Lectures on Complex Functions

Pedro G. Ferreira
Astrophysics
University of Oxford,
DW Building, Keble Road
Oxford OX1 3RH, UK

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Preamble

The following set of lectures was designed for the Oxford Undergraduate course and are given during the BA course. These notes are not particularly original and are based on a number of books and lectures. In particular I have used

- *Mathematical Methods for Physics and Engineers*, 3rd Edition, K.F. Riley, M.P. Hobson & S.J. Bence (CUP)
- *Complex Variables and Applications*, 5th Edition, R. V. Churchill & J.W.Brown (McGraw Hill).
- *Complex Analysis*, 3rd Edition, L.V.Ahlfors (McGraw Hill).
- *Introduction to Complex Analysis*, 2nd Edition, H.A.Priestley, (OUP)
- *Schaum's Outlines: Complex Variables*, 2nd Edition, M.R.Spiegel, S.Lipschutz, J.J.Schiller & D.Spellman (McGraw Hill).

You can find many texts and lecture notes out there that you can consult.

I would like to thank Graham Ross and John Wheater for passing me their lecture notes; I based these notes on theirs. I have used the `tikz` latex package to produce these plots, adapting multiple examples that can be found online. I would greatly appreciate it if you could email me at `pedro.ferreira@physics.ox.ac.uk` with any corrections. I would like to thank T. Acharya, R. Bedford, G. Gajic, M. Gazi, S. Gunatilleke, G. Jandu, M. Hutt, S. Jia, M. Myszkowski, S. Moore, J. Newnham, R. Nuckchady, D. Siretanu, K. Vasiliou, O. Witteveen, A. Wojcik and J. Wood for pointing out mistakes in an earlier draft of these notes.

1 Complex numbers and functions.

By now you will have learnt what seem like two distinct fields of mathematics: complex numbers and vector calculus. You may have guessed that there is a connection between the two. In these lectures I am going to show you that there is, and more. I hope to convince you that the magical properties of complex functions can help you do unexpected things (in math, of course).

Let us begin. Consider the 2-D vector, with elements x and y and represent it as

$$(x, y) \tag{1}$$

We know how to add two such vectors:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \tag{2}$$

But what about multiplication? Let me propose the following multiplication rule

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) \tag{3}$$

It looks a bit odd but has all the right properties (it is associative, commutative, etc). It might look familiar if we rewrite our vector as

$$(x, y) = x\hat{e}_1 + y\hat{e}_2 \tag{4}$$

with basis vectors $\hat{e}_1$, $\hat{e}_2$ and imbue them with the following multiplication rules

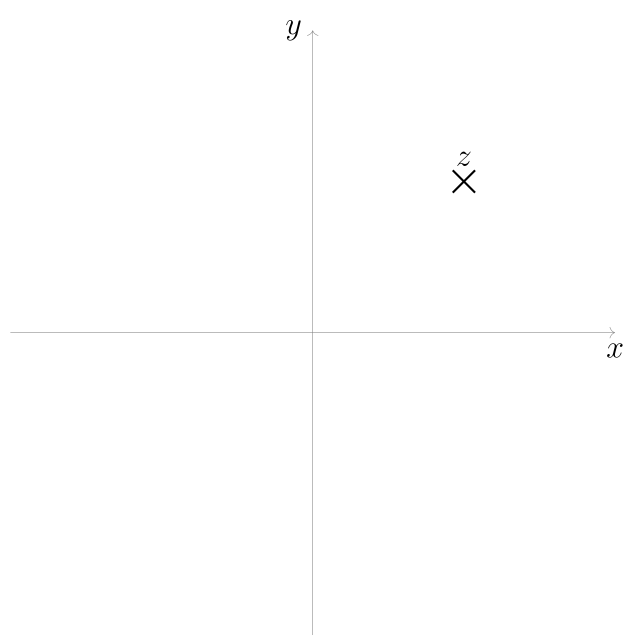
$$\begin{aligned}
\hat{e}_1 \cdot \hat{e}_1 &= \hat{e}_1 \\
\hat{e}_1 \cdot \hat{e}_2 &= \hat{e}_2 \\
\hat{e}_2 \cdot \hat{e}_1 &= \hat{e}_2 \\
\hat{e}_2 \cdot \hat{e}_2 &= -\hat{e}_1
\end{aligned} \tag{5}$$

