Symmetries of differential equations

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Introduction

What are symmetries of differential equations?

- Largest group of transformations acting on independent and dependent variables with the property that it **transforms solutions to other solutions**

Solutions to differential equations describe (ideal) phenomena that could possibly happen. So symmetries **relate physical / chemical / biological phenomena**.

Example

$$\frac{d^2y}{dt^2} = -g$$

Ball falling to the ground from rest

Example

$$\frac{d^2y}{dt^2} = -g$$

Symmetries

Time translation: $y(t) \to y(t+\varepsilon)$

Position translation: $y(t) \to y(t) + \varepsilon$

Why do we care?

- Noether's theorem: Gives us invariants, in turn make equations integrable ...

Example

$\ddot{y} = -y$ is not directly integrable, but $E = \frac{1}{2}\dot{y}^2 + \frac{1}{2}y^2$ gives you an integrable equation.

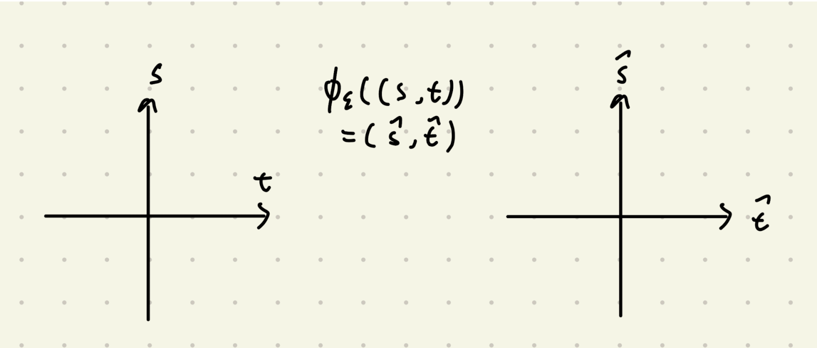
- **How are solutions / phenomena related to each other?**
  - Generate new solutions from old ones
  - Why do those relationships exist?
  - Especially relevant for PDEs

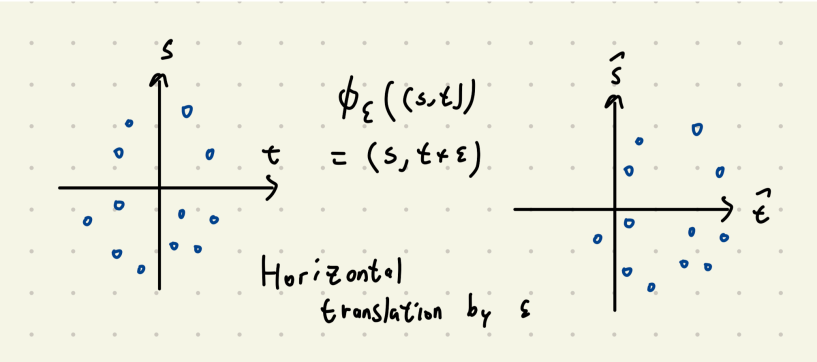
Straight lines

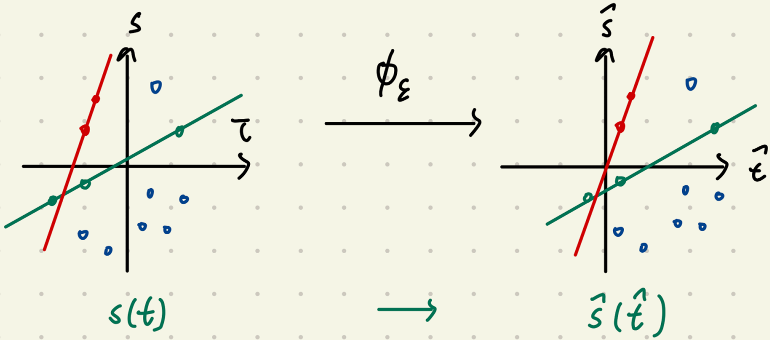
What are the symmetries of $\frac{d^2s}{dt^2} = 0$? The solutions are $s = vt + s_0$ for constants v, s_0, i.e. straight lines.

Most of the symmetries are straightforward

$(\hat{t}, \hat{s})$	Interptation
$(t + \epsilon, s)$	Horizontal translation
$(t, s + \epsilon)$	Vertical translation
$(e^{\epsilon} t, s)$	Horizontal scaling
$(t, e^{\epsilon} s)$	Vertical scaling
$(t + \epsilon s, s)$	Horizontal shearing
$(t, s + \epsilon t)$	Vertical shearing


s
t
φε((s,t))
= (ŝ, t̂)
ŝ
t̂


s
t
$\phi_\varepsilon((s,t))$
$= (s, t+\varepsilon)$
Horizontal
translation by ε
$\hat{s}$
$\hat{t}$



We say ϕ_ε is a symmetry of $\frac{d^2 s}{dt^2} = 0$

if $\frac{d^2 s}{dt^2} = 0$ implies $\frac{d^2 \hat{s}}{d\hat{t}^2} = 0$

Straight lines

What are the symmetries of $\frac{d^2 s}{dt^2} = 0$? The solutions are $s = vt + s_0$ for constants v, s_0, i.e. straight lines.
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Vtr. Field	$(\hat{t}, \hat{s})$	Interptation
$(1, 0)$	$(t + \epsilon, s)$	Horizontal translation
$(0, 1)$	$(t, s + \epsilon)$	Vertical translation
$(t, 0)$	$(e^{\epsilon} t, s)$	Horizontal scaling
$(0, s)$	$(t, e^{\epsilon} s)$	Vertical scaling
$(s, 0)$	$(t + \epsilon s, s)$	Horizontal shearing
$(0, t)$	$(t, s + \epsilon t)$	Vertical shearing

