The Isoperimetric Inequality
Part B Extended Essay

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Abstract. We discuss some results in the study of the isoperimetric inequality and curve shortening flow that share a similar theme. We discuss the relationship between the isoperimetric inequality, Wirtinger's inequality and Sobolev inequality. We also discuss the support function method and the level set method of attacking curve shortening flow.

Preface

The isoperimetric inequality states that among all simple closed plane curves of given length L, the area it encloses, A, is bounded as follows

$$L^2 \geq 4\pi A.$$

Meanwhile, curve shortening flow modifies simple closed curves by moving its points along the normal direction with speed proportional to its curvature. It is remarkably the gradient flow for arclength and it also shrinks the area enclosed by the curve at a constant rate. This property quickly leads to a proof of the isoperimetric inequality.

Due to the intimate relationship between curve shortening flow and the isoperimetric inequality outlined above, it is expected that a lot of methods used to tackle the isoperimetric inequality would appear in similar forms to tackle the curve shortening flow. We would highlight this in the essay. Along the way, we would expose common tools used in geometric analysis.

The target audience would be third year undergraduates who are interested in the geometry underpinning the study of geometric flows / geometric analysis.

Acknowledgements

We thank Jason D. Lotay for supervising the essay and providing comprehensive comments—picking up countless errors we would never have spotted ourselves. We also thank the Mathematical Institute for administrative support and Joshua Lau for comments.

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CHAPTER 1

Introduction

Motivation

The isoperimetric inequality goes back to ancient times but continues to fuel mathematical research today. This is because several crucial inequalities in geometric analysis could be shown to be equivalent to the isoperimetric inequality, such as the Sobolev inequality and Poincaré inequality. Its ubiquity has made proving it a good toy problem for new techniques and theories.

Meanwhile, curve shortening flow could be viewed as one of the simplest examples of geometric flows. Much of the theory about modern geometric flows, which have been used to solve important problems in geometry, has roots back in curve shortening flow.

The fact that curve shortening flow could result in a quick proof of the isoperimetric inequality hints at a deep relationship between them. This relationship is perhaps best exemplified by the Gage-Hamilton theorem in [GH86] that states for closed convex planar curves evolving under curve shortening flow, the isoperimetric ratio L^2/A approaches 4π and as such it makes convex curves circular and shrink to a point in finite time. Gage introduced an inequality similar to the isoperimetric inequality that plays a part in the proof of the theorem. We would discuss this in chapter 5.

Literature

Extensive literature exists for both the isoperimetric inequality and curve shortening flow.

For the isoperimetric inequality, we refer mostly to expository texts by Osserman [Oss78] and Gromov [MSG01, Appendix I]. Other than that, there's an excellent introductory exposition on the isoperimetric problem by Blåsjö [Blå05] which discusses a variety of ideas beyond the scope of this essay.

For curve shortening flow, we refer mostly to [ES91] for the level set method and [Zhu02] for the support function method. We use [And+20] as well.

Overview

Chapter 2. We briefly introduce the isoperimetric inequality and curve shortening flow and a quick proof of the isoperimetric inequality using the flow.

Chapter 3. We make use of some theory in integral geometry to prove the isoperimetric inequality and some properties of curve shortening flow.

Chapter 4. We make use of some theory in the geometry of surfaces and geometric measure theory to understand the level set method of curve shortening flow, as well as mention the proof of Sobolev inequality from isoperimetric inequality which shares similar ideas.

Chapter 5. We briefly discuss the monotonicity of the isoperimetric ratio under curve shortening flow and inequalities related to the isoperimetric inequality.

Independent work

We wish to draw attention to the following calculations and proofs which are our own independent work.

- Heuristics for the evolution of arclength and area under curve shortening flow [Figure 1]
- Deriving arclength from the support function using Crofton's formula [Theorem 3.7]
- Proof of the isoperimetric inequality via Wirtinger's inequality applied on the support function [Chapter 3, Section 5]
- Deduction of geometric properties of the level set method for curve shortening flow using techniques from the geometry of surfaces [Proof of theorem 4.1, 4.5]
- Proof of enclosed area decreasing at a constant rate under curve shortening flow using the co-area formula [Theorem 4.8]

CHAPTER 2

Preliminaries

This essay is concerned with the isoperimetric inequality (for curves in $\mathbb{R}^2$), which states that among all simple closed curves in $\mathbb{R}^2$ with given length L, the area enclosed A is bounded by the following inequality

$$L^2 \geq 4\pi A.$$

1. Curve shortening flow

Curve shortening flow is the evolution of a family of closed curves $\Gamma : S^1 \times [0, T) \to \mathbb{R}^2$ satisfying the following evolution equation

$$\frac{\partial \Gamma}{\partial t} = \kappa \mathbf{n}$$

where κ is the curvature of Γ with respect to the unit inward normal vector $\mathbf{n}$.

EXAMPLE 2.1 (Shrinking circles). Consider shrinking circles centred at the origin. As the curvature is uniform on all points of a circle, circles are self-similar solutions to the flow. The evolution reduces to an ODE for the radius, $\frac{d}{dt} r = -1/r$, which has solution with $r(0) = R$ given by $r(t) = \sqrt{R^2 - 2t}, t \in (-\infty, R^2/2)$. As such, the solution exists for finite time until extinction at a point.

We will assume that curve shortening flow exists and is unique. Proofs of this can be found extensively in [And+20, Chapter 3].

Through analyzing how arclength and enclosed area evolve under curve shortening flow, we can deduce the isoperimetric inequality. We could show that the arclength of Γ, L, and the enclosed area of Γ, A, evolves as follows:

$$\frac{dL}{dt} = -\int_{\Gamma_t} \kappa^2 \, ds, \qquad \frac{dA}{dt} = -\int_{\Gamma_t} \kappa \, ds = -2\pi.$$

We will show this rigorously later, but for the moment we like to show why this is the case heuristically for the case of shrinking circles. The strategy is as follows, we break down the circle into infinitesimal sectors, each one contributing to both the area and the arc length. By understanding how each infinitesimal sector evolve, we recover the total evolution of arc length and area.

