $W_{g,n,d} \cong V_{g,n,d} \oplus U$, where $V_{g,n,d}$ consists of sections vanishing at $p_W(P)$, and U is spanned by a section nonvanishing at $p_W(P)$. We saw that $\rho^{C_1}(V_{g,n,d}) = \{0\}$, and so $\rho^{C_1}(W_{g,n,d}) = \rho^{C_1}(U)$, which is 1-dimensional. Let u span this space; $u(p_W(P)) \neq 0$.

The component C_1 collapses to the single point $p_r(P)$ under p_r. Thus we have another decomposition: $H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(1)) = V_{\mathbf{P}^r} \oplus \hat{U}$, where sections in $V_{\mathbf{P}^r}$ vanish at $p_r(P) = p_r(C_1)$, and $\hat{U}$ is 1-dimensional, spanned by a section nonvanishing at $p_r(P)$. Again $\rho^{C_1}(H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(1))) = \rho^{C_1}(\hat{U})$, which is 1-dimensional. Let $\hat{u}$ span this space; $\hat{u}(p_r(P)) \neq 0$.

Thus, since $\Omega_+^{m,\hat{m}} = \mathrm{Sym}^m(W_{g,n,d} \otimes H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(\hat{m})))$, we have

$$\hat{\rho}_{m,\hat{m}}^{C_1,C}\left(\overline{\Omega}_+^{m,\hat{m}}\right) = \hat{\rho}_{m,\hat{m}}^{C_1}\left(\Omega_+^{m,\hat{m}}\right) = \hat{\rho}_{m,\hat{m}}^{C_1}(\mathrm{Sym}^m U \otimes \mathrm{Sym}^{\hat{m}} \hat{U}),$$

which is 1-dimensional, since U and $\hat{U}$ are. Finally, if $s \in \overline{\Omega}_0^{m,\hat{m}}$ then $\hat{\rho}_{m,\hat{m}}^{C_1,C}(s) = \alpha u^m \otimes \hat{u}^{\hat{m}}$ for some $\alpha \in k$. We know that $u^m \otimes \hat{u}^{\hat{m}}(P) \neq 0$. Since $s(P) = \hat{\rho}_{m,\hat{m}}^{C_1,C}(s)(P)$, it follows that if $s(P) = 0$ then $\alpha = 0$, and so $\hat{\rho}_{m,\hat{m}}^{C_1,C}(s) = 0$, and the proof of Claim 6.10 is complete. ■

We now give details of our decomposition of $H^0(C, L_{WC}^m \otimes L_{rC}^{\hat{m}})$.

Claim 6.11. Restriction to C respects the decomposition (50), and enables us to identify the spaces $\overline{\Omega}_+^{m,\hat{m}}$ and $\overline{\Omega}_0^{m,\hat{m}}$. Precisely,

(i) $H^0\left(C, L_{WC}^m \otimes L_{rC}^{\hat{m}}\right) = \overline{\Omega}_0^{m,\hat{m}} \oplus \overline{\Omega}_+^{m,\hat{m}}$;

(ii) $\overline{\Omega}_0^{m,\hat{m}} \cong H^0\left(C_0, L_{WC_0}^m \otimes L_{rC_0}^{\hat{m}}\right)$;

(iii) $\overline{\Omega}_+^{m,\hat{m}} \cong H^0\left(C_1, L_{WC_1}^m(-P)\right)$. □

Proof of Claim 6.11. (i) By restricting equation (50) to C, we see that

$$H^0\big(C, L_{WC}^m \otimes L_{rC}^{\hat{m}}\big) = \overline{\Omega}_0^{m,\hat{m}} + \overline{\Omega}_+^{m,\hat{m}}.$$

It remains to show that these spaces have zero intersection.

