

Functions of a complex variable (S1)

Answers for Problem Sheet 3

1. (a) $-1 - 2 \sum_{n=1}^{+\infty} z^n, |z| < 1$ (b) $1 + 2 \sum_{n=1}^{+\infty} z^{-n}$
2. (a) $\sum_{n=0}^{+\infty} (-1)^n [(1 - 3^{-n-1})/2] z^n$ (b) $-1/6 - (1/2) [\sum_{n=1}^{+\infty} (-1)^n 3^{-n-1} z^n + \sum_{n=1}^{+\infty} (-1)^n z^{-n}]$
 (c) $(1/2) \sum_{n=2}^{+\infty} (-1)^n (3^{n-1} - 1) z^{-n}$ (d) $\sum_{n=0}^{+\infty} (-1)^n 2^{-n-1} (z+1)^{n-1}, \quad 0 < |z+1| < 2$
3. (a) $\sum_{n=0}^{+\infty} z^{n-2}$ (b) $-\sum_{n=0}^{+\infty} z^{-n-3}$
4. (a) $\sum_{n=0}^{+\infty} (-1)^n z^{3n+1}$ (b) $\sum_{n=0}^{+\infty} (-1)^n z^{-3n-2}$
5. $1/z + 1 - z/2 - 5z^2/6 + \dots$
6. $1 - 5(z+2)^{-1} - (1/6)(z+2)^{-2} + (5/6)(z+2)^{-3} + \dots, \quad |z+2| > 0$
7. $(1/9)(z-3)^{-2} - (2/27)(z-3)^{-1} + (1/27) - (4/243)(z-3) + \dots, \quad 0 < |z-3| < 3$
8. (a) $z = i/[\pi(2n+1)], n = 0, \pm 1, \dots$: simple poles; $z = 0$: non-isolated singularity (b) no
9. (a) pole of order 2 (b) essential singularity
 (c) holomorphic (d) non-isolated singularity
10. i): (a) $1/z^3 - 1/(6z) + 7z/360 + \dots, \quad 0 < |z| < \pi$ (b) pole of order 3 (c) $-i\pi/3$
 ii): (a) $z + 1/z + 1/(2z^3) + \dots, \quad |z| > 0$ (b) essential singularity (c) $2\pi i$
 iii): (a) $1/z^5 - 1/(2z^3) + 1/(24z) - z/720 + \dots, \quad |z| > 0$ (b) pole of order 5 (c) $i\pi/12$
 iv): (a) $1 + 1/(2z^2) + 1/(24z^4) + \dots, \quad |z| > 0$ (b) essential singularity (c) 0
11. (a) $z = 1/2$ pole of order 2; $\text{Res}_{z=1/2} f = -1/4$
 (b) $z = 0$ essential singularity; $\text{Res}_{z=0} f = 0$
 (c) $z = n\pi (n = 0, \pm 1, \dots)$ poles of order 1; $\text{Res}_{z=n\pi} f = 1$
12. (a) 0 (b) $-2\pi i$ (c) $i\pi/e$
13. (a) $\pi/6$ (b) $\pi/(2e^3)$ (c) $2\pi/3$
14. $\pi e^{-|\lambda|}$
15. (a) 2π (b) 0 (c) -2π (d) 0
16. $\pi\sqrt{2}$
17. (a) $\pi^2/6$ (b) $(\pi a \coth \pi a - 1)/(2a^2)$
18. (a) $-\pi^3/4$ (b) $i) 0$ $ii) \pi^3/8$
19. $2i\sqrt{\pi}$