APPLICATIONS OF LIE GROUPS TO DIFFERENTIAL EQUATIONS

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Abstract. The symmetry group of a differential equation is the largest group of transformations acting on independent and dependent variables with the property that it transforms solutions to other solutions. The symmetry group often gives us greater insight into the differential equation. We would demonstrate the symmetry groups of the heat and wave equation.

Contents

Introduction

A symmetry group of a differential equation is a group which transforms solutions to other solutions. Typical examples include groups of translations, rotations and scaling. For example, if $y = f(x)$ is a solution to $\frac{d^2y}{dx^2} = 0$, then for any real constant ϵ, $y = f(x) + \epsilon$ and $y = e^{\epsilon} f(x)$ are solutions as well. Other less well-behaved groups such as nonlinear or locally defined groups could be subgroups of the full symmetry group as well. (e.g. Lorentz group).

Understanding the symmetry group gives us greater insight into the differential equation. Using symmetries, we could reduce the order of them, generate new solutions from existing ones (e.g. generating the fundamental solutions to the heat equation from the constant solution), or seek group-invariant solutions which are easier to compute. (e.g. rotationally invariant solutions to the Laplace's equation

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are well known). Symmetries are also deeply related to seperation of variables methods [Mil84] and first integral methods [Hyd00].

In section 1, we aim to understand how a change of coordinates affects the derivatives of functions. In section 2, we understand how choosing one-off transformations could simplify ordinary differential equations (e.g. integrating factors). In sections 3 - 4, we construct a general theory of using and finding Lie symmetries of ordinary differential equations. Our main example would be to find the symmetries of $\frac{d^2y}{dx^2} = 0$ [Example 4.5]. In sections 5 - 6, we briefly introduce variational symmetries and demonstrate how they could give us conserved quantities through Noether's theorem [Theorem 6.9]. In section 7, we extend our theory of symmetries to partial differential equations and discuss the symmetry groups of the heat and wave equation. [Example 7.9 and 7.10]

Almost all ideas (including the project's title) are borrowed from a book by Olver [Olv93]. We also make extensive references to a more accesssible book by Hydon [Hyd00]. I make no claim to the originality of the material presented here. Material in the report that is not cited should be common knowledge or simple applications / calculations of theorems previously discussed.

1. Transformations and Derivatives

Symmetries in diffeerential equations could be immensely useful. For example given $\frac{dy}{dx} = \frac{d}{dx}f(x)$, we know that $y = f(x) + C$ for some constant $C \in \mathbb{R}$. This means that if we can find a solution to the differential equation, we can find all the solutions by applying the symmetry of adding constants to the function. We could imagine that the real numbers under addition form the symmetry group of the differential equation.

More complicated differential equations are associated with more complicated symmetries as we would see later.

To apply transformations to differential equations, we first need to understand how transformations alter the derivative. Given a transformation from $\phi : (x, y) \to (\hat{x}, \hat{y})$, what's the relationship between $\frac{d\hat{y}}{d\hat{x}}$ and $\frac{dy}{dx}$?

Explicitly, given some function $y(x)$ on the (x, y) plane and some function $\phi : (x, y) \to (\hat{x}(x, y), \hat{y}(x, y))$. Suppose $\{\phi(x, y(x)) | x \in U\}$ coincides with $\{(\hat{x}, \hat{y}(\hat{x})) | \hat{x} \in V\}$ for some intervals U, V and some diff. function $\hat{y}$. Then at some point $(c, y(c))$, what is $\frac{d\hat{y}}{d\hat{x}}$ at $\phi(c, y(c))$?

Example 1.1 (Polar coordinates)**.** Suppose we're given $y = \sqrt{1 - x^2}$ we would like to find $\frac{dy}{dx}$.

Instead of calculating it directly, we observe that we're trying to find the slope on a semi-circle. As such it might be interesting to use polar coordinates via $\phi : (\theta, r) \to (r\cos\theta, r\sin\theta)$. We know that the pre-image of $\{(x, \sqrt{1 - x^2}),\ -1 \leq x \leq 1\}$ under ϕ is precisely $\{(\theta, 1),\ 0 \leq \theta \leq \pi\}$ so $r(\theta) = 1, \frac{dr}{d\theta} = 0$ for $0 \leq \theta \leq \pi$. Hence,

$$\frac{dy}{dx} = \frac{\frac{dy(r,\theta)}{d\theta}}{\frac{dx(r,\theta)}{d\theta}} = \frac{\frac{\partial y}{\partial r}\frac{dr}{d\theta} + \frac{\partial y}{\partial \theta}\frac{d\theta}{d\theta}}{\frac{\partial x}{\partial r}\frac{dr}{d\theta} + \frac{\partial x}{\partial \theta}\frac{d\theta}{d\theta}} = \frac{0 + r\cos\theta}{0 - r\sin\theta} = -\frac{x}{y} = \frac{-x}{\sqrt{1 - x^2}}$$

which aligns with our expectations.

Example 1.2 (Rotations)**.** Suppose we're given graph of $y = f(x)$ and we want to rotate the graph by θ radians anticlockwise to get the graph $\hat{y} = f(\hat{x})$ and find its derivative. Suppose that $\hat{y}$ is well-defined, intuitively we should get $\frac{d\hat{y}}{d\hat{x}}|_{\phi((c,f(c)))} = \tan(\arctan(\frac{dy}{dx}|_{(c,f(c))}) + \theta)\ \forall c$. We can verify this by the following. Using the rotation map $\phi : (x, y) \to (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$, we would get

$$\begin{aligned}\left.\frac{d\hat{y}}{d\hat{x}}\right|_{\phi((c,f(c)))} &= \frac{\frac{d}{dx}|_{x=c}\hat{y}(x,y(x))}{\frac{d}{dx}|_{x=c}\hat{x}(x,y(x))} = \left.\frac{\frac{\partial\hat{y}}{\partial x} + \frac{dy}{dx}\frac{\partial\hat{y}}{\partial y}}{\frac{\partial\hat{x}}{\partial x} + \frac{dy}{dx}\frac{\partial\hat{x}}{\partial y}}\right|_{x=c} \\ &= \frac{\sin\theta + \frac{dy}{dx}|_{x=c}\cos\theta}{\cos\theta - \frac{dy}{dx}|_{x=c}\sin\theta}\end{aligned}$$

Thus we can check that

$$\begin{aligned}\tan\Big(\arctan\left(\left.\frac{dy}{dx}\right|_{x=c}\right) + \theta\Big) &= \frac{\frac{dy}{dx}|_{x=c} + \tan\theta}{1 - \tan\theta\frac{dy}{dx}|_{x=c}} \\ &= \frac{\cos\theta\frac{dy}{dx}|_{x=c} + \sin\theta}{\cos\theta - \frac{dy}{dx}|_{x=c}\sin\theta}\end{aligned}$$

We could've deduced the expression directly by using the fact that sums of cosine / sine functions can be expressed as a single cosine / sine function.

2. Ordinary Differential Equations

Applying suitable transformations could greatly simplify differential equations we encounter. Let's start with analyzing elementary change of coordinate methods for differential equations to get used to the afforementioned framework. This section and the next follows from Chapter 1 of Hydon [Hyd00].

Example 2.1 (Linear combinations of x, y)**.** Consider the differential equation of form

$$\frac{dy}{dx} = F(ax + by + c)$$

for constants a, b, c. Then consider the transfomration $\phi : (x, y) \to (x, ax + by + c)$. We observe that

$$\begin{aligned}\frac{d\hat{y}}{d\hat{x}} &= \frac{d}{dx}(ax + by + c) \\ &= a + b\frac{dy}{dx} \\ &= a + bF(\hat{y})\end{aligned}$$

which is seperable. Finally $\hat{y}(\hat{x})|_{\hat{x}=x} = ax + by(x) + c$ so $y(x) = \frac{1}{b}(\hat{y}(x) - ax - c)$

Example 2.2. To solve $\frac{dy}{dx} = x + y(x) + 1$, we have $\frac{d\hat{y}}{d\hat{x}} = 1 + \hat{y}$ which has solution $\hat{y}(\hat{x}) = Ce^{\hat{x}} - 1$ for some constant C. So $y(x) = Ce^{x} - x - 2$.

Example 2.3. Differential equations of form $\frac{dy}{dx} = F(\frac{y}{x})$

Inspired by the previous example, we may choose the transformation $\phi : (x, y) \to (x, \frac{y}{x})$ which would give us

$$\begin{aligned}\frac{d\hat{y}}{d\hat{x}} &= \frac{d}{dx}\hat{y}(x, y(x)) \\ &= \frac{-y}{x^2} + F\left(\frac{y}{x}\right)\frac{1}{x} \\ &= \frac{1}{\hat{x}}\big(- \hat{y} + F(\hat{y})\big)\end{aligned}$$

which is seperable.