Now rename these basis vectors: $\hat{e}_1 = 1$ and $\hat{e}_2 = \mathrm{i}$. The two dimensional vector will look very familiar. Let us call it z:

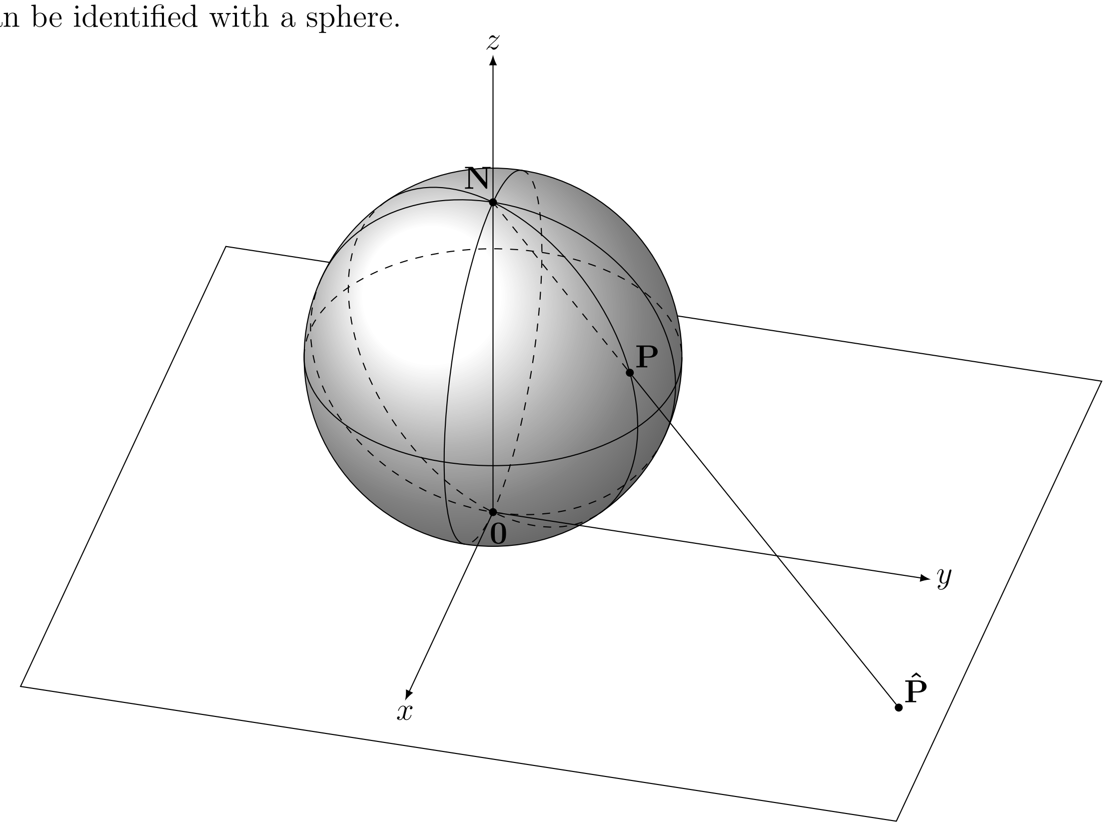
$$z = x + \mathrm{i}y \tag{6}$$

where I have dropped the 1 for obvious reasons. As has been pretty obvious, all along we have been talking about complex numbers, emphasizing the fact they live on a 2-D surface.

The 2-D surface, or plane, on which complex numbers lives is often called the *Argand* plane and is useful for seeing what happens to complex numbers when you do things to it.