We use the synonym $f = \kappa$ to denote the normal velocity of the curve for clarity while using κ to denote all other appearances of curvature. We use Γ_t to denote the circle at time t. We also use A_t, L_t to denote the area and arc length

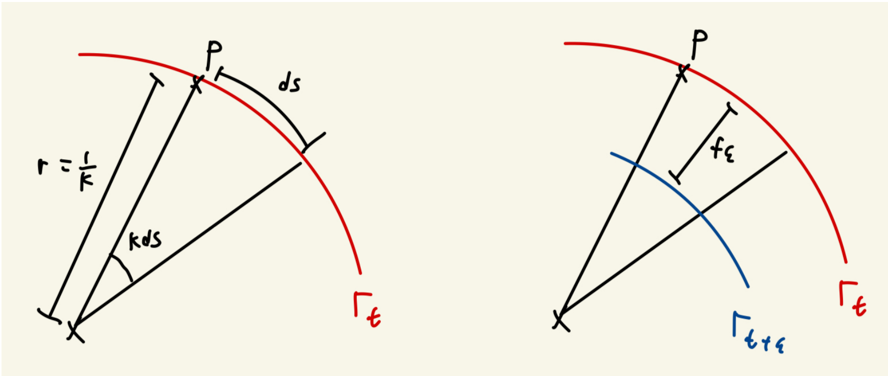

FIGURE 1.

of Γ_t. Focusing at some point p in Γ_t for some t, we consider an infinitesimal arclength element ds, the osculating circle (which is itself), and the sector of the osculating circle associated with the arc ds. The sector will have a radius of $1/\kappa$ and an angle of $\kappa\, ds$. We could see this in figure 1.

Increasing time by some small ϵ, the radius of the $\Gamma_{t+\epsilon}$ will have shrunk by approximately $f\epsilon$. The area of a sector is half of its radius squared times the angle. As such,

$$A_{t+\epsilon} - A_t \approx \int_{\Gamma_t} \left[\frac{k\, ds}{2} \left(\frac{1}{k} - f\epsilon \right)^2 - \frac{k\, ds}{2} \left(\frac{1}{k} \right)^2 \right].$$

Discarding second-order epsilon terms we have

$$A_{t+\epsilon} - A_t \approx \int_{\Gamma_t} -f\epsilon\, ds.$$

Meanwhile the arclength of a sector is its radius times the angle. Hence,

$$\begin{aligned} L_{t+\epsilon} - L_t &\approx \int_{\Gamma_t} \left[k\, ds \left(\frac{1}{k} - f\epsilon \right) - ds \right] \\ &= \int_{\Gamma_t} -f\kappa\epsilon\, ds. \end{aligned}$$

Taking the approximation $(L_{t+\epsilon} - L_t)/\epsilon \approx dL/dt$ and similarly for A, we have

$$\frac{dL}{dt} = -\int_{\Gamma_t} f\kappa\, ds, \qquad \frac{dA}{dt} = -\int_{\Gamma_t} f\, ds.$$

These arguments will still hold heuristically for curves that are not circles.

The above reveals that the quantity $\kappa\, ds$ might be of interest. We will see later that this quantity is related to the turning / normal angle of the curve. This will be further explored in the next chapter.

2. Deducing the isoperimetric inequality from curve shortening flow

With the evolution of arclength and enclosed area under curve shortening flow at hand, we could prove the isoperimetric inequality. Following an argument in [Top98, p. 50], consider

$$-\frac{d}{dt}(L^2) = -2L\frac{dL}{dt} = 2\int_{\gamma_t} ds \int_{\Gamma_t} \kappa^2\, ds \geq 2\left(\int_{\Gamma_t} \kappa\, ds\right)^2 = -4\pi\frac{dA}{dt}$$

where we have used the Cauchy-Schwarz inequality with respect to the inner product $\langle f, g\rangle := \int fg$ in the middle.

By the Gage-Hamilton theorem, we know that there exists some time T such that Γ shrinks to a point (which has zero length and area). Integrating both sides of the inequality between $t = 0$ and $t = T$, we recover the isoperimetric inequality.

3. Further assumptions

For the sake of simplicity, we will mostly concern ourselves with smooth and convex curves. Beyond simplicity, this assumption is motivated by the following. Given any curve Γ, its convex hull will have a greater area and lower perimeter. Thus we only need to consider the isoperimetric inequality restricted to convex curves. Furthermore, it is known that curve shortening flow turns any nonconvex curve convex in finite time by Grayson's theorem [And+20, Theorem 3.19].

CHAPTER 3

Integral geometry and Wirtinger's inequality

In this chapter, we will seek an intuitive and rigorous way to prove that the arclength evolves like $\frac{dL}{dt} = -\int_{\Gamma_t} \kappa^2\, ds$ under curve shortening flow. The main idea is that we first develop a better notion of calculating arclength using theory from integral geometry. In particular, we use Crofton's formula which relates the length of a curve to the expected number of times a "random" line intersects it. But what do we mean by "random" lines? We will dedicate the section below to establish a notion of random lines.

1. Invariant measures

To start off, we consider the toy problem of finding the area of a set of points, X, using a measure $\iint_X f(x,y)\, dxdy$ for some $f(x,y)$. Clearly, we expect $f(x,y)$ to be some constant, but could we deduce this from some universal property? We know that this measure should be invariant under translations and rotations of X and it turns out that this is a crucial property.