Notice that changing $\hat{x}$ would not affect $\frac{d}{dx}\hat{y}(x, y(x))$. As such we may try to construct a suitable $\hat{x}$ to eliminate the $\frac{1}{\hat{x}}$ term. Let $\phi : (x, y) \to (\hat{x}, \hat{y}) = (\log x, \frac{y}{x})$. Then,

$$\begin{aligned}\frac{d\hat{y}}{d\hat{x}} &= \frac{\frac{d}{dx}\hat{y}(x, y(x))}{\frac{d}{dx}\hat{x}(x, y(x))} = \frac{\frac{\partial \hat{y}}{\partial x} + \frac{dy}{dx}\frac{\partial \hat{y}}{\partial y}}{\frac{\partial \hat{x}}{\partial x} + \frac{dy}{dx}\frac{\partial \hat{x}}{\partial y}} \\ &= \frac{-\frac{y}{x^2} + F(\frac{y}{x})\frac{1}{x}}{\frac{1}{x} + F(\frac{y}{x}) \cdot 0} \\ &= -\frac{y}{x} + F\left(\frac{y}{x}\right) \\ &= -\hat{y} + F(\hat{y})\end{aligned}$$

So $\hat{y}(\hat{x})|_{\hat{x}=\log x} = \frac{y(x)}{x}$ hence $y(x) = x\hat{y}(\log x)$

Example 2.4 (Integrating factors)**.** Consider the differential equation of form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Consider the transformation $\phi : (x, y) \to (x, e^{\int P(x)dx}y)$. We observe that

$$\begin{aligned}\frac{d\hat{y}}{d\hat{x}} &= \frac{d\hat{y}}{dx} \\ &= P(x)e^{\int P(x)dx}y + \frac{dy}{dx}e^{\int P(x)dx} \\ &= Q(x)\hat{y}\end{aligned}$$

which is seperable and easier to solve. Finally $\hat{y}(\hat{x})|_{\hat{x}=x} = e^{\int P(x)dx}y(x)$ so $y(x) = \frac{\hat{y}(x)}{e^{\int P(x)dx}}$

Example 2.5 (Cauchy-Euler)**.** Consider the differential equation of form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} \cdots + a_0 y = 0$$

Consider the transformation $\phi : (x, y) \to (\log x, y)$. We observe that

$$\begin{aligned}\frac{d\hat{y}}{d\hat{x}} &= \frac{\frac{d}{dx}\hat{y}(x, y(x))}{\frac{d}{dx}\hat{x}(x, y(x))} = \frac{\frac{\partial \hat{y}}{\partial x} + \frac{dy}{dx}\frac{\partial \hat{y}}{\partial y}}{\frac{\partial \hat{x}}{\partial x} + \frac{dy}{dx}\frac{\partial \hat{x}}{\partial y}} \\ &= \frac{0 + \frac{dy}{dx}}{\frac{1}{x} + 0} = x\frac{dy}{dx}\end{aligned}$$

and

$$\begin{aligned}\frac{d^2\hat{y}}{d\hat{x}^2} &= \frac{\frac{\partial}{\partial x}(\frac{d\hat{y}}{d\hat{x}}) + \frac{dy}{dx}\frac{\partial}{\partial y}(\frac{d\hat{y}}{d\hat{x}})}{\frac{\partial \hat{x}}{\partial x} + \frac{dy}{dx}\frac{\partial \hat{x}}{\partial y}} \\ &= \frac{\frac{\partial}{\partial x}(x\frac{dy}{dx}) + \frac{dy}{dx}\cdot 0}{\frac{1}{x}+0} \\ &= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}\end{aligned}$$

We can see that for all i

$$\frac{d^{i+1}\hat{y}}{d\hat{x}^{i+1}} = x\frac{d}{dx}\left(\frac{d^i\hat{y}}{d\hat{x}^i}\right)$$

So

$$\begin{bmatrix} \hat{y} \\ \frac{d\hat{y}}{d\hat{x}} \\ \frac{d^2\hat{y}}{d\hat{x}^2} \\ \dots \\ \frac{d^n\hat{y}}{d\hat{x}^n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & * & * & \dots & 1 \end{bmatrix} \begin{bmatrix} y \\ x\frac{dy}{dx} \\ x^2\frac{d^2y}{dx^2} \\ \dots \\ x^n\frac{d^ny}{dx^n} \end{bmatrix}$$

The matrix is lower triangular with determinant 1. So it's invertible and hence we can find unique coefficients $b_0, \dots, b_n$ such that the differential equation becomes

$$b_n\frac{d^n\hat{y}}{d\hat{x}^n} + b_{n-1}\frac{d^{n-1}\hat{y}}{d\hat{x}^{n-1}}\dots + b_0\hat{y} = 0$$

which is easier to solve.

We could summarise the examples by the following.

Type	$(\hat{x}, \hat{y})$
$\frac{dy}{dx} + P(x)y = Q(x)$	$(x, e^{\int P(x)dx}y)$
$\frac{dy}{dx} = F(ax+by+c)$	$(x, ax+by+c)$
$\frac{dy}{dx} = F(y/x)$	$(\log x, \frac{y}{x})$
Cauchy-Euler	$(\log x, y)$

3. Lie Symmetries of First Order ODEs

We shall consider differential equations of form

$$\frac{dy}{dx} = w(x,y)$$

throughout the entire section. We are interested in symmetries $\phi : (x,y) \to (\hat{x}, \hat{y})$ that are diffeormorphisms and keep the set of solution curves invariant. I.e. if $y = f(x)$ is a solution, then $(\hat{x}(x, f(x)), \hat{y}(x, f(x)))$ as a curve in $(\hat{x}, \hat{y})$ plane written in parametric form is a solution to

$$\frac{d\hat{y}}{d\hat{x}} = w(\hat{x}, \hat{y})$$

This is very different from section 1 where our modus operandi was to make $\frac{d\hat{y}}{d\hat{x}}$ into a different and simpler expression to $w(\hat{x},\hat{y})$. Although change of coordinates would re-appear in the discussion of canonical coordinates [Section 3.4], these coordinates are derived from continuous symmetries.

3.1. **The One-paramter Lie Group.** We are particularly interested in symmetries that form a group. This motivates the following.

Definition 3.1 (One-paramter Lie group, [Hyd00] p. 4)**.** Suppose an object occupying a subset of $\mathbb{R}^N$ has an infinite set of symmetries $\phi_\epsilon : x^s \to \hat{x}^s(x^1, \dots, x^N; \epsilon)$ where ϵ is a real paramater, and that the following conditions are satisfied.

(L1) ϕ_0 is the trivial symmetry / the identity
(L2) ϕ_ϵ is a symmetry for every ϵ in some neighbourhood of zero
(L3) $\phi_\delta \circ \phi_\epsilon = \phi_{\delta+\epsilon}$ for δ, ϵ sufficiently close to zero
(L4) Each $\hat{x}^s$ may be represented as a Taylor series in ϵ

Example 3.2. $\phi_\epsilon : (x,y) \to (x, y+\epsilon)$ forms a one-paramter Lie group (check!), and is a set of symmetries for the differential equation

$$\frac{dy}{dx} = w(x)$$

This is because given $y = f(x)$ s.t. $\frac{dy}{dx} = w(x)$, we have $(\hat{x}(x, f(x)), \hat{y}(x, f(x))) = (x, f(x) + \epsilon)$ so $\frac{d\hat{y}}{d\hat{x}} = w(\hat{x})$

Note the general solution to the equation is $\int w(x)dx + c$ for some constant c. Geometrically, the symmetries raises and lowers the solution curves on the (x, y) plane vertically.

Example 3.3. $\phi_\epsilon : (x,y) \to (e^\epsilon x, e^\epsilon y)$ forms a one-paramter Lie group, and is a set of symmetries for the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

This is because given $y = f(x)$ s.t. $\frac{dy}{dx} = \frac{-x}{y}$, we have $(\hat{x}(x, f(x)), \hat{y}(x, f(x))) = (e^\epsilon x, e^\epsilon f(x))$ so $\frac{d\hat{y}}{d\hat{x}} = \frac{d\hat{y}}{dx} \cdot (\frac{d\hat{x}}{dx})^{-1} = -\frac{x}{y} = \frac{\hat{x}}{\hat{y}}$

Note that the general solution to the equation is $y(x) = \pm\sqrt{c^2 - x^2}\ \forall c \in \mathbb{R}$, circles of radius c centered on the origin. Geometrically, the symmetries map circles cenetered on the origin to circles centered on the origin.

Example 3.4. $\phi_\epsilon : (x,y) \to (x\cos\epsilon - y\sin\epsilon, x\sin\epsilon + y\cos\epsilon)$ forms a one-paramter Lie group, and is a set of symmetries for the differential equation of example 3.3.

Geometrically, the symmetries rotate circles centered on the origin.

3.2. **Tangent Vector Field.** ([Hyd00] p. 22)

It turns out that there is a local one-to-one correspondence between one-paramter Lie group and its tangent vector field. The explanation of which is beyond the scope of this article. This prompts us to place focus solely on the tangent vector field of Lie symmetries we encounter. This greatly eases computations as we would see later on. However, having a global picture of the action of the Lie group is still immensely helpful at gaining geometric intuition.

Let's introduce some notation, given a one paramter Lie group, $\phi_\epsilon : (x,y) \to (\hat{x},\hat{y})$, we say that the tangent vector at (x,y) is the vector $(\xi(x,y), \eta(x,y)) :=$

$\frac{d}{d\epsilon}|_{\epsilon=0}(\hat{x}(x,y),\hat{y}(x,y))$. We may also use the notation $\xi\partial_x + \eta\partial_y$ to represent the tangent vector.