Suppose $s \in \overline{\Omega}_0^{m,\hat{m}} \cap \overline{\Omega}_+^{m,\hat{m}}$. Since $s \in \overline{\Omega}_+^{m,\hat{m}}$, it follows by Claim 6.10(i) that $\hat{\rho}_{m,\hat{m}}^{C_0,C}(s) = 0$; in particular, as $P \in C_0$, we see that $s(P) = 0$. We know in addition that $s \in \overline{\Omega}_0^{m,\hat{m}}$, so Claim 6.10(ii) implies that $\hat{\rho}_{m,\hat{m}}^{C_1,C}(s) = 0$. Thus s restricts to zero on the whole of C, and we conclude that $s = 0$, proving (i).

(ii) Recall that the morphism

$$\hat{\rho}_{m,\hat{m}}^{C_0,C} : H^0\big(C, L_{WC}^m \otimes L_{rC}^{\hat{m}}\big) \to H^0\big(C_0, L_{WC_0}^m \otimes L_{rC_0}^{\hat{m}}\big)$$

is surjective. However, if $s \in \overline{\Omega}_+^{m,\hat{m}}$ then by Claim 6.10(i), we know that $\hat{\rho}_{m,\hat{m}}^{C_0,C}(s) = 0$. Thus, since $H^0(C, L_{WC}^m \otimes L_{rC}^{\hat{m}}) \cong \overline{\Omega}_0^{m,\hat{m}} \oplus \overline{\Omega}_+^{m,\hat{m}}$, we conclude that $\hat{\rho}_{m,\hat{m}}^{C,C_0}|_{\overline{\Omega}_0^{m,\hat{m}}}$ is surjective.

On the other hand,

$$\hat{\rho}_{m,\hat{m}}^{C_0,C}|_{\overline{\Omega}_0^{m,\hat{m}}} : \overline{\Omega}_0^{m,\hat{m}} \to H^0\big(C_0, L_{WC_0}^m \otimes L_{rC_0}^{\hat{m}}\big)$$

is injective. For if $s \in \overline{\Omega}_0^{m,\hat{m}}$ and $\hat{\rho}_{m,\hat{m}}^{C_0,C}(s) = 0$, then $s(P) = 0$; then by Claim 6.10(ii), it follows that $\hat{\rho}_{m,\hat{m}}^{C_1,C}(s) = 0$. Thus $s = 0$, so $\hat{\rho}_{m,\hat{m}}^{C_0,C}|_{\overline{\Omega}_0^{m,\hat{m}}}$ has zero kernel. We conclude that $\hat{\rho}_{m,\hat{m}}^{C_0,C}|_{\overline{\Omega}_0^{m,\hat{m}}}$ is an isomorphism of vector spaces.

(iii) To start with, note that if $M \in \Omega_+^{m,\hat{m}}$ then M has at least one factor from $V_{0,1,1}$, so M vanishes at P by definition; hence

$$\hat{\rho}_{m,\hat{m}}^{C_1,C}\big(\overline{\Omega}_+^{m,\hat{m}}\big) \subset H^0\big(C_1, L_{WC_1}^m(-P)\big).$$

Moreover, the map $\hat{\rho}_{m,\hat{m}}^{C_1,C}|_{\overline{\Omega}_+^{m,\hat{m}}}$ is injective. For if $s \in \overline{\Omega}_+^{m,\hat{m}}$, then $\hat{\rho}_{m,\hat{m}}^{C_0,C}(s) = 0$, and so if in addition $\hat{\rho}_{m,\hat{m}}^{C_1,C}(s) = 0$, then we must conclude that $s = 0$.

On the other hand, the unrestricted morphism

$$\hat{\rho}_{m,\hat{m}}^{C_1,C} : \overline{\Omega}_+^{m,\hat{m}} \oplus \overline{\Omega}_0^{m,\hat{m}} \to H^0\big(C_1, L_{WC_1}^m\big)$$

is surjective. In particular, the subspace $H^0(C_1, L_{WC_1}^m(-P)) \subset H^0(C_1, L_{WC_1}^m)$ is in its image. So for arbitrary $s \in H^0(C_1, L_{WC_1}^m(-P))$, we may write $s = \hat{\rho}_{m,\hat{m}}^{C_1,C}(s_0 + s_+)$, where $s_0 \in \overline{\Omega}_0^{m,\hat{m}}$ and