We can complicate things just a little bit by adding the improper number ∞ to the plane to, in some sense, "close it off". A good way of seeing this is to think of a unit sphere sitting on the plane, with the south pole on the point $(0,0,0)$, $\mathbf{0}$, and the north pole at $(0,0,2)$, $\mathbf{N}$. Now, at each point on the sphere, $\mathbf{P}$, consider a line that also intersects with the north pole, $\mathbf{N}$ and the $x-y$ plane, at a point $\hat{\mathrm{P}}$. We will then have a unique mapping from the plane onto the sphere for all points. Of most interest will be the north pole, $\mathbf{N}$: *any* point at infinity on the plane will be identified with the north pole $\mathbf{N}$. I.e. in this way, the complex plane plus ∞ can be identified with a sphere.



We can find explicit expressions as follows: if (x,y,z) are coordinates on the sphere and (X,Y) are coordinates on the plane, we have that the mapping and its inverse is

$$\begin{aligned} (X,Y) &= \left(\frac{2x}{2-z}, \frac{2y}{2-z}\right) \\ (x,y,z) &= \left(\frac{4X}{4+X^2+Y^2}, \frac{4Y}{4+X^2+Y^2}, 1+\frac{-4+X^2+Y^2}{4+X^2+Y^2}\right) \end{aligned} \tag{7}$$

This is known as the *stereographic* projection; the sphere is known as the *Riemann* sphere. In some sense it brings ∞ into the fold, on par with other complex numbers. This will be important later on in this course.

There are a number of interesting things you can do, based on the fact that a complex number lives on the plane. The first thing is you can reflect it across the x axis. This corresponds to flipping the sign in front of the y component. This is the process of conjugation and is denoted by

$$\bar{z} = x - \mathrm{i}y \tag{8}$$

It should be clear that this is quite a special operation and that $\bar{z}$ is not proportional to z. In other words, we have that:

$$\bar{z} \neq cz \tag{9}$$

where c is a another complex number.

We can combine z and $\bar{z}$ to find the length of the vector $|z|$:

$$|z|^2 = \bar{z} \cdot z \tag{10}$$

This also means that we can represent z in a completely different way, in terms of a length $r = |z|$ and and angle θ:

$$z = re^{\mathrm{i}\theta} \tag{11}$$

In particular we have *Euler's formula*:

$$e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta \tag{12}$$

The argument of z is often expressed as

$$\mathrm{Arg}\, z = \theta \tag{13}$$

This representation has a number of interesting properties. For a start we can see that θ can be multivalued, i.e.

$$\mathrm{Arg}\, z = \theta + 2n\pi, \tag{14}$$

where n is an integer, all give the same z. Furthermore it can be used to derive *DeMoivre's formula*

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)], \tag{15}$$

This expression is of particular interest if we want to look at the roots of z. We then have

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \left(\cos\frac{\theta + 2k\pi}{n} + \mathrm{i}\sin\frac{\theta + 2k\pi}{n} \right), \quad \mathit{for} \quad k \in [0, n-1] \tag{16}$$

This shows how multiple roots arise but even more interestingly, shows that they lie on a polygon with n vertices on the complex plane.

Let us now turn to functions on the plane, $f(x, y)$. The simplest case is when f is real-valued, i.e. the output is one real number. Our focus here will be on functions that map the plane onto itself, i.e.

$$f : (x, y) \to (u, v) \tag{17}$$

where $u(x, y)$ and $v(x, y)$. A priori we have that u and v can be any functions of (x, y).