THEOREM 3.1. *If $\int_X f(x,y)\, dxdy$ is invariant under translations and rotations, i.e. given any translation / rotation map τ the following holds*

$$\int_X f(x,y)\, dxdy = \int_{\tau(X)} f(x^*,y^*)\, dx^*dy^*, \tag{3.1}$$

then $f(x,y)$ is a constant.

PROOF. Let $\tau : (x,y) \to (a + x^* \cos\alpha - y^* \sin\alpha, b + x^* \sin\alpha + y^* \cos\alpha)$ for some x, y and α in $\mathbb{R}$ be a member of the group of translations and rotations in $\mathbb{R}^2$. We can compute that the Jacobian $\frac{\partial(x,y)}{\partial(x^*,y^*)} = 1$. So by change of variables

$$\int_X f(x,y)\, dxdy = \int_{\tau(X)} f(x(x^*,y^*), y(x^*,y^*))\, dx^*dy^*. \tag{3.2}$$

Now by equation 3.1 and 3.2, we have

$$\int_{\tau(X)} f(x(x^*,y^*), y(x^*,y^*))\, dx^*dy^* = \int_{\tau(X)} f(x^*,y^*)\, dx^*dy^*.$$

As this is true for any set $\tau(X)$ it must be the case that $f(x,y) = f(x^*,y^*)$. Now since (x,y) can be translated to any other point (x^*,y^*), $f(x,y)$ must be a constant. $\square$

We will use the same strategy for straight lines. We first introduce some coordinates of straight lines and show that they give rise to a natural invariant measure.

DEFINITION 3.2 (Normal coordinates of straight lines [SK04, Equation 2.1]). A straight line l on a plane can be determined by its normal coordinates $p, \varphi \in \mathbb{R}$. The equation of l is

$$x \cos \varphi + y \sin \varphi = p.$$

THEOREM 3.3. *Let X be a set of straight lines. If $\int_X f(p, \varphi)\, dpd\varphi$ is invariant under translations and rotations, i.e. given any translation / rotation map τ the following holds*

$$\int_{\tau(X)} f(p, \varphi)\, dpd\varphi = \int_X f(p, \varphi)\, dpd\varphi.$$

then $f(p, \varphi)$ is a constant.

PROOF (SKETCH). Following the strategy of the proof of theorem 3.1, let $\tau : (x, y) \to (a + x^* \cos \alpha - y^* \sin \alpha, b + x^* \sin \alpha + y^* \cos \alpha)$ for some x, y and α in $\mathbb{R}$ be a member of the group of translations and rotations in $\mathbb{R}^2$. The straight line $x \cos \phi + y \sin \phi = p$ is mapped to $x^* \cos(\phi - \alpha) + y^* \sin(\phi - \alpha) - (p - a \cos \phi - b \sin \phi) = 0$ under τ. Using the normal coordinates we see that

$$\begin{aligned} p^* &= p - a \cos \phi - b \sin \phi, \\ \phi^* &= \phi - \alpha. \end{aligned}$$

and we can check that the Jacobian $\frac{\partial(p^*, \phi^*)}{\partial(p, \phi)} = 1$. Following similar arguments in the proof of theorem 3.1, we can deduce that $f(p, \phi)$ is a constant. □

Hence, normal coordinates are "natural" coordinates for straight lines.

2. Crofton's formula

THEOREM 3.4 (Crofton's formula [SK04, Equation 2.13]). *Given a line l determined by its normal coordinates, let $n_\Gamma(l)$ be the number of points at which Γ and l intersect. Integrating over all lines with respect to their normal coordinates, we have*

$$L = \frac{1}{2} \int_{\varphi=0}^{2\pi} \int_{p=0}^{\infty} n_\Gamma(\varphi, p)\, dpd\varphi.$$

EXAMPLE 3.5 (Arc length of line segments). Consider the curve $\Gamma = \{(x, 0) | x \in [0, 1]\}$. Lines with normal coordinates

$$\varphi \in (0, \pi/2) \cup (3\pi/2, 2\pi), p \in [0, \cos \varphi]$$

will intersect with the curve exactly once, while lines parallel to Γ form a null set and could be ignored in the integral. All other lines will not intersect with the curve. As such by Crofton's the arc length is

$$\frac{1}{2}\int_{\varphi=0}^{\pi/2} \cos\varphi \, dp + \frac{1}{2}\int_{\varphi=3\pi/2}^{2\pi} \cos\varphi \, dp = 1.$$

A rigorous proof of Crofton's formula could be found in [SK04]. Heuristically, as both sides of Crofton's formula are additive over the concatenation of curves and invariant under rotations and translations by theorem 3.3, we could approximate smooth curves with line segments to justify the formula.

3. Support function

Crofton's formula motivates parametrization of smooth closed strictly convex curves using the support function, enabling a cleaner derivation of arclength.

Definition 3.6 (Support function [Fla68]). Let Γ be a smooth closed strictly convex curve. The support function of Γ is a function $S : S^1 \to \mathbb{R}$ given by

$$S(\theta) = \Gamma(\theta) \cdot (\cos\theta, \sin\theta)$$

where $\Gamma(\theta)$ is the point on Γ such that its unit outward normal vector is $(\cos\theta, \sin\theta)$. θ is called the normal angle at $\Gamma(\theta)$.

Remark. Sometimes, the support function is defined on points of the curve Γ rather than on the normal angle. This is the approach taken in [And+20, Equation 2.31].

The point $\Gamma(\theta)$ exists and is unique due to the fact that Γ is a smooth closed strictly convex[1] curve. This suggests that we can parametrize Γ with respect to its normal angle as well. Writing $\Gamma : S^1 \to \mathbb{R}^2$ as (Γ^1, Γ^2), we have $S(\theta) = \Gamma^1(\theta)\cos\theta + \Gamma^2(\theta)\sin\theta$.

We now connect the support function with Crofton's formula.