$s_+ \in \overline{\Omega}_+^{m,\hat{m}}$. But then

$$\hat{\rho}_{m,\hat{m}}^{C_1,C}(s_0) = s - \hat{\rho}_{m,\hat{m}}^{C_1,C}(s_+) \in H^0\big(C_1, L_{WC_1}^m(-P)\big),$$

so s_0 vanishes at P. However by Claim 6.10(ii), it follows that $\hat{\rho}_{m,\hat{m}}^{C_1,C}(s_0) = 0$, and so we conclude that $s = \hat{\rho}_{m,\hat{m}}^{C_1,C}(s_+)$. In other words, the morphism $\hat{\rho}_{m,\hat{m}}^{C_1,C}|_{\overline{\Omega}_+^{m,\hat{m}}}$ is also surjective, showing

$$\hat{\rho}_{m,\hat{m}}^{C_1,C}|_{\overline{\Omega}_+^{m,\hat{m}}} : \overline{\Omega}_+^{m,\hat{m}} \cong H^0\big(C_1, L_{WC_1}^m(-P)\big),$$

which completes the proof of Claim 6.11. ■

Back to the proof of Proposition 6.9. We wish to show that $(C_0, x_1, \ldots, x_n)$ is GIT semistable, with respect to some $l \in \mathbf{H}_{M_{g,n,d}}(\overline{J}_{g,n,d})$. Therefore, let λ' be a 1-PS of $SL(W_{g,n,d})$ acting on $(C_0, x_1, \ldots, x_n) \subset \mathbf{P}(W_{g,n,d}) \times \mathbf{P}^r$. Let $w_0, \ldots, w_{N_{g,n,d}}$ be a basis for $W_{g,n,d}$ diagonalizing the action of λ', so that λ' acts with weights $r_0, \ldots, r_{N_{g,n,d}}$. Let r_{i_j} be the minimal λ'-weight at x_j, so

$$r_{i_j} := \min\{r_i | w_i(x_j) \neq 0\}.$$

We wish to show that

$$\mu^{L_{m,\hat{m},m''}}((C_0, x_1, \ldots, x_n), \lambda') = \mu^{L_{m,\hat{m}}}(C_0, \lambda') + m'' \sum_{j=1}^{n} r_{i_j} \leq 0$$

for some m'' such that $(m, \hat{m}, m'') \in M_{g,n,d}$.

We have identified $W_{g+1,n-1,d}$ with $W_{g,n,d} \oplus V_{1,1,0}$. Extend $w_0, \ldots, w_{N_{g,n,d}}$ to a basis $w_0, \ldots, w_{N_{g+1,n-1,d}}$ for $W_{g+1,n-1,d}$ respecting this decomposition. We define λ to be the extension of λ' over $W_{g+1,n-1,d}$, which acts with weight r_{i_n} on all of $V_{1,1,0}$. Note that λ is a subgroup of $GL(W_{g+1,n-1,d})$, but not necessarily of $SL(W_{g+1,n-1,d})$, so we shall have to use the formula in Lemma 4.5. We already calculated $\dim V_{1,1,0} = (a-1)$, so we know the sum of the weights

$$\sum_{i=0}^{N_{g+1,n-1,d}} w_\lambda(w_i) = (a-1) r_{i_n}.$$

Let $\hat{B}_{m,\hat{m}}$ be a basis of $H^0(\mathbf{P}(W_{g+1,n-1,d}) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W_{g+1,n-1,d})}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m}))$ consisting of monomials of bidegree $(m, \hat{m})$ such that the degree m part is a monomial in $w_0, \ldots, w_{N_{g+1,n-1,d}}$. Semistability of $(C, x_1, \ldots, x_{n-1})$ implies that there exists a subset $\mathcal{B} \subset \hat{B}_{m,\hat{m}}$, such that

$$\overline{\mathcal{B}} := \left\{ \hat{\rho}^{C}_{m,\hat{m}}(M) : M \in \mathcal{B} \right\}$$

is a basis for $H^0(C, L^m_{WC} \otimes L^{\hat{m}}_{rC})$, with

$$\left(\sum_{M \in \mathcal{B}} w_\lambda(M) + m' \sum_{j=1}^{n-1} r_{i_j} \right) (e_{g+1,n-1,d} - g) \\ - (m(e_{g+1,n-1,d}\, m + d\hat{m} - g) + (n-1)m')(a-1) r_{i_n} \leq 0. \tag{51}$$

