On Physics and M2

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Introduction

- A lot of the formulae given in the HKDSE Physics formulae sheet can be deduced from calculus
- We're going to do that for selected equations, be curious and try to understand other equations as well!
- Aiming for fuller understanding of physics
- Contents
  - Rectilinear motion
  - Motion on the 2D plane (Circular / Projectile)
  - Waves

Introduction

- Rectilinear motion is one-dimensional motion along a straight line
- Consider a ball with constant mass m. From M2 we know

$$\begin{aligned} \text{Displacement} &= r(t) \\ \text{Velocity} &= v(t) = \frac{dr}{dt} \\ \text{Acceleration} &= a(t) = \frac{d^2r}{dt^2} \end{aligned}$$

Newton's Laws

Assuming mass is constant from now on

- Newton's first law

$$\text{Momentum} \quad = p(t) = mv(t)$$

- Newton's second law

$$\text{Force} \quad = F(t) = ma(t)$$

Assumptions

- In DSE physics, force is usually assumed to be **constant**. (e.g. gravitational force). This is crucial.
- By repeated indefinite integration we have

$$\frac{d^2r}{dt^2} = a$$
$$\frac{dr}{dt} = at + C_1$$
$$r(t) = \frac{1}{2}at^2 + C_1 t + C_2$$

- What are those constants?

Constants

We see that

$$v(0) = a \cdot 0 + C_1 = C_1$$
$$r(0) = \frac{1}{2}a \cdot 0 + C_1 \cdot 0 + C_2 = C_2$$

So putting it all together

$$v(t) = at + v(0)$$
$$r(t) = \frac{1}{2}at^2 + v(0)t + r(0)$$

Does this look familiar?

Consevation of energy

Let's make some definitions

$$\text{Kinetic energy} = T(t) = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2$$

$$\text{Potential energy} = V(r(t)) = -mar(t)$$

- We would like to show that $T(t) + V(r(t))$ is constant
- Note how potential energy is dependent on displacement only. So if energy is conserved, you could know the kinetic energy just by knowing the displacement.
- This is clearly not true in general

Conservation of energy

For the case of constant acceleration, we can do a direct but unsatisifying calculation

$$
\begin{aligned}
&(T(t) + V(r(t))) \\
&= \left[\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 - mar\right] \\
&= m\left[\frac{1}{2}\left(at + v(0)\right)^2 - a(\frac{1}{2}at^2 + v(0)t + r(0))\right] \\
&= m\left[\frac{1}{2}a^2t^2 + v(0)at + \frac{1}{2}v(0)^2 - \frac{1}{2}a^2t^2 - av(0)t - ar(0)\right] \\
&= \frac{1}{2}mv(0)^2 - mar(0) = T(0) + V(r(0))
\end{aligned}
$$

What about other systems?

Conservation of energy

Conservative force

If there exists a potential function $V(r)$ such that $F(r) = -\frac{d}{dr}V(r)$, then energy is conserved, i.e. $T(t) + V(r(t))$ is constant

- A force that satisfies the above conditions is called a conservative force
- Note that a conservative force has to be dependent on the displacement only
- Potential functions vary up to a constant

Examples	$F(r)$	$V(r)$
Spring / Harmonic Oscillator	$-kr$	$\frac{k}{2}r^2$
Constant	$-g$	gr
Gravitational / Coulomb force	$\frac{1}{r^2}$	$\frac{1}{r}$

Conservation of energy

Proof

Consider the derivative of $T(t) + V(r(t))$

$$\frac{d}{dt}\left[\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + V(r(t))\right]$$

$$= \frac{1}{2}m \cdot 2\frac{d^2r}{dt^2} \cdot \frac{dr}{dt} + \frac{dV}{dr}\frac{dr}{dt} \qquad \text{product and chain rule}$$

$$= m\frac{d^2r}{dt^2} \cdot \frac{dr}{dt} - m\frac{d^2r}{dt^2} \cdot \frac{dr}{dt} \qquad -m\frac{d^2r}{dt^2} = -F(r) = \frac{dV}{dr}$$

$$= 0$$

The only function with zero derivative is the constant function.