Example 3.5. The tangent vector of $\phi_\epsilon : (x,y) \to (x, y+\epsilon)$ at (x,y) is $(0,1)$ i.e. ∂_y

Example 3.6. The tangent vector of $\phi_\epsilon : (x,y) \to (e^\epsilon x, e^\epsilon y)$ at (x,y) is (x,y) i.e. $x\partial_x + y\partial_y$

3.3. **Recontructing Lie Groups from the Tangent Vector Field.** Given an infinitesimal generator X, we could do so using the formulae $\hat{x} = e^{\epsilon X}x, \hat{y} = e^{\epsilon X}y$ to deduce the Lie group structure [[Hyd00] p. 40]

Example 3.7. Given $X = x\partial_y$, we can calculate

$$\begin{aligned} Xx &= x\partial_y(x) = 0 \\ Xy &= x\partial_y(y) = x \\ \therefore X^2y &= 0 \end{aligned}$$

So

$$\begin{aligned} \hat{x} &= e^{\epsilon X}x = \sum_{k=0}^{\infty}\frac{1}{k!}\epsilon^k X^k(x) = x \\ \hat{y} &= e^{\epsilon X}y = \sum_{k=0}^{\infty}\frac{1}{k!}\epsilon^k X^k(y) = y + \epsilon x \end{aligned}$$

Alternatively, we could solve $\frac{\partial\hat{x}}{\partial\epsilon} = \xi(\hat{x},\hat{y}), \frac{\partial\hat{y}}{\partial\epsilon} = \eta(\hat{x},\hat{y})$ subject to the initial conditions $\hat{x}|_{\epsilon=0} = x, \hat{y}|_{\epsilon=0} = y$ to deduce the expressions for $\hat{x},\hat{y}$ instead.

Example 3.8. As a more general example of constructing a Lie group from a Lie algebra we have If $X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ then for real ϵ we have $e^{\epsilon X}$ to be

$$\begin{aligned} &\sum_{k=0}^{\infty}\frac{1}{k!}\epsilon^k X^k \\ &= \sum_{k=0}^{\infty}\left\{\frac{\epsilon^{4k}}{(4k)!}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\epsilon^{4k+1}}{(4k+1)!}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \frac{\epsilon^{4k+2}}{(4k+2)!}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{\epsilon^{4k+3}}{(4k+3)!}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\} \\ &= \begin{bmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{bmatrix} \end{aligned}$$

which forms the 2D rotation group, SO(2).

This is analogous to Euler's identity of $e^{i\epsilon} = \cos\epsilon + i\sin\epsilon$ by considering the isomorphism between the group of unit complex numbers under multiplication, U^1, and SO(2) by $e^{i\theta} \leftrightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

The linearisation about the identity of the Lie group typically yields the Lie algebra that generates the group. You could find a more thorough discussion on Lie algebras and Lie groups in [Olv93].

3.4. **Canonical Coordinates.** Given a one-paramter Lie group of symmetries ϕ_ϵ with a tangent vector field $(\xi(x,y),\eta(x,y)) := \frac{d}{d\epsilon}|_{\epsilon=0}(\hat{x},\hat{y})$, we want to find coordinates $(r(x,y),s(x,y))$ such that $(\hat{r},\hat{s}) = (r(\hat{x},\hat{y}),s(\hat{x},\hat{y})) = (r,s+\epsilon)$. We also want the change of coordinates to be invertible in some neighbourhood of (x,y) so we have the Jacobian nonzero. In essence, we want to find some change of coordinates map θ such that we have the following

$$\begin{tikzcd}
(x,y) \arrow[r, "\phi_\epsilon"] \arrow[d, "\theta"'] & (\hat{x},\hat{y}) \arrow[d, "\theta"] \\
(r,s) \arrow[r, "\theta\circ\phi_\epsilon\circ\theta^{-1}"'] & (\hat{r},\hat{s}) = (r,s+\epsilon)
\end{tikzcd}$$

Example 3.9. The simplest example of canonical coordinates is that (x,y) are the canonical coordinates to the symmetries of form $\phi_\epsilon : (x,y) \to (x,y+\epsilon)$

Fitting with our theme of analyzing Lie symmetries by their assocaiated tangent vector fields, our condition that $(\hat{r},\hat{s}) = (r,s+\epsilon)$ is locally equivalent to that of

$$\left.\frac{d\hat{r}}{d\epsilon}\right|_{\epsilon=0} = 0, \qquad \left.\frac{d\hat{s}}{d\epsilon}\right|_{\epsilon=0} = 1$$

Letting $X = \xi\partial_x + \eta\partial_y$ and using the chain rule, the above is equivalent to

$$Xr = 0, \qquad Xs = 1$$

X is called the infinitesimal generator of the Lie symmetry.

To solve these equations, we could use the method of chararacteristics. For example, the characteristics for the PDE $Xr = 0$ are the following

$$\frac{dx}{dt} = \xi(x,y)$$
$$\frac{dy}{dt} = \eta(x,y)$$
$$\frac{dr}{dt} = 0$$

We can similarly write down the characteristics for $Xs = 1$ as well.

Intuitively, $Xr = 0$ is equivalent to $(\xi,\eta)\cdot\nabla r = 0$, i.e. r is constant along streamlines along the vector field (ξ,η). Another way of thinking about it is that we want to find a "first integral" to the differential equation $\frac{dy}{dx} = \frac{\xi}{\eta}$: a function $r(x,y)$ that is constant on the solution curves $y(x)$.

Canonical coordinates are useful because they give us an differential equation solvable by quadrature. If we could find such coordinates, then note that

$$\frac{ds}{dr} = \frac{s_x + w(x,y)s_y}{r_x + w(x,y)r_y} = \Omega(r,s) \tag{3.1}$$

for some function Ω. However as the differential equation above is invariant under symmetries of form $\phi_\epsilon : (r,s) \to (\hat{r},\hat{s}) = (r,s+\epsilon)$. From the symmetry condition we obtain that Ω has no dependence on r. As such our resulting differential equation is of form

$$\frac{ds}{dr} = \Omega(r)$$

which is now reduced to quadrature.

Example 3.10. We want to rediscover the substituition used in example 2.1. We could easily check $\phi_\epsilon : (x, y) \to (x + b\epsilon, y - a\epsilon)$ forms a symmetry to that equation. To find the canonical coordinates for ϕ_ϵ, we want to solve

$$\frac{dx}{dt} = b$$
$$\frac{dy}{dt} = -a$$
$$\frac{dr}{dt} = 0$$

which gives a general solution $(x, y, r) = (bt + A, -at + B, C)$ where A, B, C are constants. By observation we could choose $(r, s) = (ax + by + c, x)$

The calculation of $\frac{ds}{dr}$ has been done on 2.1 and it has no dependence on s as expected.

Example 3.11 ([Hyd00] p. 25)**.** We want to rediscover the substituition used in example 2.3. We could easily check $\phi_\epsilon : (x, y) \to (e^\epsilon x, e^\epsilon y)$ forms a symmetry to that equation. To find the canonical coordinates for ϕ_ϵ, we want to solve

$$\frac{dx}{dt} = x$$
$$\frac{dy}{dt} = y$$
$$\frac{dr}{dt} = 0$$

which gives us a general solution $(x, y, r) = (Ae^t, Be^t, C)$ where A, B, C are constants. We would want to express r as a nontrivial function of x and y. By observation we could choose $r = y/x = B/A$ for $x \neq 0$

For s, we would get the general solution $(x, y, s) = (Ae^t, Be^t, t+C)$ where A, B, C are constants. As such we can choose $s = \ln |x|$.

These canonical coordinates fail on the line $x = 0$. Near $x = 0$ excpet on the line $y = 0$ we could use the canonical coordinates $(r, s) = (\frac{x}{y}, \ln |y|)$ instead.

No canonical coordinates exist at the point $(0, 0)$. This is because $\xi(0, 0) = \eta(0, 0) = 0$ and so it's impossible for $Xs = 1$ to have any solution. Points (x, y) where $\xi(x, y) = \eta(x, y) = 0$ are called invariant points and it's impossible to define canonical coordinates on them.

The calculation of $\frac{ds}{dr}$ has been done in example 2.3 and it has no dependence on s as expected.

4. Lie Symmetries of Higher Order ODEs

The theory above can be readily generalised to higher dimensional ODEs. Given a diffeomorphism $\Gamma : (x, y) \to (\hat{x}, \hat{y})$, it maps smooth planar curves to other smooth planar curves. In particular its action induces an action on the derivatives $y^{(k)} := \frac{d^k y}{dx^k}$ by

$$\Gamma : (x, y, y', \dots, y^{(n)}) \to (\hat{x}, \hat{y}, \hat{y}', \dots, \hat{y}^{(n)})$$

where

$$\hat{y}^{(k)} := \frac{d^k \hat{y}}{d\hat{x}^k}, \qquad k = 1, \dots, n$$

This mapping is called the nth prolongation of Γ

Example 4.1. Throughout section 1 we have been interested in the 1st prolongation of various diffeomorphisms, mainly because the 1st prolongation gives us sufficient information to understand how it acts on a 1st order ordinary differential equations.