We relate this to a weight for $(C_0, x_1, \ldots, x_n) \in \mathbf{P}(W_{g,n,d}) \times \mathbf{P}^r$. The basis $\hat{B}_{m,\hat{m}}$ respects the decomposition

$$H^0\big(\mathbf{P}(W_{g+1,n-1,d}) \otimes \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W_{g+1,n-1,d})}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})\big) = \Omega^{m,\hat{m}}_0 \oplus \Omega^{m,\hat{m}}_+;$$

hence the basis $\overline{\mathcal{B}}$ respects the decomposition

$$H^0\big(C, L^m_{WC} \otimes L^{\hat{m}}_{rC}\big) = \overline{\Omega}^{m,\hat{m}}_0 \oplus \overline{\Omega}^{m,\hat{m}}_+$$

of Claim 6.11(i). We split $\overline{\mathcal{B}}$ into two parts, $\overline{\mathcal{B}}_0$ and $\overline{\mathcal{B}}_+$, following this decomposition. We will calculate the contribution to the weight coming from the corresponding subsets $\mathcal{B}_0$ and $\mathcal{B}_+$ of $\mathcal{B}$.

First consider $\mathcal{B}_0$. We show that the weight of this collection of monomials is a λ'-weight for C_0. Recall that the map $\hat{\rho}^{\mathbf{P}(W_{g,n,d})}_{m,\hat{m}}$ which restricts sections to a linear subspace is surjective; this map is zero on $\Omega^{m,\hat{m}}_+$, so

$$\hat{\rho}^{\mathbf{P}(W_{g,n,d})}_{m,\hat{m}}\big|_{\Omega^{m,\hat{m}}_0} : \Omega^{m,\hat{m}}_0 \to H^0\big(\mathbf{P}(W_{g,n,d}) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W_{g,n,d})}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m})\big)$$

is an isomorphism of vector spaces (note we are speaking of the spaces without a bar, those that have not been restricted to C). Moreover, if $M \in \Omega^{m,\hat{m}}_0$ is in $\mathcal{B}_0$, then M is a monomial whose degree m part is in the basis $w_0, \ldots, w_{N_{g,n,d}}$ for $W_{g,n,d}$; we may interpret $\hat{\rho}^{\mathbf{P}(W_{g,n,d})}(M)$ as the same monomial in this basis. Then, since the actions of λ' and λ are identical on $W_{g,n,d}$,

$$w_{\lambda'}\big(\hat{\rho}_{m,\hat{m}}^{\mathbf{P}(W_{g,n,d})}(M)\big) = w_\lambda(M). \tag{52}$$

Now $\{\hat{\rho}^{\mathbf{P}(W_{g,n,d})}(M)|M \in \mathcal{B}_0\}$ is a collection of monomials of bidegree $(m, \hat{m})$ in $H^0(\mathbf{P}(W_{g,n,d}) \times \mathbf{P}^r, \mathcal{O}_{\mathbf{P}(W_{g,n,d})}(m) \otimes \mathcal{O}_{\mathbf{P}^r}(\hat{m}))$ such that when we restrict them to C_0, we obtain

$$\big\{\hat{\rho}_{m,\hat{m}}^{C_0,\mathbf{P}(W_{g,n,d})} \circ \hat{\rho}_{m,\hat{m}}^{\mathbf{P}(W_{g,n,d})}(M)\big|M \in \mathcal{B}_0\big\} = \big\{\hat{\rho}_{m,\hat{m}}^{C_0,C} \circ \hat{\rho}_{m,\hat{m}}^{C}(M)\big|\hat{\rho}_{m,\hat{m}}^{C}(M) \in \overline{\mathcal{B}}_0\big\},$$