As examples, we can consider first the following functions

$$f_1 : (x, y) \to (x^2 - y^2, 2xy) \tag{18}$$

$$f_2 : (x, y) \to (x^2 + y^2, 0) \tag{19}$$

$$f_3 : (x, y) \to (x^3 + xy, 0) \tag{20}$$

An interesting exercise is to note that we can represent vectors on the plane in terms of z and its conjugate $\bar{z}$ (recall that $\bar{z}$ is not collinear with z so suits us well). We have that

$$\begin{aligned} x &= \frac{1}{2}(z+\bar{z}) \\ y &= -\frac{i}{2}(z-\bar{z}) \end{aligned} \tag{21}$$

which means that we can rewrite

$$f_1 : (z,\bar{z}) = z^2 \tag{22}$$

$$f_2 : (z,\bar{z}) = \bar{z}z \tag{23}$$

$$f_3 : (z,\bar{z}) = \frac{1}{8}(z^3+3z^2\bar{z}+3z\bar{z}^2+\bar{z}^3) - \frac{\mathrm{i}}{4}(z^2-\bar{z}^2) \tag{24}$$

The first two cases look simple, simpler than if we had written the function in terms of x and y while I hope you will agree that the third case looks a mess.

What I would like to focus on, in these lectures, is functions which *only* depend on z and not on $\bar{z}$, i.e. functions like f_1 in the set of examples above. In other words, we will look at functions that satisfy

$$\frac{\partial f}{\partial \bar{z}} = 0 \tag{25}$$

Let us look at a few examples. A particularly simple one is

$$f : (x,y) \to (x,y) \tag{26}$$

i.e.

$$f(z) = z \tag{27}$$

Slightly more complicated are

$$f(z) = z^n \quad \text{with} \quad n \in \mathbb{Z} \tag{28}$$

f_1 is a particular example of these We can build more complicated functions using series expansions to get, for example

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \\ f(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = \cos(z) \\ f(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = \sin(z) \end{aligned} \tag{29}$$

Note that some functions have interesting properties: they are multivalued. Such is the case of

$$f(z) = z^{\frac{1}{n}} \quad \text{with} \quad n \in \mathbb{N} \tag{30}$$

and

$$\ln z = \ln |z| + \mathrm{i}\mathrm{Arg}\, z = \ln r + \mathrm{i}\theta + \mathrm{i}2\pi k \quad \text{with} \quad k \in \mathbb{Z} \tag{31}$$

There are some simple functions which very clearly do not belong to this class. A notable example is f_2 that we showed above which we re-expressed as

$$f(z, \bar{z}) = z\bar{z} \tag{32}$$

In other words, there are functions which may look simple if expressed in terms of x and y which are not *analytic*.

2 Complex functions and the Cauchy-Riemann conditions.

A function $f(x, y)$ is called *analytic* when df/dz is well defined. This is a non-trivial statement, if you realize that z lives on the Argand plane and can take any direction you want. For df/dz to be well defined, you must obtain the same value, whatever direction you choose to take the derivative.

Consider two different directions, one along the x axis and another along the y axis. Recall that

$$f(x, y) = u(x, y) + \mathrm{i}v(x, y) \tag{33}$$

The two choices of directions give us

$$\begin{aligned} \frac{df}{dz} &= \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} \\ \frac{df}{dz} &= \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\mathrm{i}\delta y} = \frac{1}{\mathrm{i}}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \end{aligned} \tag{34}$$

Equating these two expressions we find the *Cauchy Riemann* conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{35}$$

It is interesting that we can obtain the Cauchy-Riemann condition, simply from the fact that f is a function of z alone. To see this, let us explore the fact that

$$\frac{\partial f}{\partial \bar{z}} = 0 \tag{36}$$

we have that

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right) \tag{37}$$

We then have

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)(u + \mathrm{i}v) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{\mathrm{i}}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0 \tag{38}$$

We can see that, for the expression to satisfy the last equality, the Cauchy-Riemann conditions must be satisfied.

Let us consider some examples. First of all

$$f = x^2 - y^2 + \mathrm{i}2xy \tag{39}$$

We have that

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x \quad \frac{\partial v}{\partial y} = 2x \\ \frac{\partial u}{\partial y} &= -2y \quad \frac{\partial v}{\partial x} = 2y \end{aligned} \tag{40}$$

Clearly it does satisfy the Cauchy-Riemann conditions. And indeed, we have that

$$f(z) = z^2 \tag{41}$$

Now consider

$$f = x^3 + 3y^2x + \mathrm{i}(3yx^2 - y^3) \tag{42}$$

We have that

$$\begin{aligned} \frac{\partial u}{\partial x} &= 3x^2 + 3y^2 \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \\ \frac{\partial u}{\partial y} &= 6yx \quad \frac{\partial v}{\partial x} = 6yx \end{aligned} \tag{43}$$

Clearly it *doesn't* satisfy the Cauchy-Riemann conditions.

