AAHK MATHS AND PHYSICS MINI-COURSE PROBLEM SHEET

TOBY LAM

Introduction

Please attempt every question. (No probs if you have simply ran out of time!) Even if you don't know the answer, please write down your ideas (what you tried to do, why that didn't succeed etc). If you're stuck on one subpart, you're encouraged to try the next subpart taking the previous subparts as given. More difficult questions are marked with asterisks. Don't feel bad if you can't do them!

Please submit your answers to the Google Classroom by August 19th. LaTeX or scanned pdf files are welcomed. Please fill in the scheduling excel file as well.

Feel free to contact me at `toby.lam@accessabroadhk.org` if you have any questions.

Pre-requisities

You should have a good working knowledge of derivatives and their rules (Sum, Product, Chain ...) and some knowledge on basic integrals. You can find an excellent series on calculus by 3b1b here: `https://www.youtube.com/watch?v=WUvTyaaNkzM&list=PL0-GT3co4r2wlh6UHTUeQsrf3mlS2lk6x`

We also want to do calculations with partial derivatives. They are very simlar with normal derivatives. A geometric understanding is not necessary. You can read more about partial derivatives here: `https://www.youtube.com/watch?v=WUvTyaaNkzM&list=PL0-GT3co4r2wlh6UHTUeQsrf3mlS2lk6x`

Rectilinear and 2D Motion

Exercise 0.1. *At time 0, a person stands on $(0, 5)$ and throws a ball horizontally with the velocity vector being $(2, 0)$. As the ball is under the effect of gravity and a bit of wind, we take the acceleration vector to be $(1, -10)$ at all times.*

(1) *Find a parametrization $\mathbf{r}(t)$. Hence find the displacement vector at $t = 1$. [Hint: Note that there's now a constant acceleration in the x-direction as well. You'd have to change the parametrization provided in the lecture notes slightly.]*
(2) *Find $\mathbf{r}'(t)$. Hence find the velocity vector at $t = 1$ and its magnitude.*
(3) *When does the projectile hit the ground?*

The following exercise is inspired by a question in the first problem sheet of the first year dynamics course. You're encouraged to skim the dynamics course lecture notes in the google classroom up til page 8 for more ideas.

Exercise 0.2. *A cannon at the origin O fires a shell with speed V at an angle α to the horizontal. The shell experiences a constant acceleration vector $(0, -g)$ due to gravity.*

(1) *Write down Newton's second law for the shell, with suitable initial conditions. Solve the differential equation (by repeated integration) to show that the trajectory of the shell is given by*

$$x(t) = tV\cos\alpha, \qquad y(t) = -\frac{1}{2}gt^2 + tV\sin\alpha$$

(2) *Express t in terms of x, α and hence show*

$$y = \frac{-g}{2V^2\cos^2\alpha}x^2 + x\tan\alpha$$

(3) *Treating x as constant, show that the derivative of y with respect to α is*

$$\frac{\partial y}{\partial\alpha} = \frac{x}{\cos^2\alpha}\left(\frac{-gx\tan\alpha}{V^2} + 1\right)$$

(4) *** Suppose that V is fixed but the angle α can be varied. Show that upper boundary curve $u = u(x)$ of the set of points in the (x, y) plane which it is possible to hit with a shell is*

$$u(x) = \frac{-g}{2V^2}x^2 + \frac{V^2}{2g}$$

[Hint: Use part 3, what does $\frac{\partial y}{\partial\alpha} = 0$ signify?]

Exercise 0.3. *Consider some point particle of mass 1 moving along a straight line. You're given that*

$$F(r) = -kr$$

where $k > 0$.

(1) *Find out $V(r)$ up to a constant*
(2) *You're given that the only functions $r(t)$ that satisifies $\frac{d^2r}{dt^2} = -kr$ is of form $r(t) = A\sin(\sqrt{k}t + \psi)$ where A, ψ are real constants. Suppose you're also given that $A = r(0) = v(0) = k = 1$. What are the possible values of ψ?*
(3) *Find out $V(r(t))$ and $\frac{1}{2}|r'(t)|^2$ using the constants from (2). Sketch the graph of $r(t), V(r(t)), \frac{1}{2}|r'(t)|^2$ against t for $0 < t < 2\pi$*
(4) ** Plot displacement $r(t)$ against velocity $r'(t)$. What do you observe?*
    [Hint: Refer to uniform cicular motion]

[Remark: The question above is about Hooke's law / simple harmonic motion]

