On statements and proofs

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Point of a lecture

- Main difference between High School and University is that most of the learning takes place **outside the classroom**!

Sentences

- In mathematics you have dealt with sentences all the time
- We combine sentences together to form more complicated sentences using **connectives**.

Example

Complicated sentence: If x is odd and y is odd, then x times y is odd.
3 atomic sentences: "x is odd", "y is odd", "x times y is odd"
2 connectives: "If ... then", "and"

Sentences

Example

If ABC is a triagle, then the sum of the angles is 180 degrees.
2 atomic sentences: "ABC is a triangle",
"$\angle ABC + \angle BCA + \angle CAB = 180^\circ$"
1 Connective: "If ... then"

Propositional Connectives

- Let's try to understand this abstractly
- We shall use the alphabet to stand in for sentences
- We shall only consider **truth-functional combinations** : Truth or Falsity of new sentence is determined by truth or falsity of its component sentences.
- There are 5 propositional connectives: $\neg, \wedge, \vee, \implies, \Leftrightarrow$. We want to use these connectives to form **truth-functional combinations**.

Example

Example Sentences: "Paris is the capital of France", "Rome is the cpital of France"

Negation

- If A is a sentence, $\neg A$ denotes the negation of A
- T and F respectively denote the truth values True and False

A	$\neg A$
T	F
F	T

Figure: Truth table of $\neg A$

Conjunction

- If A and B are sentence, $A \wedge B$ denotes A and B
- $A \wedge B$ is true when and only when both A and B are true
- There are four rows to account for all possible assignments of truth values to A and B.

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Figure: Truth table of $A \wedge B$

Disjunction

- If A and B are sentences, $A \vee B$ denotes A and B
- "At least one of A and B is true"
- $A \vee B$ is false when and only when both A and B are false

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Figure: Truth table of $A \vee B$

Conditional

- If A and B are sentences, $A \implies B$ means "If A, then B"

A	B	$A \implies B$
T	T	T
T	F	F
F	T	T
F	F	T

Figure: Truth table of $A \implies B$

- This is a truth-functional interpretation of "If A, then B"

Conditional

Example

Under our definitions. The following sentences are true.

- If $1 + 1 = 2$, then University of Oxford is in England.
- If $1 + 1 = 3$, then University of Oxford is in England.
- If $1 + 1 = 3$, then pigs can fly.

Conditional

- There are other **non-truth-functional** interpretations of the conditional sentence
- We only concern the **truth-functional interpretation**

Casuality

If you put gold into cold water at time t, the gold would dissolve.

Counterfactual conditional

If you heated a piece of butter at 150 Celsius yesterday, the butter would not have melted

If and only If

- If A and B are sentences, $A \Leftrightarrow B$ means "A if and only if B"
- $A \Leftrightarrow B$ is true when and only when A and B have the same truth value.

A	B	$A \Leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

Figure: Truth table of $A \Leftrightarrow B$

Aside

Prove that $A \iff B$ is equivalent to $(A \implies B) \land (B \implies A)$

Statements

Recall the following.

Definition

$\neg, \vee, \wedge, \implies, \Leftrightarrow$ are called propositional connectives.

Any statements built up by application of these connectives has a truth value that depends on truth values of the constituent sentences.

Definition

Statement form are expressions built up from the statement letters (A, B, C ...) by appropriate application of propositional connectives.

Statement letters must have one and only one truth value, true or false.

Statement forms

There are 3 rules with statement forms.

1. All statement letters (A, B, C ...) are statement forms
2. If β and $\mathscr{C}$ are statement forms, then so are $(\not\beta), (\beta \vee \mathscr{C}), (\beta \wedge \mathscr{C}), (\beta \implies \mathscr{C})$, and $(\beta \Leftrightarrow \mathscr{C})$
3. All statement forms must be generated by statement letters and propositional connectives.

Tautology

Definition - Tautology

A statement form that is always true regardless of the truth values of its statement letters.