The Cauchy-Riemann conditions can be used to show that, if f is analytic, then u and v are *harmonic* functions, i.e. they satisfy Laplace's equation in 2 dimensions. To see this, we can take one of the Cauchy-Riemann conditions and take another derivative

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \tag{44}$$

i.e.

$$\nabla^2 u = 0 \tag{45}$$

We can use the same process to show that

$$\nabla^2 v = 0 \tag{46}$$

This means that we can easily build analytic functions from *harmonic* functions but, more interestingly, we can easily find solutions to Laplace's equation by constructing analytic functions. For example, if we take

$$\ln z = \ln|z| + \mathrm{i}\,\mathrm{Arg}\, z \tag{47}$$

we have that both

$$\begin{aligned} u &= \ln|z| \\ v &= \operatorname{Arg} z \end{aligned} \tag{48}$$

are solutions to Laplace's equation. We will see that this fact is extremely useful for solving problems with appropriate symmetries.

The Cauchy-Riemann conditions link u and v somehow. We can see this by considering curves where we fix them to be constant. In otherwords, consider the lines in the Argand plane that satisfy

$$\begin{aligned} u(x,y) &= c_1 \\ v(x,y) &= c_2 \end{aligned} \tag{49}$$

where c_1, c_2 are constants. The normal vectors to these curves, $\mathbf{n}_u$ and $\mathbf{n}_v$ are

$$\begin{aligned} \mathbf{n}_u &= \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \\ \mathbf{n}_v &= \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) \end{aligned} \tag{50}$$

If we now take the dot product between these two vectors we have

$$\mathbf{n}_u \cdot \mathbf{n}_v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0 \tag{51}$$

where we have applied the Cauchy-Riemann conditions. As you can see, the two normal vectors are perpendicular to each other. In other words, curves of constant u and v are orthogonal to each other, wherever the function is analytic.

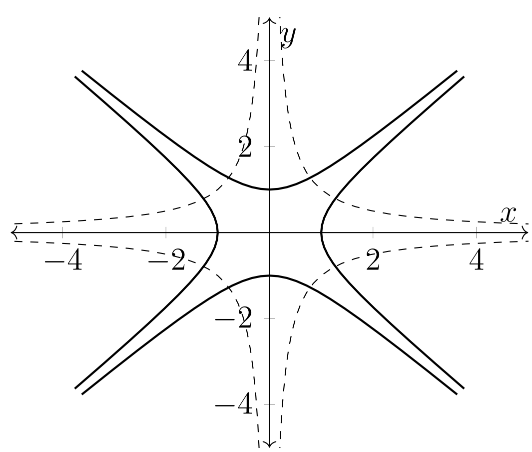
Let us look at an example and consider the function

$$f(z) = z^2 \tag{52}$$

We then have that

$$\begin{aligned} u(x,y) &= x^2 - y^2 \\ v(x,y) &= 2xy \end{aligned} \tag{53}$$

Lines of constant u and v are thus hyperbolae which are orthogonal to each other whenever they intersect.



Often we are given a part of $f(x,y)$ and we want to figure out what the corresponding $f(z)$. Sometimes it is easy and we can simply guess. For example, suppose we are given

$$v(x,y) = (x^2 - y^2) + y \tag{54}$$

and

$$f(0) = 0 \tag{55}$$

We can see that the first part is simply $\mathrm{Re}(z^2)$ while the second part is $\mathrm{Im}(z)$. We can guess that

$$f(z) = \mathrm{i}z^2 + z + c \tag{56}$$

where $c = 0$ to satisfy the condition at $z = 0$. While this is the easiest method, more generally we can use a systematic method. From the Cauchy-Riemann conditions we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -2y + 1 \tag{57}$$

which we can integrate to give

$$u = -2xy + x + h(y) \tag{58}$$

We now use Cauchy-Riemann again to find

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2x \tag{59}$$

to find

$$-2x + \frac{dh}{dy} = -2x \tag{60}$$

so

$$h(y) = c \tag{61}$$

and thus

$$u = -2xy + x \tag{62}$$

as we found above.

