# Non-locality and gauge freedom in Deutsch and Hayden's formulation of quantum mechanics

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## Abstract

Deutsch and Hayden have proposed an alternative formulation of quantum mechanics which is completely local. We argue that their proposal must be understood as having a form of 'gauge freedom' according to which mathematically distinct states are physically equivalent. Once this gauge freedom is taken into account, their formulation is no longer local.

## I. INTRODUCTION

Unitary quantum mechanics (that is, quantum mechanics without collapse of the wave function) has local interactions: the quantum state of a system (e.g. a qubit, or a spacetime region in quantum field theory) is affected only by influences which propagate via the quantum states of its immediate past light cone.[5]

As conventionally presented, though, QM does not have local *states*: if $S_1$ and $S_2$ are systems with quantum states $\rho_1$ and $\rho_2$, then because of entanglement the state of the composite system $S_1 \times S_2$ is not necessarily $\rho_1 \otimes \rho_2$.

Deutsch and Hayden[1] argue that this 'state nonlocality' is an artifact of the normal way in which we represent quantum states, and that it disappears in an alternative formalism which they propose. Their formalism is derived from the Heisenberg picture of quantum mechanics, in which the unitary time evolution is applied to the observables rather than to the state vector. In the normal understanding of that formalism, though, the state vector is still taken to express the physical state of the system (via its role in calculating expectation values) and the algebra of observable quantities is regarded as mathematical 'superstructure', used to help us to calculate those observables.

Deutsch and Hayden reverse this 'normal understanding'. They regard the state vector $|0\rangle$ as fixed, once and for all and independent of the physical state of the system, and they regard the state of a quantum system as literally given by the associated observables (so that the state of a qubit, for instance, is given by a triple of Heisenberg picture operators $S_x, S_y, S_z$ pertaining to the spin observables of that qubit). The dynamics of this theory are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{X}_i = \frac{-i}{\hbar}\left[\widehat{H}(\widehat{X}_1, \ldots \widehat{X}_n), \widehat{X}_i\right] \tag{1}$$

(where $\widehat{X}_1, \ldots \widehat{X}_n$ are the observables of the theory). It is easy to see that the theory is local in both the interaction and the state senses, apparently vindicating Deutsch and Hayden's claims. (For further detail on the formalism and interpretation of Deutsch and Hayden's approach, see[4].)

## II. QUANTUM GAUGE TRANSFORMATIONS

Suppose $\widehat{V}(t)$ is a function from times to unitary operators, and suppose that for each $t$, $\widehat{V}(t)\,|0\rangle = \exp(-i\theta)\,|0\rangle$ (for arbitrary phase factor $\theta$). Then if the state is represented, according to Deutsch and Hayden, by observables $\widehat{X}_1, \ldots \widehat{X}_n$, suppose that we make the transformation

$$\widehat{X}_i(t) \longrightarrow \widehat{X}_i'(t) = \widehat{V}^\dagger(t)\widehat{X}_i(t)\widehat{V}(t). \tag{2}$$

If $\widehat{V}(t)$ is not a constant then this changes the dynamics to

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{X}_i' = \frac{-i}{\hbar}\left[\widehat{H}(\widehat{X}_1', \ldots \widehat{X}_n'), \widehat{X}_i'\right] + \frac{-i}{\hbar}\left[\widehat{V}^\dagger(t)\frac{\mathrm{d}}{\mathrm{d}t}\widehat{V}(t), \widehat{X}_i'\right]. \tag{3}$$

It does not, however, change anything observable, since everything observable is given by the expectation values of observables with respect to $|0\rangle$, and clearly

$$\langle 0|\,\widehat{X}_i'\,|0\rangle = \langle 0|\,\widehat{X}_i\,|0\rangle\,. \tag{4}$$

To understand the significance of these 'quantum gauge transformations', it is useful to consider an analogous example: electromagnetism in the context of the Aharonov-Bohm (A-B) effect.[2] Recall: the electromagnetic potential $\mathbf{A}$ couples to electron wavefunctions via the rule

$$\widehat{P} \longrightarrow \widehat{P} + e\mathbf{A}. \tag{5}$$

If an electron beam is split, passed on either side of a solenoid, and recombined, there will be interference between the beams, and as the field in the solenoid is varied the interference fringes will shift by an amount proportional to the line integral of $\mathbf{A}$ around the electron's path. This occurs despite the fact that the magnetic field outside the solenoid is zero, or nearly so.

The A-B effect makes clear that the electromagnetic potential $\mathbf{A}$, and not just the fields $\mathbf{E}$ and $\mathbf{B}$, must be regarded as physically significant; however, all observable quantities (including the A-B effect itself) are invariant under gauge transformations

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \nabla f \tag{6}$$

for arbitrary smooth functions $f$ (along with an associated transformation of the wavefunction).

It is generally accepted that the correct response to this observation is to regard gauge-equivalent $\mathbf{A}$s as describing the same physical situation, so as not to burden our theory with massive indeterminism (caused by the possibility of arbitrary *time-dependent* gauge transformations) and with an excess of unobservable properties (caused by the fact that the observable data *right now* only fixes the state up to a gauge transformation).