Theorem 3.7. *Let Γ be a smooth, strictly convex and closed curve, then its arclength L is given by*

$$L = \int_0^{2\pi} S(\varphi) d\varphi.$$

Proof via Crofton's formula. Firstly, notice that the integral of $S(\phi)$ over $[0, 2\pi]$ is invariant under translations of Γ: If Γ is translated by (a, b) to form $\tilde{\Gamma}$, then the integral of the support function of $\tilde{\Gamma}$ over $[0, 2\pi]$ is

$$\int_0^{2\pi} (\Gamma^1 + a)\cos\varphi + (\Gamma^2(\varphi) + b)\sin\theta = \int_0^{2\pi} S(\varphi).$$

[1]As the support function is constant on line segments, strict convexity is required to ensure that there are no line segments in the curve and hence the injectivity of the support function

As such, without loss of generality assume that the origin is contained within Γ. This implies that the support function $S(\varphi)$ is nonnegative for all φ. Now we could express $n_\Gamma(\varphi, p)$ as follows.

$$n_\Gamma(\varphi, p) = \begin{cases} 2 & \text{if } S(\varphi) > p > 0 \\ 1 & \text{if } p = S(\varphi) \\ 0 & p > S(\varphi) \end{cases}$$

Hence by Crofton's formula, the arclength is

$$L = \frac{1}{2} \int_{\varphi=0}^{2\pi} \int_{p=0}^{S(\varphi)} 2\, dp d\varphi = \int_0^{2\pi} S(\varphi) d\varphi.$$

□

4. Curvature and enclosed area

Consider the support function and its derivative

$$\begin{aligned} S(\theta) &= \Gamma^1(\theta)\cos\theta + \Gamma^2(\theta)\sin\theta, \\ \frac{dS}{d\theta} &= -\Gamma^1(\theta)\sin\theta + \Gamma^2(\theta)\cos\theta. \end{aligned}$$

This implies we could express the coordinates of Γ in terms of θ by

$$\begin{aligned} \Gamma^1 &= S\cos\theta - \frac{dS}{d\theta}\sin\theta, \\ \Gamma^2 &= S\sin\theta + \frac{dS}{d\theta}\cos\theta. \end{aligned}$$

So all geometric properties of Γ could be expressed by the support function and its derivatives. This theoretically justifies that we could express the curvature and enclosed area of Γ in terms of the support function by substitution. However, geometric intuition will be lost. We will use more geometric arguments adapted from [Fla68].

The idea is that we can relate the curvature, the arc length and the turning angle in one succinct formula $\frac{d\theta}{ds} = \kappa$ and exploit it to express the curvature in terms of the support function.

Letting $\mathbf{t} = (-\sin\theta, \cos\theta)$ be the unit tangent (counterclockwise) vector and $\mathbf{n} = (-\cos\theta, -\sin\theta)$ be the unit inward normal vector. From the Frenet-Serret formulas we have

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n}, \quad \frac{d\mathbf{n}}{ds} = -\kappa\mathbf{t}.$$

If we parametrized Γ with θ instead, then we get

$$\frac{d\mathbf{t}}{d\theta} = \mathbf{n}, \quad \frac{d\mathbf{n}}{d\theta} = -\mathbf{t}.$$

as such $d\theta = \kappa ds$. We could see this in figure 2 similar to figure 1. We also note that

$$\frac{d\Gamma}{d\theta} = \frac{ds}{d\theta}\frac{d\Gamma}{ds} = \frac{1}{\kappa}\mathbf{t}.$$

The support function is now

$$S(\theta) = \langle \Gamma(\theta), -\mathbf{n} \rangle$$

and we can check that

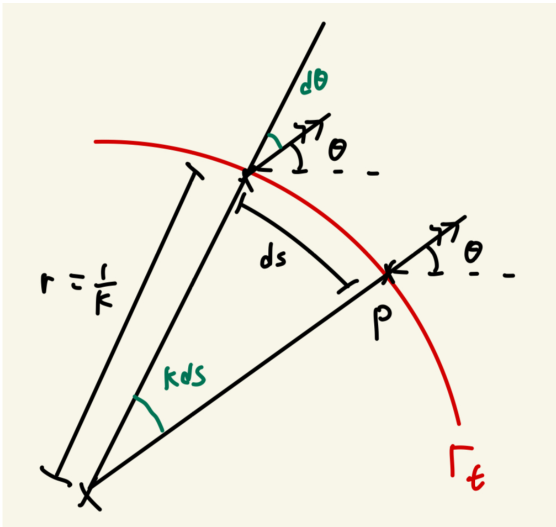

Figure 2.

$$\frac{d^2S}{d\theta^2} + S = \frac{1}{\kappa}. \tag{3.3}$$

Finally from Green's theorem, we can deduce a formula for the enclosed area.

Theorem 3.8 ([Fla68, p. 2.10]).

$$A = -\frac{1}{2}\oint \Gamma(s)\cdot \mathbf{n}\, ds = \frac{1}{2}\int_0^{2\pi} \frac{S}{\kappa}\, d\theta = \frac{1}{2}\int_0^{2\pi} S^2 - \left(\frac{dS}{d\theta}\right)^2 d\theta.$$

Remark. Along similar means, we could derive the expression for arclength in theorem 3.7 without the use of Crofton's formula.

5. Wirtinger's inequality

Now we introduce an analytic inequality that will lead to the isoperimetric inequality.