3 Mapping and transformation in the complex plane

Let us look at a generic complex function again:

$$f : (x,y) \to (u,v) \tag{63}$$

We can think of it as a mapping on the complex plane (or Riemann sphere). In vector calculus, we learnt different properties about mappings. For example, we know that it is one-to-one if the Jacobian is non-singular:

$$|J| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right| > 0 \tag{64}$$

Note that, if f is analytic, we can use the Cauchy-Riemann conditions to see that

$$|J| = \left|\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right| = \left|\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right| \tag{65}$$

Now, we have that

$$\begin{aligned} f'(z) \equiv \frac{df}{dz} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right) \\ &= \frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + \mathrm{i}\frac{\partial v}{\partial x} \end{aligned} \tag{66}$$

where we have used the Cauchy Riemann conditions in the last two equalities. Which means that

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \tag{67}$$

So we have that

$$|J| = |f'|^2 \tag{68}$$

There are a few transformations which are easy to vizualize:

- Translation:

$$f(z) = z + a \tag{69}$$

- Rotation:

$$f(z) = e^{\mathrm{i}\theta} z \tag{70}$$

- Stretching:

$$f(z) = cz \tag{71}$$

  where c is real.

- Inversion:

$$f(z) = \frac{1}{z} \tag{72}$$

All of these transformations map the full Argand plane onto itself.

It is interesting to look at other functions and see how they transform the plane. The function

$$f(z) = z^{\frac{1}{2}} \tag{73}$$

transforms the complex plane into the upper half plane, while the function

$$f(z) = z^{\frac{1}{4}} \tag{74}$$

transforms the complex plane into the first quadrant[1]. The function

$$f(z) = \log(z) = \log r + i\theta \tag{75}$$

transforms the complex plane into an infinite horizontal strip with boundaries at $y = 0$ and 2π.

A particularly interesting transformation is the *fractional* transformation or the *Möbius* map:

$$f(z) = \frac{az + b}{cz + d} \tag{76}$$

with $ad - bc \neq 0$. The *cross ratio*

$$\frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)} \tag{77}$$

is invariant under this transformation. Furthermore, if we combine two *Möbius* maps we end up with a *Möbius* map. For any trio of points z_1, z_2 and z_3 we can find a *Möbius* transformation that transforms

$$\begin{aligned} z_1 &\to 0 \\ z_2 &\to 1 \\ z_3 &\to \infty \end{aligned} \tag{78}$$

One particularly interesting example maps the half plane onto a circle:

$$f(z) = e^{\mathrm{i}\theta_0} \frac{z - z_0}{z - \bar{z}_0} \tag{79}$$

The point z_0 will be mapped onto the origin while $-\infty$ and $+\infty$ will be mapped onto the point $e^{\mathrm{i}\theta_0}$

Finally it is important to realize that f has a particularly important property: it preserves angles. To see this, consider two infinitesimal changes to z which are at an angle θ: $z_1 = z + \epsilon e^{\mathrm{i}\theta_1}$ and $z_2 = z + \epsilon e^{\mathrm{i}\theta_2}$ such that $(z_2 - z)/(z_1 - z) = e^{\mathrm{i}\theta}$. We now transform the two points

$$\begin{aligned} w_1 &= f(z_1) \\ w_2 &= f(z_2) \end{aligned} \tag{80}$$

[1]It should be clear that we are considering the principal root here.

