Functions of a complex variable (S1)

Problem sheet 4

I. Residue calculus (part 2)

1. (a) Use complex contour integration to compute

$$I = \int_0^{\infty} \frac{1}{1+x^3}\, dx \quad .$$

[Hint: Evaluate the integral of the complex-valued function $f(z) = 1/(1+z^3)$ round the contour Γ in Fig. 1 using residue theorem. Relate this result for $R \to \infty$ to the given integral I.]

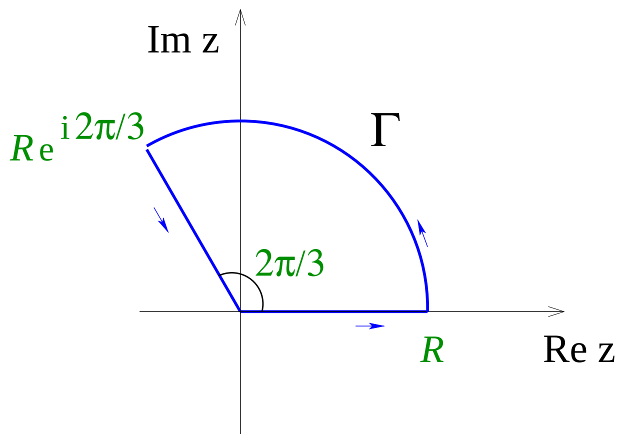

Fig.1

(b) Generalize the computation in (a) to calculate

$$I_n = \int_0^{\infty} \frac{1}{1+x^n}\, dx$$

for any $n \geq 2$. [Modify the contour in Fig. 1 so that the circular arc goes from R to $Re^{2\pi i/n}$]

2. (a) Use complex contour integration to compute

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x}\, dx \quad .$$

[Suggestion: Integrate the complex-valued function $f(z) = \exp(iz)/z$ round the contour Γ in the complex plane depicted in Fig. 2. Evaluate this integral using Cauchy theorem, then relate it to the given real-variable integral. Observe that while the contribution from the semicircle of radius R in Fig. 2 vanishes for $R \to \infty$, the contribution from the semicircle of radius r is non-vanishing for $r \to 0$.]

(b) Apply similar methods to compute

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\, dx \quad .$$

[Suggestion: Take the complex-valued function $f(z) = [\exp(2iz) - 1]/z^2$ and integrate along the same contour Γ as in Fig. 2.]

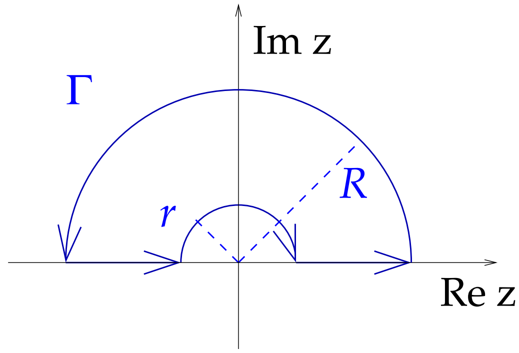

Fig.2

3. Calculate the integral of the functions

$$(a)\ \frac{\sin z}{z^4}\ ,\qquad (b)\ \frac{1}{z^4 \sin z}\ ,\qquad (c)\ \frac{\cos z}{z^4}\ ,\qquad (d)\ \frac{\tan z}{z^4}$$

round the circle with centre at the origin and radius 1 in the complex z plane.

4. Let C be the circle $|z| = 4$ in the complex z plane. Consider the functions

$$(a)\ f(z) = \frac{z+1}{z}\quad ,\qquad (b)\ f(z) = \frac{1}{z^2}\quad .$$

For each of these functions f, determine the winding number of the image of C through f about the origin

$$J = \frac{1}{2\pi}\ \Delta_C\ \arg f(z)\ \ .$$

Then determine the number of poles P and number of zeros N of f inside C, including their multiplicity, and verify the argument principle, $J = N - P$.

5. Determine the number of roots of the equations

$$(a)\ \ z^7 - 4z^3 + z - 1 = 0\ \ ,\qquad (b)\ \ 3z^6 = e^z$$

in the disk $|z| < 1$.

6. Determine the number of roots of the equation

$$2 + z^2 - e^{iz} = 0$$

in the upper half plane.

7. Calculate the sum of the following series

$$(a)\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\ \ ,\qquad (b)\ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$$

using complex contour integration methods.

8. Apply complex-plane techniques for integration in the presence of branch cuts to evaluate the following real integrals:

$$(a)\ \int_0^{\infty} \frac{\sqrt{x}}{x^2+1}\, dx\ \ ,\qquad (b)\ \int_0^{\infty} \frac{\sqrt[3]{x}}{x^2+x}\, dx\ \ .$$

9. Apply complex-plane techniques for integration in the presence of branch cuts to evaluate the following real integrals:

$$(a)\ \int_0^{\infty} \frac{\ln x}{(1+x^2)^2}\, dx\ \ ,\qquad (b)\ \int_0^{\infty} \frac{\ln(1+x^2)}{1+x^2}\, dx\ \ .$$

II. Integral transforms

10. Calculate the integral

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\omega - i\varepsilon}\,d\omega$$

for real negative t and for real positive t, with ε a positive real constant. Verify that this provides an integral representation for the step function $\Theta(t)$ [$\Theta(t) = 1$ for $t > 0$, $\Theta(t) = 0$ for $t < 0$].