For example, $\gamma : (x, y) \to (x, ax+by+c)$ featured in 2.1 has the first prolongation

$$\gamma : (x, y, y') \to (x, ax + by + c, a + by')$$

and second prolongation

$$\gamma : (x, y, y', y'') \to (x, ax + by + c, a + by', by'')$$

We calculate $\hat{y}^{(k)}$ recursively as follows

$$\hat{y}^{(k)} = \frac{\frac{d}{dx}\hat{y}^{(k-1)}(x, y, y', \dots)}{\frac{d}{dx}\hat{x}(x, y, y', \dots)} = \frac{D_x \hat{y}^{(k-1)}}{D_x \hat{x}}$$

where D_x is the total derivative with respect to x:

$$D_x = \partial_x + y'\partial_y + y''\partial_{y'} + \dots \tag{4.1}$$

For simplicity, we shall only consider ODEs of form

$$y^{(n)} = w(x, y, y', \dots, y^{(n-1)}) \tag{4.2}$$

and as such the symmetry condition would be

$$\hat{y}^{(n)} = w(\hat{x}, \hat{y}, \dots, \hat{y}^{(n-1)}) \tag{4.3}$$

when equation 4.2 holds.

4.1. Linearised Symmetry Condition for first order ODEs. ([Hyd00] p. 30)

Let's start with the one-dimensional case. Consider the differential equation $y' = w(x, y)$

Recall that by definition all lie symmetries are of form

$$\hat{x} = x + \epsilon\xi(x, y) + O(\epsilon^2)$$
$$\hat{y} = y + \epsilon\eta(x, y) + O(\epsilon^2)$$

From this we obtain

$$\begin{bmatrix} \hat{x}_x & \hat{y}_x \\ \hat{x}_y & \hat{y}_y \end{bmatrix} = \begin{bmatrix} 1 + \epsilon\xi_x + O(\epsilon^2) & \epsilon\eta_x + O(\epsilon^2) \\ \epsilon\eta_y + O(\epsilon^2) & 1 + \epsilon\eta_y + O(\epsilon^2) \end{bmatrix}$$

Substituting it to the symmetry condition

$$w(\hat{x}, \hat{y}) = \frac{d\hat{y}}{d\hat{x}} = \frac{\hat{y}_x + w(x, y)\hat{y}_y}{\hat{x}_x + w(x, y)\hat{x}_y}$$

Expanding each side as a Taylor series about $\epsilon = 0$, assuming convergence and using $\frac{1}{1-x} = 1 - x + x^2 + \dots$ for $|x| \leq 1$ we would get

$$w + \epsilon(\xi w_x + \eta w_y) + O(\epsilon^2) = w + \epsilon[\eta_x + w\eta_y - (\xi_x w + \xi_y w^2)] + O(\epsilon^2) \tag{4.4}$$

$$= w + \epsilon(D_x\eta - wD_x\xi) + O(\epsilon^2) \tag{4.5}$$

where w is a shorthand for $w(x, y)$

Equating the $O(\epsilon)$ terms we get the linearised symmetry condition

$$\xi w_x + \eta w_y = \eta_x + (\eta_y - \xi_x)w - \eta_y w^2$$

i.e. we want

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} w(\hat{x},\hat{y}) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \frac{d\hat{y}}{d\hat{x}}$$

Example 4.2. Consider the differential equation $y' = 0$, the linearised symmetry condition is $\eta_x = 0$.

4.2. **Linearised Symmetry Condition for higher order ODEs.** The calculation is similar for higher order ODEs. Considering the prolongation of the Lie symmetries, they are going to be of form

$$\begin{aligned}\hat{x} &= x + \epsilon\xi + O(\epsilon^2)\\ \hat{y} &= y + \epsilon\eta + O(\epsilon^2)\\ \hat{y}^{(k)} &= y^{(k)} + \epsilon\eta^{(k)} + O(\epsilon^2), \qquad k \geq 1\end{aligned}$$

for some smooth functions $\eta^{(k)}$ (Superscript in $\eta^{(k)}$ is an index and does not denote derivatives)

We can calculate $\eta^{(k)}$ as follows. For $k = 1$, we have

$$\hat{y}^{(1)} = \frac{D_x\hat{y}}{D_x\hat{x}} = \frac{y' + \epsilon D_x\eta + O(\epsilon^2)}{1 + \epsilon D_x\xi + O(\epsilon^2)} = y' + \epsilon(D_x\eta - y'D_x\xi) + O(\epsilon^2)$$

This is simply equation 4.5 and so

$$\eta^{(1)} = D_x\eta - y'D_x\xi \tag{4.6}$$

Similarly

$$\hat{y}^{(k)} = \frac{D_x\hat{y}^{(k)}}{D_x\hat{x}} = \frac{y^{(k)} + \epsilon D_x\eta^{(k-1)} + O(\epsilon^2)}{1 + \epsilon D_x\xi + O(\epsilon^2)} \tag{4.7}$$

$$\therefore \eta^{(k)}(x, y, \dots, y^{(k)}) = D_x\eta^{(k-1)} - y^{(k)}D_x\xi \tag{4.8}$$

and the lineraised symmetry condition (deduced from 4.3) is going to be

$$\eta^{(n)} = \psi w_x + \eta w_y + \eta^{(1)}w_{y'} + \cdots + \eta^{(n-1)}w_{y^{(n-1)}} \tag{4.9}$$

when 4.2 holds. The right hand side is the 1st degree term of the taylor expansion of w with respect to ϵ.

Example 4.3. For a vector field of form $\xi(x)\partial_x$, the first prolongation has form

$$\eta^{(1)} = -y'\frac{d\xi}{dx}$$

For example the vector field $x\partial_x$ has the first prolongation be $x\partial_x - y'\partial_{y'}$.

Example 4.4 ([Hyd00] p. 47)**.**

$$\begin{aligned}\eta^{(2)} &= D_x\eta^{(1)} - y^{(1)}D_x\xi \\ &= D_x(\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2) - y'D_x\xi && \text{[Eq 4.5]}\\ &= (\partial_x + y'\partial_y + y''\partial_{y'})(\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2) && \text{Def of } D_x\text{[Eq 4.1]}\\ &\quad - y'(\partial_x + y'\partial_y)\xi \\ &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 \\ &\quad + [\eta_y - 2\xi_x - 3\xi_y y']y''\end{aligned} \tag{4.10}$$

Example 4.5 ([Hyd00] p.47)**.** Consider

$$\frac{d^2y}{dx^2} = 0 \tag{4.11}$$

which is of form $y'' = w(x, y, y')$ with $w = 0$. So the linearised symmetry condition is

$$\eta^{(2)} = 0$$

when $y'' = 0$ Using equation 4.10 we have

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 = 0$$

As ξ and η are independent of y', the linearised symmetry condition splits into 4 determining equations. Omitting steps, they lead to the following form for ξ and η

$$\xi = c_1 + c_3x + c_5y + c_7x^2 + c_8xy$$
$$\eta = c_2 + c_4y + c_6x + c_7xy + c_8y^2$$

Therefore every symmetry generator X is a linear combination of $X_1, \dots, X_8$ as follows.

Treating x as the time variable, we notice that the differential equation (Eq 4.11) is Newton's first law on a one dimensional point mass moving with zero net force.

This implies that some of the symmetries below can be interpeted as Galilean transformations.

Symmetry	Vector field	$(\hat{x}, \hat{y})$	$\hat{y}(\hat{x})$ if $y(x) = ax + b$	Interptation
X_1	∂_x	$(x + \epsilon, y)$	$a\hat{x} + (b - \epsilon a)$	Time translation
X_2	∂_y	$(x, y + \epsilon)$	$a\hat{x} + b + \epsilon$	Position translation
X_3	$x\partial_x$	$(e^\epsilon x, y)$	$e^{-\epsilon}a\hat{x} + b$	
X_4	$y\partial_y$	$(x, e^\epsilon y)$	$e^\epsilon a\hat{x} + e^\epsilon b$	
X_5	$y\partial_x$	$(x + \epsilon y, y)$	$\frac{a}{1+\epsilon a}\hat{x} + \frac{b}{1+\epsilon a}$	
X_6	$x\partial_y$	$(x, y + \epsilon x)$	$(a + \epsilon)x + b$	Galilean boost
X_7	$x^2\partial_x + xy\partial_y$	$(\frac{x}{1-\epsilon x}, \frac{y}{1-\epsilon x})$	$(a + \epsilon b)\hat{x} + b$	
X_8	$xy\partial_x + y^2\partial_y$	$(\frac{x}{1-\epsilon y}, \frac{y}{1-\epsilon y})$	$\frac{a}{1-\epsilon b}\hat{x} + \frac{b}{1-\epsilon b}$	

Table 1. Symmetries of $\frac{d^2y}{dx^2} = 0$

Example 4.6 ([Hyd00] p. 54)**.** The equation $\frac{d^3y}{dx^3} = 0$ has a Lie symmetry group of dimension 7.

