Structures closed under intersections or unions

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Introduction

In a mathematical structure closed under intersections, we can define a minimal structure generated by a set using intersections.

Examples

- Subgroups generated by a set
- Closure (of a set)
- σ-algebra generated by a set

Subgroup generated by a set

Given a group G, let F denote the family of subgroups of G.
For any subset A of G, we can define the subgroup generated by A to be

$$\langle A\rangle = \bigcap_{H\in F, A\subset H} H$$

It has three properties

- $\langle A\rangle$ is a subgroup
- A is contained in $\langle A\rangle$
- If H is a subgroup and A is contained in H, then $\langle A\rangle$ is contained in H

Intersections of subgroups are subgroups (Not true for unions!)

Closure of a set

Given a topological space X, let T' denote the family of closed sets of X.

For any subset A of X, we can define the closure of A to be

$$\text{clo}(A) = \bigcap_{V \in T', A \subset V} V$$

It has three properties

- $\text{clo}(A)$ is closed
- A is contained in $\text{clo}(A)$
- If V is a closed set and A is contained in V, then $\text{clo}(A)$ is contained in V

Intersections of closed sets are closed (Not true for unions!)

Generalisation

Let X be a set and F be a family of subsets of X, with the property that it's closed under intersections

$$F_i \in F \, \forall i \in I \implies \bigcap_{i \in I} F_i \in F$$

Then for some subset A of X, we can define

$$\langle A \rangle = \bigcap_{S \in F, A \subset S} S$$

with the following properties

1. $\langle A \rangle \in F$
2. $A \subset \langle A \rangle$
3. If $S \in F$ and $A \subset S$, then $\langle A \rangle \subset S$

Generalisation

Let X be a set and F be a family of subsets of X, with the property that it's closed under unions

$$F_i \in F \, \forall i \in I \implies \bigcup_{i \in I} F_i \in F$$

Then for some subset A of X, we can define

$$\langle A \rangle = \bigcup_{S \in F, S \subset A} S$$

with the following properties

1. $\langle A \rangle \in F$
2. * $\langle A \rangle \subset A$
3. * If $S \in F$ and $S \subset A$, then $S \subset \langle A \rangle$

Key example: Interior of subset of a topological space

Subspace topology

Given a set Y, let Ω_Y denote the family of topologies of Y.
For any subset A of (X, τ_X), we can define the subspace topology of A to be

$$\tau_A = \bigcap_{\tau \in \Omega_A, \mathrm{id}^{-1}(\tau_X) \subset \tau} \tau$$

$\mathrm{id}^{-1}(\tau_X) \subset \tau$ means that the inclusion map from (A, τ) to (X, τ_X) is continuous.
It has three properties

- τ_A is a topology
- $f : (A, \tau_A) \to (X, \tau_X)$ is continuous
- If τ is a topology such that $f : (A, \tau) \to (X, \tau_X)$ is continuous, then $\tau_A \subset \tau$

Intersections of topologies are a topology (Not true for unions!)

Connected components

Let X be a topological space, F be the family of connected sets of X.
For some subset A of X, we can define

$$C_A = \bigcup_{S \in F, A \subset S} S$$

with the following properties

1. $C_A \in F$
2. $A \subset C_A$
3. * If $S \in F$ and $S \subset A$, then $S \subset C_A$

Union of connected sets with nonempty intersection is connected (Not true in general!)

Conclusion

In a mathematical structure closed under intersections, we can define a minimal structure generated by a set using intersections.

Questions

- Is there any structure with property 1, 2* and 3?

Related ideas

- Closure operator
- Field of sets

Acknowledgements

- Thanks to Stuart White for pointing the idea out
- I do not claim originality of any of the ideas above!