11. Consider the rectangular pulse in Fig. 3a

$$(a)\;\; f(x) = \Theta(1 - |x|)$$

and the triangular pulse in Fig. 3b

$$(b)\;\; f(x) = (1 - |x|)\;\Theta(1 - |x|)\;\; .$$

In each case, evaluate the Fourier transform

$$\widetilde{f}(\omega) = \int_{-\infty}^{+\infty} dx\; e^{-i\omega x}\; f(x)\;\; .$$

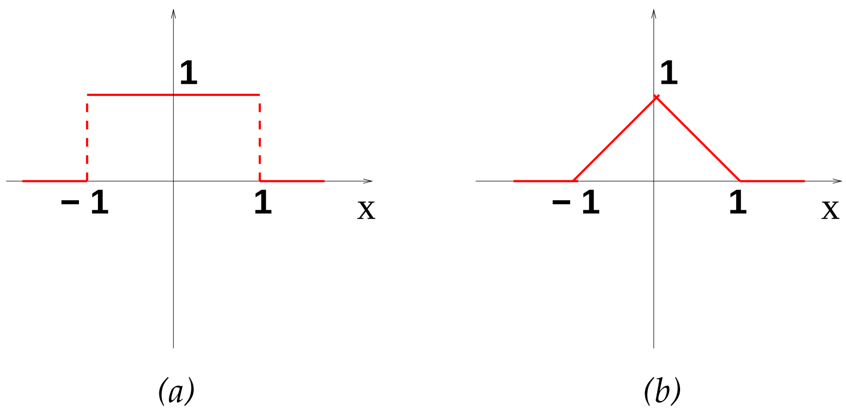

Fig.3

12. Calculate the Laplace transform

$$F(z) = \int_{0}^{+\infty} dt\; e^{-zt}\; f(t)$$

of the functions

$$(a)\;\; f(t) = \sin t\;\; , \qquad (b)\;\; f(t) = t\sin t\;\; , \qquad (c)\;\; f(t) = \cosh t\;\; .$$

13. Take the function

$$F(z) = \frac{1}{\sqrt{z}}\;\; .$$

Calculate the integral in the complex z plane

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{zt}\; F(z)\; dz \qquad (a > 0)\;\; ,$$

which defines the inverse Laplace transform of $F(z)$. The integral runs along the straight line parallel to the imaginary axis and with real part a ($a > 0$). [Hint: Evaluate the integral round the closed path Γ in Fig. 4 by residue theorem, and relate it to the integral that defines $f(t)$.]

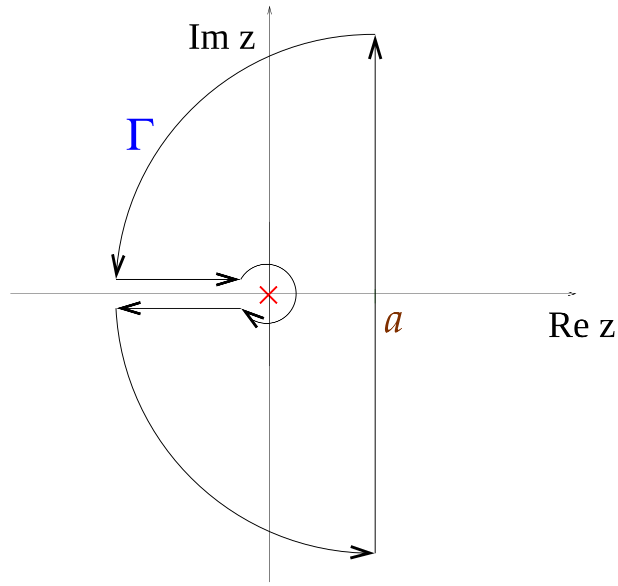

Fig.4

14. Suppose the function $u(x,t)$ satisfies the differential equation

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2}$$

for $-\infty < x < \infty$ and $t \geq 0$, with the initial-value condition

$$u(x,0) = h(x) \ \ ,$$

where λ is a real positive constant (diffusion coefficient) and $h(x)$ is a given function of x. Consider the Fourier transform of u with respect to x

$$\widetilde{u}(p,t) = \int_{-\infty}^{+\infty} dx \ e^{-ipx} \ u(x,t) \ \ .$$

(a) Obtain the differential equation for $\widetilde{u}(p,t)$ and the corresponding initial-value condition.
(b) Write the solution of the initial-value problem for $\widetilde{u}(p,t)$.
(c) Determine the solution $u(x,t)$ of the starting equation as a Fourier convolution integral over the initial distribution $h(x)$.

15. Apply the Laplace transformation to determine the function $y(t)$ satisfying the equation

$$y(t) + \int_0^t dt' \ (t - t') \ y(t') = \sin 2t \ \ , \qquad t > 0 \ \ .$$

(a) First, Laplace-transform the equation and show that the second term on the left hand side of the equation gives $\widetilde{y}(z)/z^2$, where $\widetilde{y}$ is the Laplace transform of y,

$$\widetilde{y}(z) = \int_0^{+\infty} dt \ e^{-zt} \ y(t) \ \ .$$

Determine the explicit expression for $\widetilde{y}(z)$.
(b) Next, find the solution for the function $y(t)$ by evaluating the inverse Laplace transform of $\widetilde{y}(z)$.