Symmetry	Vector field	$(\hat{x}, \hat{y})$	$\hat{y}(\hat{x})$ if $y(x) = ax^2 + bx + c$
X_1	∂_y	$(x, y + \epsilon)$	$a\hat{x}^2 + \hat{x} + c + \epsilon$
X_2	$x\partial_y$	$(x, y + \epsilon x)$	$\hat{x}^2 + (b + \epsilon)\hat{x} + c$
X_3	$x^2\partial_y$	$(x, y + \epsilon x^2)$	$(a + \epsilon)\hat{x}^2 + b\hat{x} + c$
X_4	∂_x	$(x + \epsilon, y)$	$a(\hat{x} - \epsilon)^2 + b(\hat{x} - \epsilon) + c$
X_5	$y\partial_y$	$(x, e^\epsilon y)$	$e^\epsilon(a\hat{x}^2 + b\hat{x} + c)$
X_6	$x\partial_x$	$(e^\epsilon x, y)$	$ae^{-2\epsilon}\hat{x}^2 + be^{-\epsilon}x + c$
X_7	$x^2\partial_x + 2xy\partial_y$	$(\frac{x}{1-\epsilon x}, \frac{y}{(1-\epsilon x)^2})$	$(a + \epsilon b + \epsilon^2 c)\hat{x}^2 + (b + 2\epsilon c)\hat{x} + c$

Table 2. Symmetries of $\frac{d^3y}{dx^3} = 0$

The general pattern of the Lie group of symmetries of $\frac{d^n y}{dx^n} = 0$ for $n \geq 3$ are a $n+4$ dimmensional group generated by $\partial_y, x\partial_y \ldots, x^{n-1}\partial_y, \partial_x, x\partial_x, x^2\partial_x + (n-1)xy\partial_y$. [[Hyd00] p. 54] For $n = 2$ (as in the previous example) the above "pattern" predicts there to be 6 symmetries instead of the 8 we observe. However we can generate the other two symmetries by swapping x and y, in the sense that $\phi : (x, y) \to (y, x)$ is a discrete symmetry of the differential equation $\frac{d^n y}{dx^n} = 0$ for positive integers n if and only if $n = 2$.

In fact, Lie provided a geometric proof that a nth order ordinary differential equation can have at most $n + 4$ point symmetries. [Lie93]

4.3. **Lie Algebra of Point Symmetry Generators.** This subsection directly follow from section 5.2 from Hydon [[Hyd00] p. 79] / standard texts on Lie algebras. We breifly introduce rigorous definitions of Lie Algebra / Lie bracket to highlight how that theory fits into symmetries of differential equations.

Suppose we have a differential equation of form 4.2. Let $\mathcal{L}$ be the set of all infinitesimal generators of the one-paramter Lie groups of point symmetries of that differential equation. It's clear that ξ is a vector space. The dimension, R, of the vector space is the number of arbitrary constants that appear in the general solution of the linearised symmetry condition. [Direct quote, [Hyd00] p. 50]

It turns out that the order of the ODE places restrictions upon R. For second-order ODEs, R is 0, 1, 2, 3 or 8. Moreover, R is 8 if and only if the ODE either is linear, or is linearizable by a point transformation. [Direct quote, [Hyd00] p. 51]

Example 4.7. We've seen in example 4.5 that the symmetry group of $y'' = 0$, a linear ODE, is 8 dimensional.

We're interested in how these infitestimal generators interact with each other. Suppose we have two infitestimal generators X_1, X_2 where

$$X_i = \xi_i(x, y)\partial_x + \eta_i(x, y)\partial_y, \qquad i = 1, 2$$

The product X_1X_2 would be a second order partial differntial operator

$$X_1X_2 = \xi_1\xi_2\partial_x^2 + (\xi_1\eta_2 + \eta_1\xi_2)\partial_x\partial_y + \eta_1\eta_2\partial_y^2 + (X_1\xi_2)\partial_x + (X_1\eta_2)\partial_y$$

Example 4.8. If $X_1 = \partial_x$ and $X_2 = \partial_y$ then

$$X_1X_2 = \partial_x\partial_y = X_2X_1$$

Example 4.9. If $X_1 = \partial_x$ and $X_2 = x\partial_x$ then

$$X_1X_2 = \partial_x + x\partial_x^2$$
$$X_2X_1 = x\partial_x^2 \neq X_1X_2$$

Note that the second order terms of X_1X_2 are identical to X_2X_1. This motivates the following definiton.

Definition 4.10. The commutator of X_1 with X_2 is defined to be

$$[X_1, X_2] = X_1X_2 - X_2X_1$$

We could see that the commutator is a first-order operator that describes the failure of X_1 and X_2 to commute. Specifically,

$$[X_1, X_2] = (X_1\xi_2 - X_2\xi_1)\partial_x + (X_1\eta_2 - X_2\eta_1)\partial_y$$

We could show that the Lie bracket satisfy the following properties:

- $[\cdot,\cdot]$ is bilinear
- $[\cdot,\cdot]$ is skew-symmetric: $[X,Y] = -[Y,X]$ for all $X, Y \in \mathcal{L}$
- The Jacobi Identity holds:
$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$
for all $X, Y, Z \in \mathcal{L}$

Interestingly, the commutator between two elements in $\mathscr{L}$ is also in $\mathscr{L}$

Theorem 4.11 ([Hyd00] p. 83)**.** *$\mathscr{L}$ is closed under the commutator*

Proof. Omitted. Involves the linearised symmetry condition and algebraic manipulation □

As such commutator of any two generators in the basis is a linear combination of the basis generators:
$$[X_i, X_j] = \sum_{k=1}^{R} c_{ij}^k X_k$$
The constants c_{ij}^k are called structure constants. If $[X_i, X_j] = 0$, the generators X_i, X_j are said to commute.

Example 4.12. Consider example 4.5, there are 8 generators and we could work out the commutation relations as follows. The cell at row i and column j contains $[X_i, X_j]$

		X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
∂_x	X_1	0	0	X_1	0	0	X_2	$2X_3 + X_6$	X_5
∂_y	X_2		0	0	X_2	X_1	0	X_6	$2X_4 + X_5$
$x\partial_x$	X_3			0	0	0	X_6	X_7	0
$y\partial_y$	X_4				0	X_5	0	0	X_8
$y\partial_x$	X_5					0	$X_4 - X_3$	X_8	0
$x\partial_y$	X_6						0	0	X_7
$x^2\partial_x + xy\partial_y$	X_7							0	0
$xy\partial_x + y^2\partial_y$	X_8								0

Table 3. Commutation relations of infinitesimal generators of Lie point symmetries of $y'' = 0$

We could deduce the rest of the table using skew-symmetry of the commutator and read off the structure constants.

5. Introduction to Calculus of Variations

As a large class of physical systems arise from variational principles, they are of deep interest. Particularly, Noether showed that for "systems arising from a variational principle, every conservation law of the system comes from a corresponding symmetry property" [[Olv93] p. 242]. This suggests that ideas about symmetries developed in the previous sections would be of use. In this section, we would briefly recap elementary concepts in calculus of variation and in the next consider some basic variational systems.

Theorem 5.1 (Euler-Lagrange equation for natural boundary condition)**.** *Let* $\mathscr{L}[u]$ *be the functional*

$$\mathscr{L}[u] = \int_a^b L(x, u, u')dx$$

for some smooth function L*. Then the minimizer* $u(x)$ *of* $\mathscr{L}$ *satisify*

$$\frac{\partial L}{\partial u} - \frac{d}{dx}\frac{\partial L}{\partial u'} = 0$$

and

$$\frac{\partial}{\partial u'}L|_{x=a} = \frac{\partial}{\partial u'}L|_{x=b} = 0$$

Proof. The proof is ommitted, but in essence you consider some transformation $\phi : (x, u) \to (x, u + \epsilon\eta)$ for some smooth function η ... □

Remark. $\frac{d}{dx}$ represents the **total derivative** with respect to x. We have previously used D_x to denote this. (Refer to equation 4.1)

This idea is similar to finding critical points in a C^2 function $f : \mathbb{R} \to \mathbb{R}$, where a necessary condition (but not necessary) for a local minimum at $c \in \mathbb{R}$ is $f'(c) = 0, f''(c) \geq 0$ and a sufficient (but not necessary) condition is $f'(c) = 0, f''(c) > 0$. [[GH04] p. 6]

We could have more complicated Lagrangians that depends on higher derivatives (e.g. $L(x, u, u', u'')$), the corresponding Euler-Lagrange equation would expectedly involve higher derivatives of u similar to how we constructed the symmetry conidtion for higher order ODEs [equation 4.7].

Example 5.2 (Calc of Var Lecture Notes)**.** To find the shortest path between (a, b) and (c, d) on the coordinate plane, we want to minimize

$$\mathscr{L}[u] = \int_a^c L(x, u, u')dx$$

over the space of differentiable functions $u = f(x)$ where $L(x, u, u') := \sqrt{1 + u'^2}$ subject to $u(a) = b, u(c) = d$

Since $\frac{\partial L}{\partial u} = 0$, the Euler-Lagrange equation becomes

$$\frac{d}{dx}\frac{\partial L}{\partial u'} = \frac{d}{dx}\frac{u'}{\sqrt{1 + u'^2}} = 0$$

So u' is constant and hence we have a straight line.

6. Variational Symmetries of ODEs

This section largely follows from Chatper 4 of [Olv93] and chapter 2 of [Sun14]

We start off with a toy example: kinematics in one dimension. We would have r representing the position and t representing time.

Assuming we have a point particle under zero external force, we would have the equation $r'' = 0$ from Newton's second law. Note we have already considered a similar equation $\frac{d^2y}{dx^2} = 0$ in example 4.5. Particularly this is the Euler-Lagrange equation with Lagrangian $L(t, r, r') := \frac{1}{2}m(r')^2$.