which by Claim 6.11(ii) is a basis for $H^0(C_0, L_{W C_0}^m \otimes L_{r C_0}^{\hat{m}})$. It follows that the restricted monomials, $\{\hat{\rho}^{\mathbf{P}(W_{g,n,d})}(M)|M \in \mathcal{B}_0\}$, give a λ'-weight of C_0, that is,

$$\mu^{L_{m,\hat{m}}}(C_0, \lambda') \leq \sum_{M\in\mathcal{B}_0} w_{\lambda'}\big(\hat{\rho}^{\mathbf{P}(W_{g,n,d})}(M)\big) = \sum_{M\in\mathcal{B}_0} w_\lambda(M),$$

where we have used equation (52) to relate this back to λ.

Now consider $\mathcal{B}_+ \subset \Omega_+^{m,\hat{m}}$. If $M \in \mathcal{B}_+$ then by Claim 6.10(i), the restriction $\hat{\rho}_{m,\hat{m}}^{C_0}(M) = 0$; basis elements are nonzero, so $\hat{\rho}_{m,\hat{m}}^{C_1}(M) \neq 0$. The monomial M has bidegree $(m, \hat{m})$, where the degree m part is a monomial in the basis $w_0, \ldots, w_{N_{g+1,n-1,d}}$, which respects the decomposition $W_{g+1,n-1,d} = W_{g,n,d} \oplus V_{1,1,0}$. Any factor in M from $W_{g,n,d}$ must be nonzero on C_1, so by the proof of Claim 6.10(ii), these are nonzero at P. Hence the λ-weight of such factors is bounded below by r_{i_n}. Meanwhile, the remaining factors in M come from $V_{1,1,0}$, and all have λ-weight r_{i_n}. We conclude that for such M,

$$w_\lambda(M) \geq r_{i_n} m.$$

We count the number of such M using Claim 6.11(ii),

$$\#\mathcal{B}_+ = \#\overline{\mathcal{B}}_+ = h^0\big(C_1, L_{W C_1}^m(-P)\big) = am - 1.$$

Thus

$$\sum_{M\in\mathcal{B}_+} w_{\lambda'}(M) \geq (am^2 - m)r_{i_n}.$$

We may now insert these estimates into inequality (51),

$$\begin{aligned}
&(\mu^{L_{m,\hat{m}}}(C_0,\lambda') + (am^2-m)r_{i_n} + m'\big(r_{i_1}+\cdots+r_{i_{n-1}}\big)(e_{g+1,n-1,d}-g) \\
&\le (m(e_{g+1,n-1,d}m + d\hat{m} - g) + (n-1)m')(a-1)r_{i_n} \\
&= \left(e_{g+1,n+1} + d\frac{\hat{m}}{m} + (n-1)\frac{m'}{m^2} - \frac{g}{m}\right)(a-1)m^2 r_{i_n}.
\end{aligned}$$

Recall that we set $\frac{\hat{m}}{m} = \frac{ca}{2a-1}$ and $\frac{m'}{m^2} = \frac{a}{2a-1}$. Further, one may expand: $e_{g+1,n-1,d} - g = (2a-1)g + a(n-1+cd)$. Thus we have shown that

$$\begin{aligned}
&\mu^{L_{m,\hat{m}}}(C_0,\lambda') + (am^2-m)r_{i_n} + \frac{am^2}{2a-1}\big(r_{i_1}+\cdots+r_{i_{n-1}}\big) \\
&\le \left(1 + \frac{g + d\frac{ca}{2a-1} + (n-1)\frac{a}{2a-1} - \frac{g}{m}}{(2a-1)g + a(n-1+cd)}\right)(a-1)m^2 r_{i_n} \\
&= \left(1 + \frac{1}{2a-1}\right)(a-1)m^2 r_{i_n} - S_{14} m r_{i_n},
\end{aligned}$$

where $S_{14} := \frac{g(a-1)}{(2a-1)g+a(n-1+cd)} < \frac{1}{2}$. In conclusion,

$$\mu^{L_{m,\hat{m}}}(C_0,\lambda') + \frac{am^2}{2a-1}\big(r_{i_1}+\cdots+r_{i_n}\big) - (1-S_{14})mr_{i_n} \le 0. \tag{53}$$