But, given that the function is analytic, we can Taylor expand to get

$$\begin{aligned} w + \delta w_1 &= f(z) + f'(z)\epsilon e^{\mathrm{i}\theta_1} \\ w + \delta w_2 &= f(z) + f'(z)\epsilon e^{\mathrm{i}\theta_2} \end{aligned} \tag{81}$$

and so the angle between the two transformed points is

$$\mathrm{Arg}\left(\frac{\delta w_2}{\delta w_1}\right) = \mathrm{Arg}\left(\frac{f'(z)\epsilon e^{\mathrm{i}\theta_2}}{f'(z)\epsilon e^{\mathrm{i}\theta_1}}\right) = \theta \tag{82}$$

Such angle preserving transformations are known as *conformal* transformations.

4 Solutions to Laplace's equation

We know that the two parts of an analytic function are harmonic functions, i.e.

$$\begin{aligned} \nabla^2 u &= 0 \\ \nabla^2 v &= 0 \end{aligned} \tag{83}$$

This means that the components of an analytic function are solutions to the 2-D Laplace equation, which is ubiquitous in physical problems. Let us look at a few.

In electrostatics, we have that that the electrical field is given by

$$\vec{E} = -\vec{\nabla}\Phi \tag{84}$$

where Φ is the electrostatic potential. In a source free region, we have that Φ satisfies

$$\nabla^2\Phi = 0 \tag{85}$$

which, in 3D can be solved by the Coulomb potential

$$\Phi = \frac{q}{\kappa r} \tag{86}$$

for a charge q at $r = 0$.

In 2-D, with radial symmetry, Laplace's equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) = 0 \tag{87}$$

which we can solve to give us

$$\Phi = -2q\ln r \tag{88}$$

upto an integration constant. This is the electrostatic field due to a line charge of q per unit length. Gauss' law tells us that

$$\oint \vec{E}.d\vec{S} = E2\pi r = 4\pi q \tag{89}$$

where we know, from symmetry that the electric field is radial. But note that, in 2D, we can identify the electrostatic potential as one component of an analytic function

$$f(z) = \Phi + \mathrm{i}v \tag{90}$$

and recall that this Φ arises if we consider

$$f(z) = -2q \ln z \tag{91}$$

Building on what we found previously, we have that the equipotential lines, $\Phi =$ constant are orthogonal to $v =$ constant which in turn correspond to field lines for the electric field (given that $\nabla\Phi$ is orthogonal to equipotential lines).

It is interesting to consider another two examples where harmonic functions play a role. First of all, in the case of heat flow, we can define the heat flow across a surface, $\vec{Q}$ as given by

$$\vec{Q} = -\kappa \vec{\nabla} T \tag{92}$$

where T is the temperature. If there is no heat source, then we have that

$$\vec{\nabla} \cdot \vec{Q} = \nabla^2 T = 0 \tag{93}$$

Again, we can construct an analytic function

$$f = T + \mathrm{i}v \tag{94}$$

Now the isothermal lines will be orthogonal to the flux lines, given by $v =$ constant. Second, in the case of incompressible fluid flow, we can define a velocity potential, Φ_v such that the fluid velocity is given by

$$\vec{\mathrm{v}} = \vec{\nabla} \Phi_v \tag{95}$$

Now we have that

$$f = \Phi_{\mathrm{v}} + \mathrm{i}v \tag{96}$$

where the lines of constant v are the streamlines.

Let us now construct solution for particular geometries. Consider first the 2-D electrostatic problem of a uniform electric field along the x-axis. A solution to Laplace's equation which obeys the correct symmetries

$$\Phi = -Ex \tag{97}$$

corresponds to a uniform electrical field which is parallel to the x-axis:

$$\vec{E} = (E, 0) \tag{98}$$

But, with our new found knowledge of complex functions, we can construct a complex potential

$$f = \Phi + \mathrm{i}v \tag{99}$$

and note that

$$f = -Ez \tag{100}$$

is the solution to our problem. There is added information in the complex potential, f. While constant slices of the real part of f give us equipotential lines, constant slices of the imaginary part of f (i.e. of v) describe the field lines.