6.1. **Homogeneity of Time and Energy Conservation.** [[Sun14] p. 35]

If the Lagrangian has no explicit depedence on time, i.e. $\frac{\partial L}{\partial t} = 0$, then we can get a conserved quantity: the Hamiltonian $H = L - r' \frac{\partial L}{\partial r'}$. To see this, consider the total derivative of the Lagrangian with respect to time

$$\begin{aligned}
\frac{dL}{dt} &= 0 + \frac{\partial L}{\partial r} r' + \frac{\partial L}{\partial r'} r'' && \text{chain rule} \\
\frac{dL}{dt} &= \left[\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial r'} + \frac{d}{dt}\frac{\partial L}{\partial r'}\right] r' + \frac{\partial L}{\partial r'} r'' \\
\frac{dL}{dt} &= \frac{d}{dt}\left(\frac{\partial L}{\partial r'}\right) r' + \frac{\partial L}{\partial r'} r'' && \text{Euler-Lagrange equation 5.1} \\
\frac{dL}{dt} &= \frac{d}{dt}\left(r' \frac{\partial L}{\partial r'}\right) && \text{product rule} \\
\therefore 0 &= \frac{d}{dt}\left(L - r' \frac{\partial L}{\partial r'}\right)
\end{aligned}$$

In particular with $L = \frac{1}{2}m(r')^2$ we have $L - r' \frac{\partial L}{\partial r'} = -\frac{1}{2}m(r')^2$, the kinetic energy, consant.

Remark. We could've deduced the above by expressing the Lagrangian as $L(r, \frac{dt}{dr}, \frac{d^2t}{dr^2})$ and then using theorem 6.1

6.2. **Homogeneity of Space and Momentum Conservation.** [[Sun14] p. 35]

If the Lagrnangian has no explicit position dependence, i.e. we have $\frac{\partial L}{\partial r} = 0$. Then by the theorem below, we would have $\frac{\partial L}{\partial r'}$ constant.

Theorem 6.1 (Special case)**.** *If $\frac{\partial L}{\partial r} = 0$ then $\frac{\partial L}{\partial r'}$ is constant on minimisers for the functional $\mathscr{L}$, i.e. $\frac{d}{dt}\frac{\partial L}{\partial r'} = 0$ on minimisers $r(t)$*

Proof. Immendiate from the the Euler-Lagrange eqaution [Theorem 5.1] □

In partciular with $L = \frac{1}{2}m(r')^2$ we have $\frac{\partial L}{\partial r'} = mr'$, the momentum, constant.

6.3. **Variational Symmetries.** What does this have to do with the symmetries we have been considering? Similar to chapter 1, the first step is to understand how transformations act on variational problems and deduce an appropiate symmetry condition.

Suppose we have a diffeomorphism $\phi : (t, r) \to (\hat{t}, \hat{r})$ and we prolongate it to $\phi : (t, r, r') \to (\hat{t}, \hat{r}, \hat{r}')$. We would want this diffeomorphism to preserve "the action of the lagrangian". This motivates the following.

Definition 6.2 ([Olv93] Def 4.10 p. 254 (Simplified))**.** Suppose ϕ is a diffeomorphism from (t, r) to $(\hat{t}, \hat{r})$. If for all intervals $\Omega \subset \mathbb{R}$

$$\int_{\hat{\Omega}} L(\hat{t}, \hat{r}, \hat{r}') d\hat{t} = \int_{\Omega} L(t, r, r') dt$$

where $\hat{\Omega} = \mathrm{Im}_\phi \Omega$ and that $\hat{r}$ is a single valued function defined over $\hat{\Omega}$, then ϕ is a variational symmetry of the functional $\mathscr{L}[r] = \int_\Omega L(t, r, r') dt$

Theorem 6.3. *Variational symmetries are closed under the commutator*

Proof. Omitted □

This is similar to theorem 4.11

Theorem 6.4 ([Olv93] Thm 4.12 p. 254 (Simplified) *or* [Hyd00])**.** *A one-parameter Lie group is a variational symmetry group of the functional if and only if*

$$pr(X)L + LD_t\xi = 0$$

where $(\xi(t,r), \eta(t,r)) := \frac{d}{d\epsilon}|_{\epsilon=0}(\hat{t}, \hat{r})$*, the tangent vector field;* $X := \xi\partial_t + \eta\partial_r$*, the infinitesimal generator;* $pr(X) := \xi\partial_t + \eta\partial_r + \eta^{(2)}\partial_{r_t}$*, the first prolongation of the infinitestimal generator (See 4.10)*

Proof. Omitted □

Example 6.5. With the Lie group of symmetries of form $\phi_\epsilon : (t,r) \to (t+\epsilon, r)$, we have $(\xi,\eta) = (1,0)$ and so $X = \partial_t$ and $pr(X) = \partial_t$. Hence

$$\begin{aligned} pr(X)L + LD_t\xi &= \partial_t L + LD_t(1) \\ &= \partial_t L \end{aligned}$$

So time translation is a symmetry group if and only if the Lagrangian does not depend on time explicitly.

Example 6.6. With the Lie group of symmetries of form $\phi_\epsilon : (t,r) \to (t, r+\epsilon)$, we have $(\xi,\eta) = (0,1)$ and so $X = \partial_r$ and $pr(X) = \partial_r$. Hence

$$\begin{aligned} pr(X)L + LD_t\xi &= \partial_r L + LD_t(0) \\ &= \partial_r L \end{aligned}$$

So position translation is a symmetry group if and only if the Lagrangian does not depend on position explicitly.

Theorem 6.7 ([Olv93] Thm 4.14 p. 255 (Simplified))**.** *If* G *is a variational symmetry group of the functional* $\mathscr{L}[u] = \int_{\Omega_0} L(t,r,r')dt$*, then* G *is a symmetry group of the Euler-Lagrange equations* $L_r - D_x L_{r'} = 0$

Proof. Omitted □

Example 6.8. As time translation and position translation are symmetries of the Lagrangian $\frac{1}{2}(r')^2$, they are symmetries of the Euler-Lagrange equation $r'' = 0$ as well as shown in table 2. However symmetries of Euler-Lagrange equation do not necessarily correspond to symmetries of the Lagrangian. (Consider X_7, X_8 in table 2)

Theorem 6.9 (Noether's theorem (Simplified) [Olv93] Thm 4.30 p. 274)**.** *If* $X = \xi\partial_t + \eta\partial_r$ *is an infinitesimal generator of variational symmetry of the functional* $\mathscr{L}[u] = \int_{\Omega_0} L(t,r,r')dt$ *, then*

$$P = \eta\frac{\partial L}{\partial r'} + \xi\left(L - r'\frac{\partial L}{\partial r'}\right)$$

is a conserved quantity

Proof. Omitted □

Example 6.10. $(\xi,\eta) = (1,0)$ corresponds to homogeneity of time and so $\frac{\partial L}{\partial r'}$ is conserved as in example 6.1

$(\xi,\eta) = (0,1)$ corresponds to homogenity of space and so $L - r'\frac{\partial L}{\partial r'}$ is conserved as in example 6.2

6.4. **Reduction of Order.** We've seen in elementary dynamics that conservation of energy $\frac{1}{2}mr'^2 + V(r) = E,$ constant gives us in principle the ability to solve t as a function of x as

$$t = \pm \int \frac{dr}{(\frac{2}{m}(E - V(r))^{1/2})}$$

This could be understood as a variational symmetry of the Lagrangian $L = \frac{1}{2}mr'^2 - V(r)$ giving rise to a reduction of order in the Euler-Lagrange equation $mr'' = -V'(r)$.

Example 6.11 ([Olv93] Example 4.18 p. 257)**.** If there's homogeneity of time (as in example 6.1), then we know the Hamiltonian $E = L - r'\frac{L}{r'}$ is conserved. This implicitly defines $r' = F(r, E)$ as a function of r and E. We can integrate and recover the solution of the Euler-Lagrange equation

$$\int \frac{dr}{F(r, E)} = t + c$$

which is exactly what we have above when $L = \frac{1}{2}mr'^2 - V(r)$

It is worth noting that it's rather remarkable we have reduced the order of the Euler-Lagrange equation $r'' = -V'(r)$ by two. This is a general property of variational symmetries compared to symmetries of differential equations which only reduce the order by one. [[Olv93] p. 242]

Example 6.12 ([Olv93] Example 4.18 p. 257)**.** If there's homogenity of space (as in example 6.2), then we know the momentum $p = \frac{\partial L}{\partial r'}(t, r')$ is conserved. This implicitly defines $r' = F(t, p)$ as a function of t and p. We can integrate and recover the solution of the Euler-Lagrange equation

$$r = \int F(r, p)dt + c$$

This strategy clearly doesn't work if the potential $V(r)$, but similar ideas might work in higher dimensions (e.g. perservation of angular momentum).

7. Lie Symmetries of PDEs

As in section 1, we begin by looking at how coordinate transformations act on PDEs. Transformation are very useful in understanding partial differential equations as well, as demonstrated by the following example.