How do we show a statement form is a tautology?

- If the statement form has n statement letters, draw a truth table with 2^n rows to account for all possible combinations of truth values
- Deduce the truth values for all cases
- If the truth values of the statement form are true for all cases, it's a tautology. Otherwise, it's not.

Examples

Example

Show that $(A \wedge B) \implies A$ is a tautology

A	B	$A \wedge B$	A	$(A \wedge B) \implies A$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	F	T

We determined the fifth column using the third, fourth column and the definition of $\implies$.
We see that $(A \wedge B) \implies A$ has the truth value T regardless of the truth values of A and B, hence it's a tautology.

Examples

We're going to skip this in the lectures. Look at this after the lecture!

Example

Show that $(A \wedge B) \implies (A \vee B)$ is a tautology

A	B	$A \wedge B$	$A \vee B$	$(A \wedge B) \implies (A \vee B)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Logical implication and equivalence

Let β and $\mathscr{C}$ be statement forms.

Definition - β logically implies $\mathscr{C}$

β **logically implies** $\mathscr{C}$ if and only if every **truth assignment** to the **statement letters** of β that makes β true also makes $\mathscr{C}$ true.

Definition - β and $\mathscr{C}$ logically equivalent

β and $\mathscr{C}$ **logically equivalent** if and only if β and $\mathscr{C}$ receive the same truth value under **every assignment of truth values** to the **statement letters** of β and $\mathscr{C}$

Propositions

Proposition

β logically implies $\mathscr{C}$ if and only if $\beta \implies \mathscr{C}$ is a tautology.

Proposition

β and $\mathscr{C}$ are logically equivalent if and only if $\beta \Leftrightarrow \mathscr{C}$ is a tautology.

You could find the proofs of this in the book, Introduction to Mathematical Logic, Elliot Mendelson. (Chapter 1, page 6 - 9)

Common Tautologies

- $P \vee \neg P$ (Law of excluded middle)
- $((P \implies Q) \wedge P) \implies Q$ (Modus ponens)
- $\neg(P \vee Q) \Leftrightarrow ((\neg P) \wedge (\neg Q))$ (De Morgan's)
- $\neg(P \wedge Q) \Leftrightarrow ((\neg P) \vee (\neg Q))$ (De Morgan's)

More on conditionals

Given the statement $P \implies Q$, we define the following statements.

- Converse: $Q \implies P$
- Contra-positive: $(\neg Q) \implies (\neg P)$

They are an important part of a mathematician's vocabulary.
Turns out, contra-positive has an interesting property.

Contrapositives

Proposition

$P \implies Q$ is logically equivalent to its contrapositive, $(\neg Q) \implies (\neg P)$

Proof.

P	Q	$P \implies Q$	$\neg P$	$\neg Q$	$(\neg Q) \implies (\neg P)$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

□

More Vocab

Mathematicians like to use a variety of vocabularies.
The following are all logically equivalent

- If P then Q
- Q if P
- P only if Q
- P is sufficient for Q
- Q is necessary for P

In other words,

- P if Q is the converse of P only if Q
- P is sufficient for Q is the converse of P is necessary for Q

Scope

- The logic we have covered is called **Propositional Logic**. It is not capable of dealing with non-logical objects (integers, "there exists", "for all").
- Propositional logic as we have defined it is woefully limited. We only have statement letters and statement forms!

Example

The following aren't statement forms

- A: The sentence (A) is false.
- $1 + 1 = 2$

Scope

- We would need **first-order logic** in order to rigorously define integers / addition / multiplication ...
- If you're interested, read up on chapter 2-3 of "Intro to Mathematical Logic"
- Let's take a "leap of faith" and see how basic concepts of propositional logic apply in the world of mathematics we are familiar with.
- We would use tautologies to help us prove things.

Proofs

A $\implies$ B is one of the most common statements you would see in mathematics.

Proposition

If x is even then x^2 is divisible by 4.