Theorem 3.9 (Wirtinger's inequality). *If $S(\theta)$ is a smooth function with period 2π and if $\int_0^{2\pi} S(\theta) = 0$, then*

$$\int_0^{2\pi} \left(\frac{dS}{d\theta}\right)^2 d\theta \geq \int_0^{2\pi} S^2 \, d\theta.$$

Proof of isoperimetric inequality. Given some support function $S(\theta)$ for some smooth strictly convex closed curve, consider $\tilde{S}(\theta) = S(\theta) - \frac{L}{2\pi}$ which satisfies $\int_0^{2\pi} \tilde{S}(\theta) = 0$. Now we could apply Wirtinger's inequality to $\tilde{S}$ and get

$$\int_0^{2\pi} \left(\frac{d\tilde{S}}{d\theta}\right)^2 d\theta \geq \int_0^{2\pi} \tilde{S}^2 \, d\theta.$$

Substituting $\tilde{S} = S - \frac{L}{2\pi}$ we obtain

$$\int_0^{2\pi} \left(\frac{dS}{d\theta}\right)^2 d\theta \geq \int_0^{2\pi} \left(S - \frac{L}{2\pi}\right)^2 d\theta.$$

Expanding the right-hand side and using $\int_0^{2\pi} S(\theta)\, d\theta = L$ from theorem 3.7 yields

$$\int_0^{2\pi} \left(\frac{dS}{d\theta}\right)^2 \geq \int_0^{2\pi} S^2 \, d\theta - \frac{L^2}{2\pi}.$$

Reodering the terms we get

$$\frac{L^2}{2\pi} \geq \int_0^{2\pi} S^2 - \left(\frac{dS}{d\theta}\right)^2 d\theta.$$

Finally we use theorem 3.8 to prove the isoperimetric inequality

$$L^2 \geq 4\pi A.$$

□

Remark. Wirtinger's inequality is more commonly applied to the Cartesian or polar coordinates functions of Γ to derive the isoperimetric inequality. (respectively mentioned in [Oss78, Lemma 1.2] and [Pre10, Proposition 3.2.3]).

Remark. Wirtinger's inequality could be proven using Fourier analysis. This suggests that we could write the cartesian coordinates of Γ as periodic functions, expand them as Fourier series and prove the isoperimetric inequality that way. This is indeed the strategy presented by Hurwitz from 1902. [Hur02]

Remark. A purely integral geometric proof of the isoperimetric inequality is presented in [SK04, p.38], which does not make use of Wirtinger's inequality. Meanwhile, these integral geometric techniques could also be extended to prove Minkowski's inequality for closed convex curves in [Fla68]. It is worth noting that

Minkowski's inequality could be used to prove the isoperimetric inequality which is covered extensively in [SS05, Section 3.4].

6. Curve shortening flow

We will be able to deduce the evolution of geometric quantities under curve shortening flow using the theory of support functions following arguments presented in [Zhu02, Chapter 1].

Suppose that $\Gamma : S^1 \times [0,T) \to \mathbb{R}^2$ evolves under curve shortening flow $\partial_t \Gamma = \kappa \mathbf{n}$. Using the normal angle to parametrize each curve Γ_t, we have

$$\tilde{\Gamma}(\theta,t) = \Gamma(u(\theta,t),t).$$

By chain rule we have,

$$\partial_t \tilde{\Gamma} = (\partial_u \Gamma)(\partial_t u) + \kappa \mathbf{n}.$$

As the support function satisfies $S(\theta,t) = \langle \tilde{\Gamma}(\theta,t), -\mathbf{n} \rangle$ we have

$$\partial_t S = \langle \tilde{\Gamma}, -\kappa \mathbf{t} \rangle + \langle (\partial_u \Gamma)(\partial_t u) + \kappa \mathbf{n}, -\mathbf{n} \rangle.$$

But $\tilde{\Gamma}$ is perpendicular to $\mathbf{t}$ and $\partial_u \Gamma$ is perpendicular to $\mathbf{n}$, so only $\langle \kappa \mathbf{n}, -\mathbf{n} \rangle = -\kappa$ remains. So

$$\partial_t S = -\kappa \tag{3.4}$$

and by equation 3.3

$$\partial_t S = -\frac{1}{S + S_{\theta\theta}}.$$

Thus we have reduced curve shortening flow of smooth strictly convex closed curves into a second-order parabolic equation. Now we can employ standard theories of parabolic equations to attack it. This is further elucidated in [Zhu02].

Example 3.10 (Shrinking circles). Consider shrinking circles centred at the origin, the support function is exactly the radius and we can check it satisfies the same differential equation

$$\frac{d}{dt} S = -\frac{1}{S + S_{\theta\theta}} = -\frac{1}{S}$$

as $S_\theta = 0$. If we translated the circles by some (a,b), then the new support function will be $\tilde{S} = S + a\cos\theta + b\sin\theta$ but $\tilde{S}_{\theta\theta} = -a\cos\theta - b\sin\theta$ so our analysis is unchanged.

Finally, we can deduce the evolution of arclength. Using $d\theta = \kappa\, ds$,

$$\frac{d}{dt}\int_{\Gamma_t} ds = \frac{d}{dt}\int_0^{2\pi} \frac{1}{\kappa}\, d\theta.$$

Interchange integration and differentiation in the right hand side, then by equation 3.3 we get

$$\frac{d}{dt}\int_{\Gamma_t} ds = \int_0^{2\pi} \partial_t(S + S_{\theta\theta})\, d\theta.$$

Swap ∂_t and ∂_θ, then by the evolution equation 3.4 we get

$$\begin{aligned}\frac{d}{dt}\int_{\Gamma_t} ds &= \int_0^{2\pi} -\kappa - \kappa_{\theta\theta}\, d\theta \\ &= -\int_0^{2\pi} \kappa\, d\theta \\ &= -\int_{\Gamma_t} \kappa^2\, ds.\end{aligned}$$

We could deduce further that $\frac{dA}{dt} = -2\pi$. (The calculation is done in [Zhu02, Equation 1.4]. We will deduce the evolution of the area using better methods in the next chapter). Now we could appeal to chapter 2, section 2 to obtain a proof of the isoperimetric inequality.