Example 7.1 (Classification of second-order linear PDEs)**.** In part A differential equations, we've done the classification of second order semi-linear PDEs of form

$$a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} = f(x, y, u, u_x, u_y)$$

through using some change of variables $\phi : (x, y) \to (\hat{x}, \hat{y})$ with non vanishing Jacobian. We deduced that the PDE is transformed into

$$A(\hat{x}, \hat{y})u_{\hat{x}\hat{x}} + 2B(\hat{x}, \hat{y})u_{\hat{x}\hat{y}} + C(\hat{x}, \hat{y})u_{\hat{y}\hat{y}} = F(\hat{x}, \hat{y}, u, u_{\hat{x}}, u_{\hat{y}})$$

with the following relation

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} = \begin{bmatrix} \hat{x}_x & \hat{y}_x \\ \hat{y}_y & \hat{x}_y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \hat{x}_x & \hat{y}_y \\ \hat{y}_x & \hat{x}_y \end{bmatrix}$$

so that taking determinants we obtain

$$(AC - B^2) = (ac - b^2)\left(\frac{\partial(\hat{x}, \hat{y})}{\partial(x, y)}\right)^2$$

leading to a classification of second-order linear PDEs by the sign of $ac - b^2$ as follows

- $ac < b^2$ hyperbolic e.g. wave equation
- $ac > b^2$ elliptic: e.g. Laplace equation
- $ac = b^2$ parabolic: e.g. heat equation

that is invariant under transformations ϕ

7.1. **Lie Symmetries.** In this section, we shall describe the notation established in [Olv93].

Suppose we are considering a system $\mathscr{S}$ of differential equations involving p independent variables $x = (x^1, \ldots, x^p)$ and q dependent variables $u = (u^1, \ldots, u^q)$. Let $X = \mathbb{R}^p$, with coordinates $x = (x^1, \ldots, x^p)$, be the space representating the independent variables, and let $U = \mathbb{R}^p$, with coordinates $u = (u^1, \ldots, u^q)$, represent the dependent variables. For simplicitly, we would consider only the case of $q = 1$ for the rest of this section.

Then a symmetry group of $\mathscr{S}$ would be a local group of transformations, G, acting on some open subset $M \subset X \times U$ in such a way that "G transforms solutions of $\mathscr{S}$ to other solutions of $\mathscr{S}$". [Direct quote, [Olv93] p. 90]

Somewhat rigorously, we mean that whenever $u = f(x)$ is a solution of $\mathscr{S}$ then whenever $g \cdot f$ is defined for $g \in G$, $u = g \cdot f(x)$ is also a solution of the system. [[Olv93], p. 93]

Similar to the case in ODEs, this necessitates consideration of how symmetries in $X \times U$ act on the partial derivatives.

If $f : X \to \mathbb{R}$ is a smooth function then there are $\binom{p+k-1}{k}$ numbers needed to represent all the different k-th order derivatives of the components of f at a point x by a stars and bars argument. As such, there's a total of $\sum_{k=0}^{n} \binom{p+k-1}{k} = \binom{p+n}{n}$ derivatives of order n or below.

Letting U_k be the Euclidean space consiting of k-th order partial derivatives. We define $U^{(n)}$ to be $U \times U_1 \times \cdots \times U_n$ to represent all derivatives of order n or below.

Example 7.2 ([Olv93], p. 95)**.** Consider $f : \mathbb{R}^2 \to \mathbb{R}$. Then U_1 isomorphic to $\mathbb{R}^2$ with coordinates (u_x, u_y). U_2 isomrophic to $\mathbb{R}^3$ with coordinates (u_{xx}, u_{xy}, u_{yy}). Finally $U^{(2)}$ has coordinates $u^{(2)} = (u; u_x, u_y; u_{xx}, u_{xy}, u_{yy})$.

Similar to the case of ODEs, we now aim to understand the prolongation of vector fields.

Definition 7.3 (Multi-index notation [Olv93] p. 95)**.** Given $J = (j_1, \ldots, j_k)$, an unordered k-tuple of integers, with entries $1 \leq j_k \leq p$ indicating which derivatives are being taken, we write

$$\partial_J f(x) = \frac{\partial^k f(x)}{\partial x^{j_1} \partial x^{j_2} \ldots \partial x^{j_k}}$$

The order of such a multi-index, which we dnote by $\#J = k$, indicated how many derivatives are being taken.

Example 7.4. Given a function $f(x) = f(x^1, x^2, x^3)$ and $J = (1, 2, 3)$ we have $\partial_J f(x) = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3} f(x)$. For $J = (2, 1, 1, 1)$ we have $\partial_J f(x) = \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1} f(x)$

Theorem 7.5 (General Prolongation Formula (for 1-dimensional U) [Olv93] p. 110)**.** *Let*

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \phi\frac{\partial}{\partial u}$$

be a vector field defined on an open subset $M \subset X \times U$. The n-th prolongation of $\mathbf{v}$ is the vector field

$$pr^{(n)}\mathbf{v} = \mathbf{v} + \sum_J \phi^J(x, u^{(n)})\frac{\partial}{\partial u_J} \tag{7.1}$$

defined on the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$, the second summation being over all (unordered) multi-indices $J = (j_1, \dots j_k)$, with $1 \leq j_k \leq p, 1 \leq k \leq n$. The coefficient functions ϕ^J of $pr^{(n)}\mathbf{v}$ are given by the following formula

$$\phi^J(x, u^{(n)}) = D_J\left(\phi - \sum_{i=1}^{p} \xi^i u_i\right) + \sum_{i=1}^{p} \xi^i u_{J,i} \tag{7.2}$$

where $D_J = D_{j_1}D_{j_2}\dots D_{j_k}$, $u_i = \frac{\partial u}{\partial x^i}$, and $u_{J,i} := \frac{\partial}{\partial x_i}(\partial_J u)$

Proof. Omitted $\square$

Remark. The term $\phi - \sum_{i=1}^{p} \xi^i u_i$ is called the characteristic and is of geometric interest. Refer to [Hyd00] or [Olv93] for more details.

Example 7.6. Let X be one dimensional and have coordinate x. We shall show that theorem 7.5 recovers the prolongation formulae for ODEs (equations 4.6 and 4.8).

When $J = (1)$, from equation 7.2 we have

$$\begin{aligned}
\phi^{(1)} &= D_x\left(\phi - \xi\frac{du}{dx}\right) + \xi\frac{d}{dx}\frac{du}{dx} \\
&= D_x\phi - \frac{du}{dx}D_x\xi
\end{aligned}$$

which is equation 4.6 Let J_k denote the k-tuple with k 1s, from equation 7.2 we have

$$\phi^{J_k} = (D_x)^k\left(\phi - \xi\frac{du}{dx}\right) + \xi\frac{d^{k+1}u}{dx^{k+1}} \tag{7.3}$$

We can show by induction that $\phi^{J_k} = D_x\phi^{J_{k-1}} - \frac{d^k u}{dx^k}D_x\xi$ which is equation 4.8. We have already show the base case when $J = (1)$. As such we consider

$$\begin{aligned}
\phi^{J_{k+1}} &= (D_x)^{k+1}\left(\phi - \xi\frac{du}{dx}\right) + \xi\frac{d^{k+2}u}{dx^{k+2}} \\
&= D_x\left(\phi^{J_k} - \xi\frac{d^{k+1}u}{dx^{k+1}}\right) + \xi\frac{d^{k+2}u}{dx^{k+2}} && \text{Equation 7.3} \\
&= D_x\phi^{J_k} - \frac{d^{k+1}u}{dx^{k+1}}D_x\phi
\end{aligned}$$

which concludes the proof

Example 7.7 ([Olv93] p. 114 (Simplified))**.** Let's consider some simple examples of vector fields on the Cartesian plane. Consider $X = \mathbb{R}^2$ and $U = \mathbb{R}$.

Let
$$\mathbf{v} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$
denote some vector field on $X \times U$. Note that the vector fields we are considering have no dependence on U. We can calculate the first prolongation
$$\mathrm{pr}^{(1)}\mathbf{v} = \mathbf{v} + \phi^x \frac{\partial}{\partial u_x} + \phi^y \frac{\partial}{\partial u_y}$$
as follows
$$\begin{aligned} \phi^x &= D_x(-\xi u_x - \eta u_y) + \xi u_{xx} + \eta u_{xy} \\ &= -\xi_x u_x - \eta_x u_y \end{aligned} \tag{7.4}$$
and simiarly
$$\begin{aligned} \phi^y &= D_y(-\xi u_x - \eta u_y) + \xi u_{xy} + \eta u_{yy} \\ &= -\xi_y u_x - \eta_y u_y \end{aligned} \tag{7.5}$$

As such we have
$$\begin{pmatrix} \phi^x \\ \phi^y \end{pmatrix} = -\begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = -\frac{\partial(\xi,\eta)}{\partial(x,y)}^T \begin{pmatrix} u_x \\ u_y \end{pmatrix} \tag{7.6}$$
If $\mathbf{v} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, representating scaling, we have
$$\begin{aligned} \phi^x &= -u_x \\ \phi^y &= -u_y \end{aligned}$$

Geometrically this means that if we scale up the (x, y) plane then the partial derivatives with respect to x and y are scaled down. Explicitly, we can see that the corresponding Lie group has form $\phi_\epsilon : (x, y, u) \to (\hat{x}, \hat{y}, \hat{u}) = (e^\epsilon x, e^\epsilon y, u)$ and so $\frac{\partial \hat{u}}{\partial \hat{x}} = e^{-\epsilon} u_x = u_x - u_x\epsilon + O(\epsilon^2)$

If $\mathbf{v} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, representating anticlockwise rotation, we have
$$\begin{aligned} \phi^x &= -u_y \\ \phi^y &= -u_x \end{aligned}$$

Theorem 7.8 ([Olv93] Thm 2.31 on p. 104)**.** *Suppose $\Delta(x, u^{(n)}) = 0$ is differential equation defined over $M \subset X \times U$. Under sufficiently nice conditions, if G is a local group of transformations acting on M and*
$$\mathrm{pr}\,\mathbf{v}[\Delta(x, u^{(n)})] = 0$$
whenever $\Delta(x, u^{(n)}) = 0$ for every infinitesimal generator $\mathbf{v}$ of G, then G is a symmetry group of the system.

