The Isoperimetric Inequality and Sobolev Inequality

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Isoperimetric Inequality & Sobolev Inequality

Isoperimetric:
A closed curve of length L that encloses a planar region of area A satisfies the following inequality

$$L^2 \geq 4\pi A$$

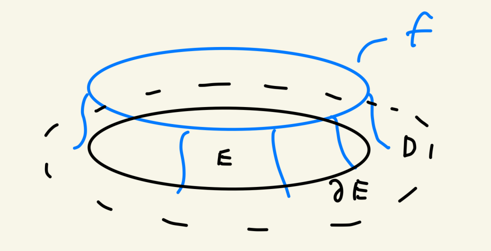
Sobolev:
Let f be a smooth function that has compact support in D. Then

$$\left(\iint_D |\nabla f|^2 \right) \geq 4\pi \iint_D f^2$$

Don't they look similar? It turns out they are **equivalent**!

Sobolev $\Longrightarrow$ Isoperimetric

Approximate the indicator function of $E \subset D$ with a smooth function f.



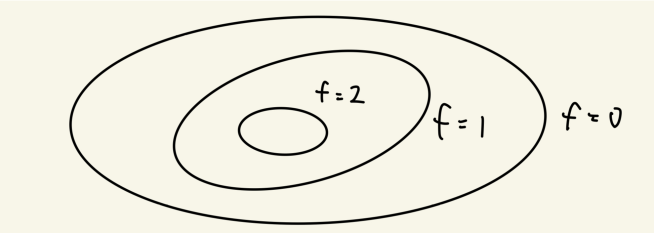
$\left(\iint_D |\nabla f|\right)^2$ picks up the arclength squared of δE while $\iint_D f^2$ picks up the area of E.

Isoperimetric $\implies$ Sobolev

Idea: Use the Isoperimtric inequality on level sets of f

Setup: Let

$$D(t) = \{\mathbf{p} \in D \text{ s. t. } |f(\mathbf{p})| > t\}$$
$$C(t) = \delta D$$
$$A(t) = \text{Area}(D)$$
$$L(t) = \text{Length}(C)$$



Isoperimetric $\implies$ Sobolev

Sketch proof:

$$\iint_D |\nabla f| \, dxdy = \int_0^\infty L(t) \, dt \qquad \text{Coarea formula} \tag{1}$$

$$\geq 2\sqrt{\pi} \int_0^\infty \sqrt{A(t)} \, dt \qquad \text{Isoperimetric} \tag{2}$$

$$\iint_D f^2 \, dxdy = \int_0^\infty 2tA(t) \, dt \qquad \text{Coarea formula} \tag{3}$$

Finally we can show that in general for decreasing functions $A(t)$, $\left(2\sqrt{\pi} \int_0^\infty \sqrt{A(t)}\right)^2 \geq \int_0^\infty 2tA(t) \, dt$.

References

All ideas come from

Robert Osserman. "The isoperimetric inequality." Bulletin of the American Mathematical Society, 84(6) 1182-1238 November 1978.