Remark. With more work, the Gage-Hamilton theorem for strictly convex closed curves could be obtained with the support function method in [Zhu02, Theorem 1.4]

CHAPTER 4

Level set method and Sobolev inequality

The level set approach of understanding mean curvature flow was introduced in [ES91]. Instead of considering $\{\Gamma_t\}_{t\geq 0}$ as a family of curves, we consider them as the level sets of some function $g(x,y)$ such that $\Gamma_t = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = t\}$. We will seek to deduce the geometric features of this approach using techniques from the geometry of surfaces.

Meanwhile, it is well known that the Sobolev inequality, a crucial tool in functional analysis, is equivalent to the isoperimetric inequality. Through applying isoperimetric inequality to level sets, we could deduce the Sobolev inequality.

These two ideas both make use of the co-area formula, a generalisation of Fubini's theorem that could be adapted to level sets. For curve shortening flow, the formula gives us a natural and direct proof of why the enclosed area decreases at a constant rate. Meanwhile, the formula is an essential ingredient in the proof of the Sobolev inequality from the isoperimetric inequality.

1. Curvature

We will begin understanding the geometric features of the level set method. One of the initial difficulties will be expressing the curvature of each of the curves Γ_t with respect to g only. We will dedicate the rest of this section to resolving it. In [ES91], it was resolved by the use of the following theorem.

Theorem 4.1. *[ES91, p.638 eq 2.2] Let ν be a smooth unit normal vector field to $\{\Gamma_t\}_{t\geq 0}$. Then*

$$-div(\nu)\nu$$

is the curvature vector field.

We could see it in action with a simple example.

Example 4.2 (Circles). Let $\Gamma_t = \{(t\cos\theta, t\sin\theta) \mid \theta \in \mathbb{R}\}$ for $t > 0$. $\nu = \frac{1}{\sqrt{x^2+y^2}}(x,y)$ on $\mathbb{R}^2/\{(0,0)\}$ is be a smooth unit normal vector field to $\{\Gamma_t\}_{t\geq 0}$. The divergence of ν is $\frac{1}{\sqrt{x^2+y^2}}$. Meanwhile the curvature of a circle of radius $\sqrt{x^2+y^2}$ is also $\frac{1}{\sqrt{x^2+y^2}}$. So we can check that

$$-\mathrm{div}(\nu)\nu = -\frac{1}{\sqrt{x^2+y^2}}\nu$$

is the curvature vector field to $\{\Gamma_t\}_{t\geq 0}$.

As the theorem was mentioned without proof, we will seek to prove it by appealing to the geometry of surfaces. Firstly, we consider the following lemma that relates the curvature of Γ_t to g directly.

LEMMA 4.3. *The curvature of the curve Γ_t at (x, y) takes the form*

$$\kappa = -\frac{1}{|\nabla g|^3} \begin{bmatrix} -g_y & g_x \end{bmatrix} \begin{bmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{bmatrix} \begin{bmatrix} -g_y \\ g_x \end{bmatrix}.$$

PROOF. Our strategy is as follows. Let X be the graph of $g(x, y)$. Focusing at some point $(x, y, g(x, y))$, we let γ be some curve $\gamma : [\epsilon, \epsilon] \to X$ such that $\gamma(0) = (x, y, g(x, y))$ and $\gamma'(0)$ projected to the xy plane is parallel to the unit tangent vector of Γ_t at (x, y). We then compute the normal curvature of γ using the second fundamental form of X. Finally, we relate the normal curvature of γ and the curvature of γ using trigonometry.

Parametrizing the surface X by $\mathbf{r} = (x, y, g(x, y))$, we have

$$\begin{aligned} \mathbf{r}_x &= (1, 0, g_x), \\ \mathbf{r}_y &= (0, 1, g_y), \\ \mathbf{n} &= \frac{1}{\sqrt{|\nabla g|^2 + 1}}(g_x, g_y, -1) \end{aligned}$$

where $\mathbf{n}$ is the normal vector of the surface. Using the above, one could calculate the second fundamental form of the surface

$$II = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = -\frac{1}{\sqrt{|\nabla g|^2 + 1}} H.$$

As such if we have some curve γ that has its unit tangent vector be $\frac{1}{|\nabla g|}(-g_y, g_x, 0)$ on the surface at the point $(x, y, g(x, y))$, we have the normal curvature of the curve γ be

$$\begin{aligned} \kappa_n &= \dot{\gamma} II \dot{\gamma}^T \\ &= -\frac{1}{\sqrt{|\nabla g|^2 + 1}} \frac{1}{|\nabla g|^2} \begin{bmatrix} -g_y & g_x \end{bmatrix} H \begin{bmatrix} -g_y \\ g_x \end{bmatrix}. \end{aligned}$$

By $\kappa_n = \kappa \cos \psi$ where ψ is the angle between the normal vector of the surface and the principal normal of γ, we have $\cos \psi = \frac{|\nabla g|}{\sqrt{|\nabla g|^2+1}}$. As such

$$\kappa = -\frac{1}{|\nabla g|^3} \begin{bmatrix} -g_y & g_x \end{bmatrix} H \begin{bmatrix} -g_y \\ g_x \end{bmatrix}.$$

□

PROOF OF THEOREM 4.1. We could construct a smooth unit normal vector field as follows

$$\nu = \frac{\nabla g}{|\nabla g|}.$$

Hence, we could calculate that

$$\begin{aligned}\nabla \cdot \nu &= \nabla \cdot \big(\frac{\nabla g}{|\nabla g|}\big)\\ &= \frac{1}{|\nabla g|}\nabla^2 g + \nabla\big(\frac{1}{|\nabla g|}\big) \cdot \nabla g\\ &= \frac{1}{|\nabla g|}\nabla^2 g - \frac{\nabla(|\nabla g|)}{|\nabla g|^2} \cdot \nabla g.\end{aligned}$$