This is analogous to the linearised symmetry condition in equation 4.9

Example 7.9. [Olv93, Example 2.41 on p. 117] The one-dimensional Heat Equation $u_t - u_{xx} = 0$

Let $\Delta(x, t, u^{(2)} = u_t - u_x x)$. Let $\mathbf{v} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}$. We could find the second prolongation of $\mathbf{v}$ to be
$$\mathrm{pr}^2\,\mathbf{v} = \mathbf{v} + \phi^x\frac{\partial}{\partial u_x} + \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}$$

and the function ϕ^x and ϕ^{tt} according to equation 7.2. Finally to find vector fields $\mathbf{v}$ that forms symmetries of the heat equation, we impose the condition $\phi^x - \phi^{tt} = 0$. One could find the detailed calculations in [Olv93]. One would find that the Lie algebra of infinitesimal symmetries of the heat equation is spanned by the seven vector fields

Vector field	$(\hat{x}, \hat{t}, \hat{u})$	$\hat{u}(\hat{x}, \hat{t})$ if $u = f(x,t)$
$\mathbf{v}_1 = \partial_x$	$(x+\epsilon, t, u)$	$f(\hat{x}-\epsilon, \hat{t})$
$\mathbf{v}_2 = \partial_t$	$(x, t+\epsilon, u)$	$f(\hat{x}, \hat{t}-\epsilon)$
$\mathbf{v}_3 = u\partial_u$	$(x, t, e^{\epsilon}u)$	$e^{\epsilon} f(\hat{x}, \hat{t})$
$\mathbf{v}_4 = x\partial_x + 2t\partial_t$	$(e^{\epsilon}x, e^{2\epsilon}t, u)$	$f(e^{-\epsilon}\hat{x}, e^{-2\epsilon}\hat{t})$
$\mathbf{v}_5 = 2t\partial_x - xu\partial_u$	$(x+2\epsilon t, t, u \cdot \exp(-\epsilon x - \epsilon^2 t))$	$e^{-\epsilon\hat{x}+\epsilon^2\hat{t}} f(\hat{x}-2\epsilon\hat{t}, \hat{t})$
$\mathbf{v}_6 = 4tx\partial_x + 4t^2\partial_t - (x^2+2t)u\partial_u$	$(\frac{x}{1-4\epsilon t}, \frac{t}{1-4\epsilon t}, u\sqrt{1-4\epsilon t}\exp(\frac{-\epsilon x^2}{1+4\epsilon t}))$	$\frac{1}{\sqrt{1+4\epsilon\hat{t}}}\exp(\frac{-\epsilon\hat{x}^2}{1+4\epsilon\hat{t}}) f(\frac{\hat{x}}{1+4\epsilon\hat{t}}, \frac{\hat{t}}{1+4\epsilon\hat{t}})$
$\mathbf{v}_\alpha = \alpha(x,t)\partial_u$	$(x, t, u+\epsilon\alpha(x,t))$	$f(\hat{x}, \hat{t}) + \epsilon\alpha(\hat{x}, \hat{t})$

TABLE 4. Symmetries of $u_t - u_{xx} = 0$

In the vector field $\mathbf{v}_\alpha$, $\alpha(x,t)$ is any other solution to the heat equation. Combined with $\mathbf{v}_3$, they reflect the linearity of the heat equation. $\mathbf{v}_1$ and $\mathbf{v}_2$ reflect the time- and space-invariance of the equation. $\mathbf{v}_4$ reflects the scaling symmetry of the heat equation: one could check that if $u = f(x,t)$ is a solution to the heat equation then

$$
\begin{aligned}
&(\partial_t - \partial_{xx})(f(e^{-\epsilon}x, e^{-2\epsilon}t)) \\
&= e^{-2\epsilon}(f_t(e^{\epsilon}x, e^{-2\epsilon}t) - f_{xx}(e^{\epsilon}x, e^{-2\epsilon}t)) \\
&= 0
\end{aligned}
$$

so $u = f(e^{\epsilon}, e^{-2\epsilon t})$ is a solution as well.

$\mathbf{v}_5$ represents a kind of Galilean boost to a moving coordinate frame. Again one could explicitly check that

$$
\begin{aligned}
&(\partial_t - \partial_{xx})(e^{-\epsilon x+\epsilon^2 t} f(x-2\epsilon t, t)) \\
&= e^{-\epsilon x+\epsilon^2 t}(f_t(x-2\epsilon t, t) - f_{xx}(x-2\epsilon t, t))
\end{aligned}
$$

$\mathbf{v}_6$ is similar to the vector field $xy\partial_x + y^2\partial_y$ in the symmetries of $y'' = 0$ (Table 2). The vector field is of particular interest as it acts on the constant solution $u(x,t) = c$ for any constant c to form fundamental solutions $\hat{u}(\hat{x}, \hat{t}) = \frac{c}{\sqrt{1+4\epsilon\hat{t}}} \exp \frac{-\epsilon\hat{x}^2}{1+4\epsilon t}$.

Example 7.10. [Olv93, Example 2.42 on p. 123 (Simplified)] The one-dimensional Wave equation $u_{tt} - u_{xx} = 0$

Following a similar calculation to the last example, we deduce that the Lie algbera of infinitestimal symmetries of the wave equation is spanned by the following vector fields

Vector field	$(\hat{x}, \hat{t}, \hat{u})$	$\hat{u}(\hat{x}, \hat{t})$ if $u = f(x,t)$
$\mathbf{v}_1 = \partial_x$	$(x+\epsilon, t, u)$	$f(\hat{x}-\epsilon, \hat{t})$
$\mathbf{v}_2 = \partial_t$	$(x, t+\epsilon, u)$	$f(\hat{x}, \hat{t}-\epsilon)$
$\mathbf{r}_{xt} = t\partial_x + x\partial_t$	$(x\cosh\epsilon + t\sinh\epsilon, x\sinh\epsilon + t\cosh\epsilon, u)$	Omitted
$\mathbf{d} = x\partial_x + t\partial_t$	$(e^{\epsilon}x, e^{\epsilon}t, u)$	$f(e^{-\epsilon}\hat{x}, e^{-\epsilon}\hat{t})$
$\mathbf{i}_x = (x^2+t^2)\partial_x + 2xt\partial_t - xu\partial_u$	Omitted	Omitted
$\mathbf{i}_t = 2xt\partial_x + (x^2+t^2)\partial_t - tu\partial_u$	Omitted	Omitted
$\mathbf{v}_3 = u\partial_u$	$(x, t, e^{\epsilon}u)$	$e^{\epsilon} f(x,t)$
$\mathbf{v}_\alpha = \alpha(x,y,t)\partial_u$	$(x, t, u+\epsilon\alpha(x,t))$	$f(\hat{x}, \hat{t}) + \epsilon\alpha(\hat{x}, \hat{t})$

TABLE 5. Symmetries of the wave equation

To properly understand the hyperbolic rotation $\mathbf{r}_{xt}$ and the inversions $\mathbf{i}_x, \mathbf{i}_t$ would involve knowledge in hyperbolic geometry / Minkowski space / Lorentz group which is beyond the scope of this article.

Example 7.11 ([Gün19], Example 2.5 on p. 12)**.** The two-dimensional Laplace Equation $u_{xx} + u_{yy} = 0$

We would show that it is invariant under the symmetry group generated by the vector field $\mathbf{v} = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ where ξ and η satisify the Cauchy-Riemann equations $\xi_x = \eta_y, \xi_y = -\eta_y$. This is expected from results established in complex analysis.

In essence we need to show that $\phi^{xx} + \phi^{yy} = 0$. By equation 7.2 we have

$$\begin{aligned}\phi^{xx} &= D_x D_x(-\xi u_x - \eta u_y) + \xi u_{xxx} + \eta u_{yxx} \\ &= D_x(-\xi_x u_x - \eta_x u_y - \xi u_{xx} - \eta u_{yx}) + \xi u_{xxx} + \eta u_{yxx} \\ &= -\xi_{xx} u_x - \eta_{xx} u_y - 2\xi_x u_{xx} - 2\eta_x u_{xy}\end{aligned}$$

Similarly $\phi^{yy} = -\eta_{yy} u_y - \xi_{yy} u_x - 2\eta_y u_{yy} - 2\xi_y u_{xy}$. So the infinitesimal criterion (Thm 7.8)

$$\phi^{xx} + \phi^{yy} = -2\xi_x(u_{xx} + u_{yy}) = 0$$

is satisified on the solution surface.

One could refer to [Gün19] for a more comprehensive discussion on the symmetries of the Laplace equation.

Further Directions

Extensions. Symmetries of ODEs could be used to derive first integrals. Solutions invariant under symmetries are also of interest. [Hyd00]

Theorem 7.5 and 7.8 readily generalises to higher dimesnions. Variational symmetries of PDEs could also be studied. [Olv93]

Other directions. Seperation of Variables techniques used in solving differential equations could be studied using Lie group theory [Mil84].

Recently, symmetries of differential equations have also been used to create model selection criterion for biological properties [OBC20][BP22].

One could also refer to an intrtoductory philosophical text on the concept of symmetries in mathematics and physics [Bel13, Chapter 8, 9].

Supplementary Notes

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