We may repeat the argument above, attaching the elliptic curve at the location of any other of the points $x_1,\ldots,x_{n-1}$. This will give us similar inequalities, but with the role x_n played by a different marked point. Adding up all such inequalities, and dividing by n, yields

$$\mu^{L_{m,\hat{m}}}(C_0,\lambda') + \left(\frac{am^2}{2a-1} - \frac{1-S_{14}}{n}m\right)\sum_{j=1}^{n} r_{i_j} \le 0.$$

By the hypotheses on m, we know that $\frac{1-S_{14}}{n}m$ is an integer, so if we set $m'' = \frac{am^2}{2a-1} - \frac{1-S_{14}}{n}m$, then this is also an integer, and we have shown that

$$\mu^{L_{m,\hat{m},m''}}((C_0,x_1,\ldots,x_n),\lambda') \le 0.$$

Thus, as we made no assumptions about the 1-PS λ',

$$(C_0,x_1,\ldots,x_n) \in \overline{J}^{ss}(L_{m,\hat{m},m''}).$$

It remains to show that $(m, \hat{m}, m'') \in M_{g,n,d}$, where $M_{g,n,d}$ is as in the statement of this proposition. We observed that $M_{g+1,n-1,d} \subset M_{g,n,d}$, and so the conditions on m and $\hat{m}$ are satisfied. We need to check that $n|\frac{m''}{m^2} - \frac{a}{2a-1}| = n\frac{1-S_{14}}{nm} \leq \frac{1}{4a-2} - \frac{3g+q_2-\bar{g}-\frac{1}{2}}{m}$. But $\frac{1-S_{14}}{m} < \frac{1}{m}$, and

$$\frac{1}{m} < \frac{1}{4a-2} - \frac{3g+q_2-\bar{g}-\frac{1}{2}}{m}$$
$$\iff m > \left(3g+q_2-\bar{g}+\frac{1}{2}\right)(4a-2).$$

This is implied by $(m, \hat{m}, m') \in M_{g+1,n-1,d}$, for the final lower bound on m is $m > (6(g+1) + 2q_2 - 2(\overline{g+1}) + 1)(2a-1)$.

The choice of $(h, x_1, \ldots, x_n) \in J$ was arbitrary. Hence $J_{g,n,d} \subseteq \overline{J}^{ss}_{g,n,d}(L_{m,\hat{m},m''})$. We proved the reverse inclusion in Theorem 5.21, and so $\overline{J}^{ss}_{g,n,d}(L_{m,\hat{m},m''}) = J_{g,n,d}$. Now by Corollary 5.22, it follows that $\overline{J}^{ss}_{g,n,d}(l) = J_{g,n,d}$ for all $l \in \mathbf{H}_{M_{g,n,d}}(\overline{J}_{g,n,d})$, and in particular for all $L_{m,\hat{m},m'}$ such that $(m, \hat{m}, m') \in M_{g,n,d}$. This completes the proof of Proposition 6.9. ■

Now our desired results are immediate corollaries. We no longer need to distinguish between different spaces of maps, so we stop using the subscripts for J and M.