Now by chain rule

$$\nabla(|\nabla g|) = \frac{1}{|\nabla g|}\nabla g^T H.$$

As such we have

$$\begin{aligned}\nabla \cdot \nu &= \frac{1}{|\nabla g|}(\nabla^2 g) - \frac{1}{|\nabla g|^3}\nabla g^T H \nabla g\\ &= \frac{1}{|\nabla g|^3}\left((g_x^2 + g_y^2)(g_{xx} + g_{yy}) - \begin{bmatrix} g_x & g_y \end{bmatrix}\begin{bmatrix} g_{xx} & g_{xy}\\ g_{xy} & g_{yy}\end{bmatrix}\begin{bmatrix} g_x \\ g_y\end{bmatrix}\right)\\ &= \frac{1}{|\nabla g|^3}\left(g_{yy}g_x^2 + g_{xx}g_y^2 + g_{xx}g_x^2 + g_{yy}g_y^2 - g_{xx}g_x^2 - 2g_{xy}g_xg_y - g_yg_{yy}^2\right)\\ &= \frac{1}{|\nabla g|^3}\big(g_{yy}g_x^2 - 2g_{xy}g_xg_y + g_{xx}g_y^2\big)\\ &= \frac{1}{|\nabla g|^3}\begin{bmatrix} -g_y & g_x \end{bmatrix}\begin{bmatrix} g_{xx} & g_{xy}\\ g_{xy} & g_{yy}\end{bmatrix}\begin{bmatrix} -g_y \\ g_x\end{bmatrix}.\end{aligned}$$

□

2. Curve shortening flow

We now obtain a condition on g such that its level sets evolve under curve shortening flow.

LEMMA 4.4. *If $\{\Gamma_t\}_{t\geq 0}$ evolves by curve shortening flow, then*

$$|\nabla g| = \frac{1}{\kappa}.$$

PROOF (SKETCH). As g is constant on each curve Γ_t, the gradient of g must point in an orthogonal direction to the curve, as such $|\nabla g| = |\frac{\partial g}{\partial t}|$. From the evolution equation of curve shortening flow, we know that $\frac{\partial \Gamma}{\partial t} = \kappa \mathbf{n}$. So if we take a time step κdt, then g will be raised by dt. Hence, $|\nabla g| = |\frac{\partial g}{\partial t}| = \frac{1}{\kappa}$. ☐

Combining lemma 4.3 and 4.4, we have the following.

THEOREM 4.5. *If $\{\Gamma_t\}_{t \geq 0}$ evolves by curve shortening flow, then*

$$|\nabla g|^2 = \begin{bmatrix} -g_y & g_x \end{bmatrix} \begin{bmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{bmatrix} \begin{bmatrix} -g_y \\ g_x \end{bmatrix}.$$

PROOF. From lemma 4.3 we have

$$\kappa = -\frac{1}{|\nabla g|^3} \begin{bmatrix} -g_y & g_x \end{bmatrix} \begin{bmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{bmatrix} \begin{bmatrix} -g_y \\ g_x \end{bmatrix}.$$

From lemma 4.4 we have $\kappa = \frac{1}{|\nabla g|}$. Substituting that into the above equation and rearranging terms we obtain the theorem. ☐

EXAMPLE 4.6 (Shrinking circles). Consider $g(x, y) = \frac{1}{2}(1 - x^2 - y^2)$ which is the corresponding level set function to shrinking circles with radius 1 at $t = 0$. We can check that the Hessian matrix of g is

$$H = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

As $\nabla g = (-x, -y)$, we have

$$\mathbf{v} := \begin{bmatrix} -g_y \\ g_x \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}.$$

Finally we see that

$$-\mathbf{v}^T H \mathbf{v} = x^2 + y^2 = |\nabla g|^2.$$

The geometric features of this method are developed in much greater detail in [ES91].

3. Evolution of area

Under the level set method, we could describe the evolving (smooth closed strictly convex) curves and the areas they enclose as follows. The strategy presented below is due to [Oss78]. Let

$$D = \{(x, y) \in \mathbb{R}^2 \, : \, g(x, y) \geq 0\},$$
$$\text{The curve } \Gamma_t = \{(x, y) \in D : \, g(x, y) = t\},$$
$$\text{Region enclosed by } \Gamma_t = \{(x, y) \in D : \, |g(x, y)| \geq t\}.$$

We also introduce these parameters for integration.

$$s = \text{parameter of arc length along } \Gamma_t$$
$$\sigma = \text{parameter of arc length along an orthogonal trajectory,}$$
$$\text{to the family} \{\Gamma_t\}_{t \geq 0}, \text{increasing with } t.$$

Heuristically, the area element $dxdy = dsd\sigma = |\nabla g|^{-1}dsdt$. The co-area formula in turn gives us a natural and rigorous justification of the evolution of area under curve shortening flow.

THEOREM 4.7 (Co-area formula [Oss78, Equation 3.4]). *For an arbitrary smooth function $h(x, y)$*

$$\iint_D h(x,y)|\nabla g|\, dxdy = \int_0^T \left[\int_{\Gamma_t} hds\right] dt$$

where T is the time the flow terminates.

The co-area formula is proven and discussed in much greater detail in [EG99, Section 3.4]. It could be thought of as Fubini's theorem for curvilinear coordinates.

THEOREM 4.8. *Under curve shortening flow, the enclosed area decreases at a constant rate of -2π*

PROOF. The area of D could be deduced as follows.

$$\begin{aligned}
&\iint_D \frac{1}{|\nabla g|}|\nabla g|\, dxdy \\
&= \int_0^T \left[\int_{\Gamma_t} \frac{1}{|\nabla g|} ds\right] dt && \text{Co-area formula} \\
&= \int_0^T 2\pi dt && |\nabla g| = \frac{1}{\kappa} \text{ \& Gauss-Bonnet} \\
&= 2\pi T.
\end{aligned}$$

By translating the level set functions $g(x, y)+t$ for some $t \in [0, T)$ and repeating the same argument, we can see that the area decreases at a constant rate of -2π. $\square$

4. Sobolev inequality

It is well known that Sobolev inequality and the isoperimetric inequality are equivalent. Interestingly, the proof of Sobolev inequality from isoperimetric inequality uses basically the same ideas we have been developing.

