# Symmetries of differential equations

Toby Lam

# Introduction

What are symmetries of differential equations?

- Largest group of transformations acting on independent and dependent variables with the property that it **transforms solutions to other solutions**

Solutions to differential equations describe phenomena that could possibly happen. So symmetries **relate physical / chemical / biological phenomena**.

# Why do we care?

- Noether's theorem: Gives us invariants, in turn make equations integrable ...

**Example**

$\ddot{y} = -y$ is not directly integrable, but $E = \frac{1}{2}\dot{y}^2 + \frac{1}{2}y^2$ gives you an integrable equation.

- **How are solutions / phenomena related to each other?**
  - Generate new solutions from old ones
  - Why do those relationships exist?
  - Especially relevant for PDEs

# Straight lines

What are the symmetries of $\frac{d^2s}{dt^2} = 0$? The solutions are $s = vt + s_0$ for constants $v, s_0$, i.e. straight lines.
Most of the symmetries are straightforward

| $(\hat{t}, \hat{s})$ | Interptation |
|---|---|
| $(t+\epsilon, s)$ | Horizontal translation |
| $(t, s+\epsilon)$ | Vertical translation |
| $(e^{\epsilon}t, s)$ | Horizontal scaling |
| $(t, e^{\epsilon}s)$ | Vertical scaling |
| $(t+\epsilon s, s)$ | Horizontal shearing |
| $(t, s+\epsilon t)$ | Vertical shearing |

# Straight lines

What are the symmetries of $\frac{d^2s}{dt^2} = 0$? The solutions are $s = vt + s_0$ for constants $v, s_0$, i.e. straight lines.

Most of the symmetries are straightforward

| Vtr. Field | $(\hat{t}, \hat{s})$ | Interptation |
|---|---|---|
| $(1, 0)$ | $(t + \epsilon, s)$ | Horizontal translation |
| $(0, 1)$ | $(t, s + \epsilon)$ | Vertical translation |
| $(t, 0)$ | $(e^{\epsilon} t, s)$ | Horizontal scaling |
| $(0, s)$ | $(t, e^{\epsilon} s)$ | Vertical scaling |
| $(s, 0)$ | $(t + \epsilon s, s)$ | Horizontal shearing |
| $(0, t)$ | $(t, s + \epsilon t)$ | Vertical shearing |

**Synonyms:**

| | |
|---|---|
| $(\hat{t}, \hat{s})$: | Transformation, Symmetry, Lie group element ... |
| Vtr. Field: | Velocity vtr. field, Lie algebra |

# Straight lines

What are the symmetries of $\frac{d^2 s}{dt^2} = 0$? The solutions are $s = vt + s_0$ for constants $v, s_0$, i.e. straight lines.
Most of the symmetries are straightforward

| Vtr. Field | $(\hat{t}, \hat{s})$ | Interptation |
|---|---|---|
| $(1, 0)$ | $(t + \epsilon, s)$ | Horizontal translation |
| $(0, 1)$ | $(t, s + \epsilon)$ | Vertical translation |
| $(t, 0)$ | $(e^\epsilon t, s)$ | Horizontal scaling |
| $(0, s)$ | $(t, e^\epsilon s)$ | Vertical scaling |
| $(s, 0)$ | $(t + \epsilon s, s)$ | Horizontal shearing |
| $(0, t)$ | $(t, s + \epsilon t)$ | Vertical shearing |

**Remark**: You could think of the vector fields as a velocity vector field and we integrate it to get the displacement. Alternatively, the vector fields forms a Lie algebra and we exponentiate it to get the Lie group element $(\hat{x}, \hat{y})$

# Straight lines

However there's two "nonlinear" transformations

| Vtr. Field | $(\hat{t}, \hat{s})$ | Interptation |
|---|---|---|
| $(t^2, ts)$ | $\left(\frac{t}{1-\epsilon t}, \frac{s}{1-\epsilon t}\right)$ | ? |
| $(ts, s^2)$ | $\left(\frac{t}{1-\epsilon s}, \frac{s}{1-\epsilon s}\right)$ | ? |

Let's see it in action!

# Straight lines

However there's two "nonlinear" transformations

| Vtr. Field | $(\hat{t}, \hat{s})$ | Interptation |
|---|---|---|
| $(t^2, ts)$ | $\left(\frac{t}{1-\epsilon t}, \frac{s}{1-\epsilon t}\right)$ | Horizontal "Panning" |
| $(ts, s^2)$ | $\left(\frac{t}{1-\epsilon s}, \frac{s}{1-\epsilon s}\right)$ | Vertical "Panning" |

**Remark:** Rather interesting we could get something extrinsic (projective linear transformations) out of finding out intrinsic symmetries.