Proof of Theorem 6.1, Corollary 6.2, and Theorem 6.3. For Theorem 6.1 and Corollary 6.2, we work over $\mathbb{C}$. By Proposition 6.7 and Proposition 6.9, we see that $\overline{J}^{ss}(l) = \overline{J}^{s}(l) = J$, as schemes over $\mathbb{C}$, for any $l \in \mathbf{H}_M(\overline{J})$. In particular, this holds for any $L_{m,\hat{m},m'}$ where $(m, \hat{m}, m') \in M$. Then by Corollary 3.5, it follows that

$$\overline{J}//_{L_{m,\hat{m},m'}} SL(W) \cong \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d),$$

and by Corollary 3.7 it follows that there exists $J_{X,\beta}$ such that

$$\overline{J}_{X,\beta}//_{L_{m,\hat{m},m'}|_{\overline{J}_{X,\beta}}} SL(W) \cong \overline{\mathcal{M}}_{g,n,d}(X, \beta).$$

For Theorem 6.3, we note that Theorem 6.3 has for convenience been phrased in terms of vector spaces over a field k, and so we work at first over k. Now the equality $\overline{J}^{ss}(l) = \overline{J}^{s}(l) = J$, for schemes over k, follows from Proposition 6.8 and Proposition 6.9, where k is any field.

It remains to check that the open subschemes $\overline{J}^{ss}(l) \subseteq \overline{J}$ and $J \subseteq \overline{J}$ are equal over $\mathbb{Z}$. As they are indeed open subschemes, it is sufficient to check that they contain the same points, since they will then automatically have the same scheme structure. If x is a point in J, then we may consider x as a point in $J \times_{\mathbb{Z}} \operatorname{Spec} k(x)$, where $k(x)$ is the residue field of x. We know that J and $\overline{J}^{ss}(l)$ are equal over $\operatorname{Spec} k(x)$, so this provides $J \subseteq \overline{J}^{ss}(l)$. The converse is shown in the same way.

Corollary 3.5 now implies again that in this case,

$$\overline{J}//_{L_{m,m'}} SL(W) \cong \overline{\mathcal{M}}_{g,n}$$

over $\mathbb{Z}$. □

Acknowledgements

It is a great pleasure for both authors to thank Frances Kirwan, who supervised Swinarski as an M.Sc. student and Baldwin as a D.Phil. student, for posing this problem, and for all her support as we worked on it. Elizabeth Baldwin would further like to thank Nigel Hitchin and Peter Newstead for their very useful comments on [2], and Carel Faber, Dan Abramovich, and Christian Liedtke for helpful advice and support. She is also grateful to the Engineering and Physical Sciences Research Council for funding her doctorate and the Mittag-Leffler Institute for a very stimulating stay there. David Swinarski would like to thank the Marshall Aid Commemoration Commission, which funded his work at Oxford through a Marshall Scholarship. He is also grateful to Brian Conrad, Ian Morrison, Rahul Pandharipande, and Angelo Vistoli for patiently answering questions.

References

[1] Alexeev, V., and G. M. Guy. "Moduli of weighted stable maps and their gravitational descendants." *Journal of the Institute of Mathematics of Jussieu* (2006): preprint math.AG/0607683.

[2] Baldwin, E. "A geometric invariant theory construction of moduli spaces of stable maps." DPhil thesis, University of Oxford, 2006.

[3] Baldwin, E. "A GIT construction of moduli spaces of stable maps in positive characteristic." *Journal of the London Mathematical Society* (2008): Advance Access doi: 10.1112/jlms/jdn014.

[4] Baldwin, E. "A GIT construction of $\overline{\mathcal{M}}_{g,n}$ as a quotient of a subscheme of $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$." Manuscript in preparation.

[5] Bayer, A., and Yu. Manin. "Stability conditions, wall-crossing and weighted Gromov–Witten invariants." (2006): preprint math.AG/0607580.

[6] Behrend, K., and Yu. Manin. "Stacks of stable maps and Gromov–Witten invariants." *Duke Mathematical Journal* 85, no. 1 (1996): 1–60.

[7] Dolgachev, I. V., and Yi Hu. "Variation of geometric invariant theory quotients." *Publications Mathematiques Institut de Hautes Etudes Scientifiques* 87 (1998): 5–56.

[8] Fulton, W., and R. Pandharipande. "Notes on stable maps and quantum cohomology." In *Algebraic geometry—Santa Cruz 1995*, 45–96. Proceedings of Symposia in Pure Mathematics 62. Providence, RI: American Mathematical Society, 1997.