THEOREM 4.9 (Sobolev inequality). *Let f be a smooth function on $\mathbb{R}^2$ with compact support. Then*

$$\left(\int_{\mathbb{R}^2} |\nabla f|\right)^2 \geq 4\pi \int_{\mathbb{R}^2} |f|^2.$$

The isoperimetric inequality for some domain V could be obtained from Sobolev inequality by constructing smooth, compactly supported approximations f to the indicator function of V. $|\nabla f|$ would "pick up" the arclength of the boundary of V while f would "pick up" the area of V. What is more relevant to our discussion is proving Sobolev inequality using the isoperimetric inequality.

THEOREM 4.10 ([Oss78, Theorem 3.1]). *The isoperimetric inequality implies the Sobolev inequality*

PROOF (SKETCH). Let

$$\begin{aligned} D(t) &= \{\mathbf{p} \in \mathbb{R}^2 \,:\, |f(\mathbf{p})| > t\}, \\ C(t) &= \partial D, \\ A(t) &= \text{Area}(D(t)), \\ L(t) &= \text{Length}(C(t)). \end{aligned}$$

We discover that

$$\iint_{\mathbb{R}^2} |\nabla f\,|\,dxdy = \int_0^\infty L(t)\,dt, \qquad \text{Co-area formula} \tag{4.1}$$

$$\geq 2\sqrt{\pi}\int_0^\infty \sqrt{A(t)}\,dt, \qquad \text{Isoperimetric inequality} \tag{4.2}$$

$$\iint_{\mathbb{R}^2} f^{\,2}\,dxdy = \int_0^\infty 2tA(t)\,dt. \qquad \text{Co-area formula} \tag{4.3}$$

Finally, as $A(t)$ is a decreasing function, the following inequality holds

$$\left(\int_0^\infty \sqrt{A(t)}\,dt\right)^2 \geq \int_0^\infty 2tA(t)\,dt. \tag{4.4}$$

Combining (1), (2), (3), (4) recovers the Sobolev inequality. □

For complete details of this argument, we refer to [Oss78].

CHAPTER 5

Isoperimetric ratio

It turns out that curve shortening flow for convex curves, the isoperimetric ratio $L^2/4\pi A$ is nonincreasing. This fact plays a part in the proof of the Gage-Hamilton theorem. Meanwhile, the proof of the monotonicity of the isoperimetric ratio uses an inequality similar to the isoperimetric inequality.

We can first compute that

$$\begin{aligned}\frac{d}{dt}\left(\frac{L^2}{A}\right) &= \frac{1}{A^2}\left(2L\frac{dL}{dt}A - L^2\frac{dA}{dt}\right) \\ &= \frac{1}{A^2}\left(-2LA\int_\Gamma \kappa^2\, ds + 2\pi L^2\right) \\ &= -2\frac{L}{A}\left(\int_\Gamma \kappa^2\, ds - \pi\frac{L}{A}\right).\end{aligned}$$

What remains is showing that $\int_\Gamma \kappa^2\, ds - \pi L/A$ is nonnegative for all convex curves Γ.

THEOREM 5.1 (Gage's inequality). *If Γ is a convex curve, then*

$$\int_\Gamma \kappa^2\, ds \geq \pi\frac{L}{A}.$$

PROOF. Omitted. Could be found in [Gag83]. □

Unfortunately, Gage's inequality is only true for convex curves. We note that for nonconvex curves, we have a weaker inequality.

THEOREM 5.2 ([And+20, Equation 3.22]). *Given some smooth curve Γ, we have*

$$\int_\Gamma \kappa^2\, ds \geq \frac{4\pi^2}{L}.$$

PROOF. By Cauchy-Schwarz inequality, we have

$$\int_\Gamma \kappa^2\, ds \int_\Gamma ds \geq \left(\int_\Gamma \kappa\, ds\right)^2.$$

We know $\int_\Gamma \kappa\, ds = 2\pi$ by Gauss-Bonnet and $\int_\Gamma ds = L$. Rearranging terms we get

$$\int_\Gamma \kappa^2\, ds \geq \frac{4\pi^2}{L}.$$

$\square$

This inequality is also useful in the study of curve shortening flow.

REMARK. For convex curves, Gage's inequality and the isoperimetric inequality recover theorem 5.2.

Conclusion

Summary

In this essay, we have analyzed the evolution of arclength and area under curve shortening flow through various viewpoints, which leads to a proof of the isoperimetric inequality.

In chapter 3, we observe how the support function reduces curve shortening flow for convex curves into a PDE that is easier to analyze. In particular, it's well suited to understand the evolution of arclength. We also discussed how the support function and Wirtinger's inequality give rise to a proof of the isoperimetric inequality for convex curves. In chapter 4, we make use of the level set method to derive a geometric clearer explanation of the evolution of area. We also discussed how the isoperimetric inequality could be used to prove Sobolev's inequality. In chapter 5, we briefly discussed some inequalities similar to the isoperimetric inequality that help us understand the curve shortening flow.

Other directions

- A lot of the theory for curve shortening flow has roots in the study of the heat equation. See [And+20, Chapter 1].
- We have omitted discussing the Brunn-Minkowski theorem and its generalisations which relate volumes of compact subsets of Euclidean space. This theorem is useful in proving the isoperimetric inequality and its generalisations. See [SS05, Section 1.5].
- A generalisation of Wirtginer's inequality is Poincaré's inequality, which is also deeply related to the isoperimetric inequality. See [MSG01, Appendix I].

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