Exercise 0.4. **** Consider* $L(t, r, \frac{dr}{dt}) = \frac{1}{2}m(\frac{dr}{dt})^2 - gr$ *(The kinetic energy minus the potential energy of a point particle of mass m under gravity, this is called the Lagrangian)*

Prove that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \frac{dr}{dt}}\right) - \frac{\partial L}{\partial r} = m\frac{d^2 r}{dt^2} + g$$

[When doing the partial derivative $\frac{\partial L}{\partial \frac{dr}{dt}}$*, pretend that* $\frac{dr}{dt}$ *is a variable independent of r]*

so that the Euler-Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \frac{dr}{dt}}\right) - \frac{\partial L}{\partial r} = 0$$

gives you Newton's second law for this system.

Calculate $\frac{\partial L}{\partial \frac{dr}{dt}}$ *and* $\frac{dr}{dt} \times \frac{\partial L}{\partial \frac{dr}{dt}} - L$*. What is the physical meaning of those quantities?*

[Remark: The idea of constructing the Lagrangian and using the Euler-Lagrange Equation to represent the law of motions is extremely important in classical and quantum mechanics]

Waves

From now on, i denotes the imgaginary number $\sqrt{-1}$

You're given Euler's identity:

$$\exp(ix) = e^{ix} = \cos x + i \sin x$$

for real x.

The following exercise is covered in the first lecture of a second year mathematics cousrse at Oxford on Quantum Mechanics.

Exercise 0.5. *** We want to "deduce" the one-dimensional Schrodinger equation used in quantum mechanics. The one-dimensional complex plane wave is described by the following function* $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$

$$\psi(x, t) = A \exp(i(kx - wt))$$

where k is a real constant. A is a real constant (the ampltitute). w is a real constant (angular frequency). Notice the similarities between this and the wave equation we covered in the lectures. You can read more about the complex plane wave here `https://en.wikipedia.org/wiki/Sinusoidal_plane_wave`

(1) *Treating t as constant, show that the derivative of* ψ *with respect to x is*

$$\frac{\partial \psi}{\partial x} = ik\psi$$

[Hint: You can expand the exponential function into real and complex parts, or use the fact that $\frac{d}{dz}e^z = e^z$ *even for complex numbers z]*

(2) *Treating x as constant, show that the derivative of* ψ *with respect to t is*

$$\frac{\partial \psi}{\partial t} = -iw\psi$$

(3) *You're given de Broglie relations, which states that the energy E and momentum p are related to w and k by the following equations.*

$$E = \hbar w$$
$$p = \hbar k$$

where $\hbar$ is a real constant (Planck's constant). Suppose you're also given that

$$E = \frac{p^2}{2m}$$

Show that

$$ih\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial^2 x}$$

This is the one-dimensional version of the Schodinger's Equation with zero potential.

[Remark: A large majority of concepts and methods in quantum mechanics are heavily inspired by classical mechanics. An appreciation of classical mechanics would go far in aiding your understanding of quantum mechanics!]

Further Group Theory

The following exercises are **optional**

Exercise 0.6. *Given a group G, prove that isomorphisms (bijective homomorphisms) from G to itself,* $\mathrm{Aut}(G)$*, form a group under composition*

[Remember to check that the binary operation is well defined! That is $g * h \in \mathrm{Aut}(G)\ \forall g, h \in \mathrm{Aut}(G)$]

Exercise 0.7. *What is* $\mathrm{Aut}((\mathbb{Z}, +))$*? What is* $\mathrm{Aut}((\mathbb{R}, +))$*?*

A left action of a group on a set S is a map

$$p : G \times S \to S$$

such that

- $p(e, s) = s$ for all $s \in S$
- $p(g, p(h, s)) = p(gh, s)$ for all $s \in S$ and $g, h \in G$

Exercise 0.8. *Let S be the set of (twice differentiable) solutions $r(t)$ to the differential equation $\frac{d^2r}{dt^2} = -g$[1]. Show that the following $p : (\mathbb{R}, +) \times S \to S$ are left group actions*

(1) $p(\epsilon, r(t)) := r(t + \epsilon)$
(2) $p(\epsilon, r(t)) := r(t) + \epsilon$
(3) $p(\epsilon, r(t)) := r(t) + \epsilon t$

[Remember to check that the map is well defined! That is $p(g, s) \in S\ \forall g \in G, s \in S$]

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[1]i.e. S is the set of functions $r(t)$ such that $\frac{d^2r}{dt^2} = -g$ is true