# Laplace Equation (Elliptic)

Symmetries of $u_{xx} + u_{yy} = 0$

- Conformal maps preserves harmonicity of functions
- Orientation preserving conformal $\iff$ Holomorphic
- Symmetries (in terms of vector fields) must come from derivatives of holomorphic functions
- As such, any vector field $\xi\partial_x + \eta\partial_y$ that satisifies the Cauchy-Riemann relations

$$\xi_x = \eta_y$$
$$\xi_y = -\eta_x$$

  forms a symmetry.

**Remark**: "Elliptic" makes you think of circles

# Heat Equation (Parabolic)

Symmetries of $u_t - u_{xx} = 0$

| Vector field | $(\hat{x}, \hat{t}, \hat{u})$ |
|---|---|
| $\mathbf{v}_1 = \partial_x$ | $(x+\epsilon, t, u)$ |
| $\mathbf{v}_2 = \partial_t$ | $(x, t+\epsilon, u)$ |
| $\mathbf{v}_3 = u\partial_u$ | $(x, t, e^{\epsilon}u)$ |
| $\mathbf{v}_4 = x\partial_x + 2t\partial_t$ | $(e^{\epsilon}x, e^{2\epsilon}t, u)$ |
| $\mathbf{v}_5 = 2t\partial_x - xu\partial_u$ | $(x+2\epsilon t, t, u\cdot\exp(-\epsilon x - \epsilon^2 t))$ |
| $\mathbf{v}_6 = 4tx\partial_x + 4t^2\partial_t - (x^2+2t)u\partial_u$ | $(\frac{x}{1-4\epsilon t}, \frac{t}{1-4\epsilon t}, u\sqrt{1-4\epsilon t}\exp(\frac{-\epsilon x^2}{1+4\epsilon t}))$ |
| $\mathbf{v}_\alpha = \alpha(x,t)\partial_u$ | $(x, t, u+\epsilon\alpha(x,t))$ |

**Remark**: There's a 2 in $\mathbf{v}_4$ due to parabolicity. (Consider $u = -x^2 - 2t$)

**Remark**: $\mathbf{v}_5$ is a "Galilean boost". $\mathbf{v}_6$ transforms constant solutions into fundamental solutions. $\mathbf{v}_\alpha$ is the principle of superposition.

# Wave Equation (Hyperbolic)

Symmetries of $u_{tt} - u_{xx} = 0$

| Vector field | $(\hat{x}, \hat{t}, \hat{u})$ |
|---|---|
| $\mathbf{v}_1 = \partial_x$ | $(x+\epsilon, t, u)$ |
| $\mathbf{v}_2 = \partial_t$ | $(x, t+\epsilon, u)$ |
| $\mathbf{r}_{xt} = t\partial_x + x\partial_t$ | $(x\cosh\epsilon + t\sinh\epsilon, x\sinh\epsilon + t\cosh\epsilon, u)$ |
| $\mathbf{d} = x\partial_x + t\partial_t$ | $(e^{\epsilon}x, e^{\epsilon}t, u)$ |
| $\mathbf{i}_x = (x^2+t^2)\partial_x + 2xt\partial_t - xu\partial_u$ | Omitted |
| $\mathbf{i}_t = 2xt\partial_x + (x^2+t^2)\partial_t - tu\partial_u$ | Omitted |
| $\mathbf{v}_3 = u\partial_u$ | $(x, t, e^{\epsilon}u)$ |
| $\mathbf{v}_\alpha = \alpha(x, y, t)\partial_u$ | $(x, t, u + \epsilon\alpha(x, t))$ |

**Remark**: $\mathbf{r}_{xt}$ is a "hyperbolic rotation". $\mathbf{i}_x, \mathbf{i}_t$ are "inversions". $\mathbf{v}_\alpha$ is the principle of superposition.

**Remark**: Hyperbolic equations treat $t$ and $x$ equally.

# Takeaways

- Symmetries are powerful things
- The more you use them, the more you'll spot them
- Find structural reasons why they appear
- They help us make sense of **models** / differential equations

# End

Hydon - Symmetry methods for differential equations

- Friendlier presentation / focuses on ODEs

Olver - Symmetries of differential equations

- Covers generalised symmetries as well

`blog: tobylam.xyz`

## Questions away!