Let's try to unpack this. The above is an $A \implies B$ statement where

- A: x is even
- B: x^2 is divisible by 4

We usually prove "A $\implies$ B" by 3 ways.

- Direct Proof
- Direct Proof of the contrapositive
- Proof by Contradiction
- *Disproof by counterexample

Direct Proof

Objective: Try to show
$(A \implies A_1) \wedge (A_1 \implies A_2) \cdots \wedge (A_k \implies B)$

Proof

x is even
$\implies$ x = 2n, where n is an integer
$\implies x^2 = 4n^2$
$\implies x^2$ has 4 as a factor
$\implies x^2$ is divisible by 4

Each of the $\implies$ are justified by our "knowledge" of integers. This is clearly quite a hand-wavy way of explaining things. But let's accept this for now!

Aside

Try to show that $(A \implies A_1) \wedge (A_1 \implies A_2) \cdots \wedge (A_k \implies B)$ logically implies $A \implies B$ as an exercise.

Direct Proof of the contrapositive

Recall that the contrapositive of $A \implies B$ is $\neg B \implies \neg A$. Since the contrapositive is logically equivalent to itself. We can try to directly prove the contrapositive instead.

- $\neg A$: x is odd
- $\neg B$: x^2 is not divisible by 4

Proposition

If x^2 is not divisible by 4, then x is odd.

Direct Proof of the contrapositive

Proposition

If x^2 is not divisible by 4, then x is odd.

Proof

x^2 is not divisible by 4

$\implies x^2$ is not divisible by 2

$\implies x \cdot x$ is odd.

We know that if a product of two integers is odd, then the two integers must be odd.

$\implies x$ is odd.

Proof by contradiction

Before we consider proof by contradiction, we shall introduce an important tautology in propositional calculus: Reductio ad absurdum

$$((\neg P \implies C) \wedge (\neg P \implies \neg C)) \implies P$$

Aside

The proof is left as an exercise to the reader :)

Proof by contradiction

What if we replace P with $A \implies B$?

Exercise

Prove that that $\neg P$ is $A \wedge \neg B$

So by reductio ad absurdum, we get

$$(((A \wedge \neg B) \implies C) \wedge ((A \wedge \neg B) \implies \neg C)) \implies (A \implies B)$$

When we are doing a proof by contradiction, we are trying to find a good "C". "C" could be anything!

Proof by contradiction

Objective: Directly prove that $A \wedge \neg B \implies C$ and $A \wedge \neg B \implies \neg C$

Proof 1 (C := B)

x is even and x^2 is not divisible by 4
$\implies$ ...(Copying our direct proof)
$\implies$ x^2 is divisible by 4 and x^2 is not divisible by 4

Proof 1 is a direct proof in disguise as $A \wedge \neg B \implies B$ logically implies $A \implies B$

Proof 2 (C := A)

x is even and x^2 is not divisible by 4
$\implies$ ...(Copying our direct proof by contrapositive)
$\implies$ x is even and x is odd.

Proof by contradiction

Asisde

Note how a proof by contradiction is completely unnecessary in the above case. This happens frequently. Always check if your proof by contradiction can be simplified into a direct proof.

Further reading

Check out the proof of irrationality of $\sqrt{2}$ for a proper proof by contradiction.

Summary

These are equivalent

- If A then B
- If not B then not A

How do you prove "If A then B" by contradiction?

- Assume A and not B
- Find C such that both C and not C is true
- Make sure it isn't just a direct proof!

Further Reading

Further Reading

- TMUA notes on Logic and Proofs
- Introduction to mathematical logic (Chapter 1 Section 1.1 to 1.3)

TMUA notes feature a much more basic explanation of the concepts above.

The earlier sections of the lecture notes are heavily based on Introduction to mathematical logic. Refer to that book if you're interested in further mathematical logic.

It was difficult to cover all the material in this short lecture. Please read Introduction to mathematical logic if you have the time.