An even better way of thinking about this is with Lagrangians

Introduction

- We now move on to motion on the 2D plane
- We need a mathematical machinery called curves

Curves

Curves

A parameterized curve is a differentiable function $\mathbf{r} : U \to \mathbb{R}^2$ where U is some interval, e.g. $U = \{t \mid a < t < b\}$

- A function that takes in time and outputs a coordinate in 2D space
- We differentiate $\mathbf{r}$ by taking derivative of its components

Example

A ball moving 1 unit along the x-axis with 1 unit per second
$\mathbf{r}(t) = (t, 0)$, for $0 < t < 1$
$\mathbf{r}'(t) = (1, 0)$

Examples

Example

A projectile under the effect of gravity, with initial velocity $(3, 4)$

$$\mathbf{r}(t) = (3t, 4t - \frac{9.81}{2}t^2), \text{for } 0 < t < 1$$
$$\mathbf{r}'(t) = (3, 4 - 9.81t)$$
$$\mathbf{r}''(t) = (0, -9.81)$$

Example

A ball uniformly rotating around the origin

$$\mathbf{r}(t) = (\cos t, \sin t)$$
$$\mathbf{r}'(t) = (-\sin t, \cos t)$$

Projectile Motion

- We make the crucial assumption that the only force acted on the projectile is gravitational force
- We then make use of Newton's second law
- Use repeated integration on the x and y coordinate

$$\mathbf{r}''(t) = (0,\ -g)$$
$$\mathbf{r}'(t) = (C_1^*,\ -gt + C_1)$$
$$\mathbf{r}(t) = (C_1^* t + C_2^*,\ -\frac{1}{2}gt^2 + C_1 t + C_2)$$

- Once again the constants correspond to initial position / velocity

$$\mathbf{r}'(0) = (C_1^*,\ C_1)$$
$$\mathbf{r}(0) = (C_2^*,\ C_2)$$

Conservation of Energy

- Why is it that energy is still conserved?
- In general, $\mathbf{r}(t) = (x(t), y(t))$ for some functions $x(t), y(t)$. We have

$$\mathbf{r}'(t) = (x'(t), y'(t))$$
$$\mathbf{r}''(t) = (x''(t), y''(t))$$

- As such we have expressions for kinetic and (gravitational) potential energy as follows

$$\text{Kinetic energy} = T(t) = \frac{1}{2}m(x'(t)^2 + y'(t)^2)$$
$$\text{Potential energy} = V(\mathbf{r}(t)) = mgy(t)$$

Proof

We now prove that the total energy is conserved for uniform projectile motion i.e. when
$\mathbf{r}(t) = (x(t), y(t)) := (C_1^* t + C_2^*,\ -\frac{1}{2}gt^2 + C_1 t + C_2)$

$$
\begin{aligned}
& T(t) + V(\mathbf{r}(t)) \\
&= \frac{1}{2}m(x'(t)^2 + y'(t)^2) + mgy(t) \\
&= \frac{1}{2}m(C_1^{*2} + (-gt + C_1)^2) + mg(-\frac{1}{2}gt^2 + C_1 t + C_2) \\
&= m\left[\frac{1}{2}C_1^{*2} + \frac{1}{2}g^2t^2 - gtC_1 + \frac{1}{2}C_1^2 - \frac{1}{2}g^2t^2 + gC_1 t + gC_2\right] \\
&= \frac{1}{2}m(C_1^{*2} + C_1^2) + mgC_2 = T(0) + V(\mathbf{r}(0)) \qquad \square
\end{aligned}
$$

Uniform Circular Motion

- Consider a ball uniformly rotating around the origin
- Two variables completely determine its behaviour: Radius and velocity
- Parametrize $\mathbf{r}(t) = (R\cos(kt),\ R\sin(kt))$ so we get

$$\mathbf{r}'(t) = (-Rk\sin(kt), Rk\cos(kt))$$
$$\mathbf{r}''(t) = (-Rk^2\cos(kt), -Rk^2\sin(kt))$$

Uniform Circular Motion

Vector	Magnitude
$\mathbf{r}(t) = (R\cos(kt),\ R\sin(kt))$	R
$\mathbf{r}'(t) = (-Rk\sin(kt), Rk\cos(kt))$	Rk
$\mathbf{r}''(t) = (-Rk^2\cos(kt), -Rk^2\sin(kt))$	Rk^2

