

Functions of a complex variable (S1)

Problem sheet 1

I. Complex numbers; elementary functions; sets in the complex plane

1. Determine all fifth roots of unity: $w = \sqrt[5]{1}$.
2. Determine all cubic roots of $z = i - 1$: $w = (i - 1)^{1/3}$.
3. Solve the equation $(z + i)^5 + (z - i)^5 = 0$.
4. Find the real and imaginary parts of (a) $\cos i$ and (b) $\sin i$.
5. Verify that for any two complex numbers z_1 and z_2

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad , \quad |z_1 + z_2| \geq \left| |z_1| - |z_2| \right| \quad (\text{“triangle inequalities”}) \quad .$$

6. Solve the equations

$$(a) \quad e^z = -1 \quad , \quad (b) \quad e^z = -2 \quad , \quad (c) \quad \sinh z = 0 \quad , \quad (d) \quad \sin z = 3 \quad .$$

7. Draw the curves in the complex z plane

$$(a) \quad |z - 1| = 1 \quad , \quad (b) \quad (\operatorname{Re} z)^2 + 2 (\operatorname{Im} z)^2 = 1 \quad , \quad (c) \quad \operatorname{Im} z^2 = 2 \quad .$$

8. Find all the values of

$$(a) \quad i^i \quad , \quad (b) \quad (1 + i)^i \quad , \quad (c) \quad (-1)^{1/\pi} \quad .$$

9. Find the real and imaginary parts of the principal branch of the logarithm of

$$(a) \quad z = 1 - i \quad , \quad (b) \quad z = 1 + i \quad , \quad (c) \quad z = i \quad .$$

10. Consider the set $S = \left\{ \frac{i}{n} : n = 1, 2, \dots \right\}$ in the complex plane. (a) What are the limit points of S ? (b) What are the interior and boundary points of S ? (c) State whether or not S is open; closed; bounded; connected; compact.

II. Complex differentiation; holomorphic functions

11. Show that the function $f(z) = z^2$ is holomorphic in the entire complex plane, while the function $f(z) = |z|^2$ is holomorphic nowhere.
12. Determine the subsets of the complex z plane in which each of the following functions is holomorphic,

$$(a) \quad \frac{1}{e^z - 1} \quad , \quad (b) \quad \frac{1}{(1 + z^2)^2} \quad , \quad (c) \quad \cos \bar{z} \quad , \quad (d) \quad \tan z \quad ,$$

and calculate the derivative of each function in its region of holomorphy.

13. Let the function f be holomorphic on the domain D in \mathbb{C} .
- (a) Show that if f is real-valued then f must be constant.
- (b) Show that if $|f|$ is constant then f must be constant.
14. (a) Determine which of the following functions $u(x, y)$ are harmonic:
- (i) $x^2 - y^2 - x$, (ii) $\sin x \cosh y$, (iii) $e^{-x} \cos y + xy$, (iv) $\sin x - \cos y$, (v) $x - y$.
- (b) For each of the harmonic functions above, find a holomorphic function of which it is the real part, and find a harmonic conjugate function $v(x, y)$.
15. Take real-valued function u on domain D in \mathbb{C} , $u : D \rightarrow \mathbb{R}$. Let u be harmonic on D . Show that if u^2 is harmonic on D then u is constant.
16. (a) Find the holomorphic function whose imaginary part is given by

$$e^{-x}(x \cos y + y \sin y)$$

and which vanishes at the origin.

- (b) Determine the family of curves in the xy plane which are orthogonal to the curves

$$e^{-x}(x \cos y + y \sin y) = \text{const.} .$$

17. Locate and classify the singular points of the following functions, and determine whether they are isolated:

$$(a) \frac{\cos z}{(z+i)^3} , \quad (b) \frac{z^2}{(z^2-1)(z^2+4)} , \quad (c) \frac{1}{\sin 1/z} .$$

18. Classify the behaviour of the following functions at $z = \infty$:

$$(a) z(1+z^2) , \quad (b) e^z , \quad (c) \frac{1+z^2}{z^2} .$$

19. Consider the mapping specified by the function

$$f : z \mapsto w = z + \frac{1}{z} .$$

- (a) Give the subset of the complex z plane in which f is holomorphic.
- (b) Determine whether the mapping is conformal in the region of holomorphy.
- (c) Onto which subset of the w plane does f map the upper half and lower half of circle $|z| = 1$?
- (d) What is the image through f of a circle $|z| = \rho$ with $\rho \neq 1$?

20. Consider the mapping specified by the function

$$f : z \mapsto w = \frac{i-z}{i+z} .$$

- (a) Is the mapping conformal?
- (b) What is the image through f of the real axis $z = x$?
- (c) Onto which subset of the w plane does f map the upper half plane $\text{Im } z > 0$?