[9] Gieseker, D. *Lectures on Moduli of Curves*. Tata Institute of Fundamental Research Lectures on Mathematics and Physics 69. Bombay, India: Tata Institute of Fundamental Research, 1982.

[10] Grothendieck, A. "Éléments de géométrie algébrique 3: Étude cohomologique des faisceaux cohérents 2." *Publications Mathematiques Institut de Hautes Etudes Scientifiques* 17 (1963): 91.

[11] Harris, J., and I. Morrison. *Moduli of Curves*. Graduate Texts in Mathematics 187. New York: Springer, 1998.

[12] Hartshorne, R. *Algebraic Geometry*. Graduate Texts in Mathematics 52. New York: Springer, 1977.

[13] Hassett, B. "Moduli spaces of weighted pointed stable curves." *Advances in Mathematics* 173, no. 2 (2003): 316–52.

[14] Hassett, B., and D. Hyeon. " Log canonical models for the moduli space of curves: First divisorial contraction." (2006): preprint math.AG/0607477.

[15] Hyeon, D., and Y. Lee. "Log minimal model program for the moduli space of stable curves of genus three." (2007): preprint math.AG/0703093.

[16] Kim, B., and R. Pandharipande. "The connectedness of the moduli space of maps to homogeneous spaces." In *Symplectic Geometry and Mirror Symmetry*, 187–201. River Edge, NJ: World Scientific, 2001.

[17] Knudsen, F. F. "The projectivity of the moduli space of stable curves 3: The line bundles on $M_{g,n}$, and a proof of the projectivity of $\overline{M}_{g,n}$ in characteristic 0." *Mathematica Scandinavica* 52, no. 2 (1983): 200–12.

[18] Morrison, I. "Projective stability of ruled surfaces." *Inventiones Mathematicae* 56, no. 3 (1980): 269–304.

[19] Mukai, S. *An Introduction to Invariants and Moduli*. Cambridge Studies in Advanced Mathematics 81. Cambridge, United Kingdom: Cambridge University Press, 2003.

[20] Mumford, D., J. Fogarty, and F. Kirwan. *Geometric Invariant Theory*, 3rd ed. Results in Mathematics and Related Areas 34. Berlin: Springer, 1994.

[21] Mustata, A., and A. Mustata. "Intermediate moduli spaces of maps." *Inventiones Mathematicae* 167, no. 1 (2007): 47–90.

[22] Newstead, P. E. *Introduction to Moduli Problems and Orbit Spaces*. Tata Institute of Fundamental Research Lectures on Mathematics and Physics 51. Bombay, India: Tata Institute of Fundamental Research, 1978.

[23] Parker, A. "An elementary GIT construction of the moduli space of stable maps." (2006): preprint math.AG/0604092.

[24] Schubert, D. "A new compactification of the moduli space of curves." *Compositio Mathematica* 78, no. 3 (1991): 297–313.

[25] Seshadri, C. S. "Quotient spaces modulo reductive algebraic groups." *Annals of Mathematics* 95, no. 2: 511–56; errata, *Annals of Mathematics* 96, no. 2 (1972): 599.

[26] Seshadri, C. S. "Geometric reductivity over arbitrary base." *Advances in Mathematics* 26, no. 3 (1977): 225–74.

[27] Swinarski, D. "GIT stability for weighted pointed curves." (2008): preprint arXiv 0801.1288.

[28] Swinarski, D. "Geometric invariant theory and moduli spaces of maps." Master's thesis, University of Oxford, 2003.

[29] Thaddeus, M. "Geometric invariant theory and flips." *Journal of the American Mathematical Society* 9, no. 3 (1996): 691–723.

[30] Vakil, R. "The moduli space of curves and its tautological ring." *Notices of the American Mathematical Society* 50, no. 6 (2003): 647–58.

[31] Vakil, R. "Murphy's law in algebraic geometry: Badly-behaved deformation spaces." *Inventiones Mathematicae* 164, no. 3 (2006): 569–90.