- $\mathbf{r}(t) \perp \mathbf{r}'(t) \perp \mathbf{r}''(t)$
- Angular velocity $=$ $2\pi \div$ time taken to go around the circle

$$\frac{2\pi}{\frac{2\pi}{k}} = k$$

Wave Equation

- Fluids in real life are studied in fluid dynamics. The equations that govern their kinematics is extraordinary complicated, such as the Navier-Stokes Equations.
- Under specific and idealised conditions, we can deduce the wave equation, which gives us an approximate understanding of waves.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

- The wave equation still involves heavy multivariable calculus and would not be discussed here
- Instead we would explore the one-dimensional sinusoidal travelling wave

$$u(t, x) = A \sin(kx - wt + \psi)$$

More questions

$$u(t, x) = A \sin(kx - wt + \psi)$$

Exercises

- Show that the wavelength $\lambda = 1/k$
- Show that the angular frequency is w
- What does ψ represent?
- Can you come up with a parametrization for a stationary wave?

Why care about group theory?

Automorphisms of X are (bijective) **structure preserving** maps from X to X

$$\mathrm{Aut}(X)$$

Figure: symmetries = automorphisms of X

$\mathrm{Aut}(X)$ is ALWAYS a group!!

Structure of X	$\mathrm{Aut}(X)$
Set	Bijections (Permutation group S_n)
Group	Isomorphisms[1]
$\mathbb{R}^n$ (area and orientation)	Special linear group
$\mathbb{R}^n$ (additive / scalar multiplicative)	General linear group (invertible matrices)
$\mathbb{R}^n$ (diff. manifold)	Diff. functions with inverse diff. (diffeomorphisms)
$\mathbb{R}^n$ (topology)	Cont. functions with inverse cont.
$\mathbb{R}^n$ (projective space)	Projective general linear group
$\mathbb{C}$ (complex analysis)	Biholomorphic (bijective infinitely differentiable) functions
$\mathbb{C}$ (projective)	Mobius transformations

[1] I mean that the isomorphisms from a group to itself forms a group. Q8 of the first problem sheet wanted you to consider $\mathrm{Aut}((Z_{11}, +))$ which is of order 110.

Symmetries

- Let's think about $\mathrm{Aut}(X)$ where X is **some ideal subset of the set of all possible physical phenomena**

Example

Does a mirrored version of the world behave the same as the mirror image of the current world?

Symmetries

- Let's think about $\mathrm{Aut}(X)$ where X is **some ideal subset of the set of all possible physical phenomena**

Example

Does a mirrored version of the world behave the same as the mirror image of the current world?
Ans: Wu experiment says no! (1957 Nobel Prize in Physics)
You could think of it as whether the "reflection" group $\mathbb{Z}_2$ belongs in $\mathrm{Aut}(X)$

Example

- If I have the differential equation

$$-g = \frac{d^2 r}{dt^2}$$

  and I have some solution $r(t)$, you can check that $r(t) + a$ and $r(t + b)$ are solutions as well for any real constant a, b, without knowing what the general solution actually is.
- In fact, you can generate all of the solutions from the solution $r(t) = -gt^2$ alone.
- What does it mean physically?
  - This differential equation / physical law doesn't change under time or space translations

Symmetries

We want to think about $\mathrm{Aut}(X)$ where

$$X := \{r(t) \text{ s.t. } r'' = -g\}$$

Solution	Interpertation	"Lie" Group
$r(t+\epsilon)$	Time translation	$\mathbb{R}$
$r(t)+\epsilon$	Position translation	$\mathbb{R}$
$r(t)+\epsilon t$	"Galilean Boost"	$\mathbb{R}$

Table: Collection of symmetries of $r'' = -g$

These are called Lie symmetries as they correspond to Lie groups. You can check out Peter E. Hydon's book "Symmetry Methods for Differential Equations"

Conclusion

We've covered more advanced mathematics in three areas of physics

- Rectilinear motion
- Projectile motion
- Waves

You would see Electromagnetism in the next two lectures.

Theory	Gauge symmetry group
Electromagnetism	$\mathrm{U}(1)$
Yang-Mills	$\mathrm{SU}(2)$