Synonyms:

$(\hat{t}, \hat{s})$: Transformation, Symmetry, Lie group element ...
Vtr. Field: Velocity vtr. field, Lie algebra

Straight lines

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Remark: You could think of the vector fields as a velocity vector field and we integrate it to get the displacement. Alternatively, the vector fields forms a Lie algebra and we exponentiate it to get the Lie group element $(\hat{x}, \hat{y})$

Straight lines

However there's two "nonlinear" transformations

Vtr. Field	$(\hat{t}, \hat{s})$	Interptation
(t^2, ts)	$\left(\frac{t}{1-\epsilon t}, \frac{s}{1-\epsilon t}\right)$	?
(ts, s^2)	$\left(\frac{t}{1-\epsilon s}, \frac{s}{1-\epsilon s}\right)$	?

Let's see it in action!

Straight lines

However there's two "nonlinear" transformations

Vtr. Field	$(\hat{t}, \hat{s})$	Interptation
(t^2, ts)	$(\frac{t}{1-\epsilon t}, \frac{s}{1-\epsilon t})$	Horizontal "Panning"
(ts, s^2)	$(\frac{t}{1-\epsilon s}, \frac{s}{1-\epsilon s})$	Vertical "Panning"

Remark: Rather interesting we could get something extrinsic (projective linear transformations) out of finding out intrinsic symmetries.

Laplace's Equation (Elliptic)

Symmetries of $u_{xx} + u_{yy} = 0$

- Angle-preserving maps preserves solutions to Laplace's equation
- For maps on the real plane: Orientation and angle-preserving $\iff$ Complex differentiable
- Symmetries (in terms of vector fields) must come from derivatives of complex differentiable functions. As such, any vector field $\xi\partial_x + \eta\partial_y$ satisifies the Cauchy-Riemann relations

$$\xi_x = \eta_y$$
$$\xi_y = -\eta_x$$

  if and only if it is a symmetry.

Remark: "Elliptic" makes you think of circles

Wave Equation (Hyperbolic)

Symmetries of $u_{tt} - u_{xx} = 0$

Vector field	$(\hat{x}, \hat{t}, \hat{u})$
$\mathbf{v}_1 = \partial_x$	$(x+\epsilon, t, u)$
$\mathbf{v}_2 = \partial_t$	$(x, t+\epsilon, u)$
$\mathbf{r}_{xt} = t\partial_x + x\partial_t$	$(x\cosh\epsilon + t\sinh\epsilon, x\sinh\epsilon + t\cosh\epsilon, u)$
$\mathbf{d} = x\partial_x + t\partial_t$	$(e^{\epsilon}x, e^{\epsilon}t, u)$
$\mathbf{i}_x = (x^2+t^2)\partial_x + 2xt\partial_t - xu\partial_u$	Omitted
$\mathbf{i}_t = 2xt\partial_x + (x^2+t^2)\partial_t - tu\partial_u$	Omitted
$\mathbf{v}_3 = u\partial_u$	$(x, t, e^{\epsilon}u)$
$\mathbf{v}_\alpha = \alpha(x,y,t)\partial_u$	$(x, t, u+\epsilon\alpha(x,t))$

Remark: $\mathbf{r}_{xt}$ is a "hyperbolic rotation". $\mathbf{i}_x, \mathbf{i}_t$ are "inversions". $\mathbf{v}_\alpha$ is the principle of superposition.

Heat Equation (Parabolic)

Symmetries of $u_t - u_{xx} = 0$

Vector field	$(\hat{x}, \hat{t}, \hat{u})$
$\mathbf{v}_1 = \partial_x$	$(x+\epsilon, t, u)$
$\mathbf{v}_2 = \partial_t$	$(x, t+\epsilon, u)$
$\mathbf{v}_3 = u\partial_u$	$(x, t, e^{\epsilon}u)$
$\mathbf{v}_4 = x\partial_x + 2t\partial_t$	$(e^{\epsilon}x, e^{2\epsilon}t, u)$
$\mathbf{v}_5 = 2t\partial_x - xu\partial_u$	$(x+2\epsilon t, t, u \cdot \exp(-\epsilon x - \epsilon^2 t))$
$\mathbf{v}_6 = 4tx\partial_x + 4t^2\partial_t - (x^2+2t)u\partial_u$	$(\frac{x}{1-4\epsilon t}, \frac{t}{1-4\epsilon t}, u\sqrt{1-4\epsilon t}\exp(\frac{-\epsilon x^2}{1+4\epsilon t}))$
$\mathbf{v}_\alpha = \alpha(x,t)\partial_u$	$(x, t, u+\epsilon\alpha(x,t))$

Remark: There's a 2 in $\mathbf{v}_4$ due to parabolicity. (Consider $u = -x^2 - 2t$)

Remark: $\mathbf{v}_5$ is a "Galilean boost". $\mathbf{v}_6$ transforms constant solutions into fundamental solutions. $\mathbf{v}_\alpha$ is the principle of superposition.

Takeaways

- Symmetries are powerful things
- The more you use them, the more you'll spot them
- Find structural reasons why they appear
- They help us make sense of **models** / differential equations

End

Hydon - Symmetry methods for differential equations

- Friendlier presentation / focuses on ODEs

Olver - Symmetries of differential equations

- Covers generalised symmetries as well

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Questions away!