However, this does come with a price: if we identify gauge-equivalent vector potentials then our theory has non-local states in the sense described above. For while the Aharonov-Bohm vector potential cannot be gauge-transformed to zero everywhere, it can be in any region which does not completely enclose the solenoid. Since a region which *does* enclose the solenoid can be decomposed into regions which do not, it follows that whether the solenoid-enclosing region induces an A-B effect is not determined by the properties of its parts.

The loop representation of $\mathbf{A}$ makes this state non-locality manifest. We replace $\mathbf{A}$ with the *loop phases*

$$C_\gamma = \int_\gamma \mathbf{A} \cdot \mathrm{d}x \tag{7}$$

where $\gamma$ is any closed loop. $\mathbf{A}$ is fixed up to gauge transformations by the $C_\gamma$, and $\mathbf{B}_i$ is given at a point $\mathbf{x}$ by the loop phase for an infinitesimal loop in a plane perpendicular to $\mathbf{e}_i$. A loop which encloses the solenoid cannot be expressed as the sum of loops which do not enclose the solenoid, so the loop representation has nonlocal states.

## III. LESSONS FOR QUANTUM MECHANICS

The same arguments which lead us to identify gauge-equivalent vector potentials should lead us to identify gauge-equivalent quantum states. Specifically:

1. The possibility of time-dependent quantum gauge transformations makes it undetermined which dynamical equations give the true dynamics for the quantum state: is it

(1) or some (3)? (1) is somewhat simpler, but it is unclear whether this is sufficient: after all, in electromagnetism

$$\Box A_\mu = 0 \tag{8}$$

is a somewhat simpler choice of dynamics than those given by many other gauges, but this does not lead us to regard it as the 'true' dynamics.

2. Even time-independent gauge transformations make the state grossly underdetermined by observable data. Provided that $\widehat{V}\,|0\rangle = \exp(-i\theta)\,|0\rangle$, nothing whatever—no observable data, no theoretical considerations—can tell us that the physical state is given by $\widehat{X}_1, \ldots \widehat{X}_n$ rather than $\widehat{V}^\dagger \widehat{X}_1 \widehat{V}, \ldots \widehat{V}^\dagger \widehat{X}_n \widehat{V}$.

(There is also a more 'philosophical' concern: in a physical theory we would normally prefer that what is 'observable' (i. e., the expectation values derived from $|0\rangle$) would emerge from a physical analysis of measurement, rather than by *fiat*.)

This suggests that we should identify Deutsch-Hayden states which differ only by a gauge transformation. But if we do so, we return to the usual representation of quantum states! For two Deutsch-Hayden states are gauge-equivalent if and only if they have the same expectation values—and of course the expectation values of all possible measurements on a given quantum system are encoded in that system's density operator. So if we do identify gauge-equivalent states, we are again left with a theory whose states are non-local.

## IV. CONCLUSION

Deutsch and Hayden's proposal secures locality of states only at the cost of a gauge freedom closely analogous to the gauge freedom of electromagnetism. However, in quantum mechanics as in electromagnetism, to avoid problems of indeterminism and state underdetermination it is necessary to identify gauge-equivalent states. In quantum mechanics as in electromagnetism, if we do make this identification then it leads to nonlocality of states.

Deutsch and Hayden argue[1] (p. 1772) that if a theory is local according to any formulation, then it is local period. But their version of quantum mechanics is only a new formulation if we do indeed identify gauge-equivalent states. If not, it is not a 'new formulation': it is a new *theory*—with novel properties such as associating many distinct states to the same in-principle-observable data— albeit one which has the same observational consequences as the old theory. (Deutsch has himself insisted on this distinction in his more foundational work, for instance in discussing the de Broglie-Bohm interpretation[3]). It is a new theory which is genuinely local, but which pays an unacceptably high price for that locality.

We conclude that Deutsch and Hayden's proposal is best understood as a gauge theory whose gauge-independent physical properties are given by the normal quantum formalism. As such, although it may well give important insights into quantum-information issues such as information flow (for a detailed analysis of this point see[4]), it does not achieve the goal of showing that quantum mechanics is completely local. Rather, quantum mechanics has only local interactions, but has nonlocal states.

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1 David Deutsch and Patrick Hayden, "Information Flow in Entangled Quantum Systems," Proc. R. Soc. Lond. **A456**, 1759–1774 (2000), available online at http://xxx.arxiv.org/abs/quant-ph/9906007.

2 Yakhir Aharonov and David Bohm, "Significance of Electromagnetic Potentials in the Quantum Theory," Phys. Rev. **115** (3), 485–491 (1959).

3 David Deutsch, "Comment on Lockwood," Brit. J. Phil. Sci. **47**, 222–228 (1996).

4 Christopher G. Timpson, "Nonlocality and Information Flow: The Approach of Deutsch and Hayden," Found. Phys. **35** (2), 313-343 (2005).

5 In QFT, this is a consequence of the requirement that spacelike separated observables must commute.