Alternatively, we could have set up a plate so that the x axis is at a constant potential Φ_0. We now have the the electrical field is along the y direction

$$\vec{E} = \left(0, \frac{\sigma}{\varepsilon_0}\right) \tag{101}$$

where σ is the charge density. The electrostatic potential is now

$$\Phi = \Phi_0 - \frac{\sigma}{\varepsilon_0} y \tag{102}$$

which corresponds to

$$f = \Phi_0 + \mathrm{i}\frac{\sigma}{\varepsilon_0} z \tag{103}$$

Consider yet another example: a conducting cylinder of radius R immersed in a uniform electric field. We have that $\Phi = 0$ on the unit circle, i.e. on $z = Re^{\mathrm{i}\theta}$. If we think about the complex potential, we have that

$$f = -E\left(z - \frac{R^2}{z}\right) \tag{104}$$

satisfies this boundary condition, i.e. it is pure imaginary on the boundary. We then have that

$$\Phi = \mathrm{Re}(f) = -E\left(x - \frac{R^2 x}{x^2 + y^2}\right) \tag{105}$$

We can use the fact that analytic functions act as conformal transformations to solve problems with more intricate boundary value problems. A crucial aspect is that the potential, Φ remain harmonic in the transformed coordinates. To see this, consider the transformation

$$f : (x, y) \to (u, v) \tag{106}$$

We now have

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial x^2} &= \frac{\partial^2 \Phi}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial \Phi}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 \Phi}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial \Phi}{\partial v}\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2 \Phi}{\partial u \partial v} \\ \frac{\partial^2 \Phi}{\partial y^2} &= \frac{\partial^2 \Phi}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial \Phi}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 \Phi}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial \Phi}{\partial v}\frac{\partial^2 v}{\partial y^2} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\frac{\partial^2 \Phi}{\partial u \partial v} \end{aligned} \tag{107}$$

Recall that

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \tag{108}$$

Taking a linear combination of Eq 107 we have

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=|f'(z)|^2\left(\frac{\partial^2\Phi}{\partial u^2}+\frac{\partial^2\Phi}{\partial v^2}\right)+\frac{\partial\Phi}{\partial u}\nabla^2 u+\frac{\partial\Phi}{\partial v}\nabla^2 v$$
$$+2\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\frac{\partial^2\Phi}{\partial u\partial v} \tag{109}$$

Applying the Cauchy-Riemann conditions, we can see that the last term goes away; we know that u and v are harmonic so the 2^{nd} and 3^{rd} term go away too. Thus, if we transform with an analytic function, we have that Φ satisfies Laplace's equation in the new coordinate system.

Consider now a plate held at a fixed potential $\Phi = \Phi_0$ on the positive x and y axis'. We can transform this problem into the one we solved originally by transforming the corner into a line with

$$w = z^2 \tag{110}$$

The boundary will now be along the x axis in the w plane – a problem we solved previously. In these new coordinates we have that

$$\vec{E} = \left(0, \frac{\sigma}{\varepsilon_0}\right) \tag{111}$$

where σ is the charge density. Integrating this we have

$$\Phi = \Phi_0 - \frac{\sigma}{\varepsilon_0}v \tag{112}$$

Now recall that $\Phi = \text{Re}(f)$ so we must have

$$f = \Phi_0 + \mathrm{i}\frac{\sigma}{\varepsilon_0}w \tag{113}$$

We can now transform back to our original coordinates to get

$$\Phi = \text{Re}(f) = \text{Re}\left(\Phi_0 + \mathrm{i}\frac{\sigma}{\varepsilon_0}z^2\right) = \Phi_0 - 2\frac{\sigma}{\varepsilon_0}xy \tag{114}$$

Therefore the equipotentials are hyberbola.

Let us take an even more contrived set up. Let us consider a long hollow cylinder made out of a thin sheet of conducting material; let us assume the cylinder is divided into two parts, the top semicircle is set at $\Phi = 0$ and the bottom half at $\Phi = 1$. We have that the electrostatic potential is a harmonic function inside the boundary set by $x^2+y^2 = 1$. We can use the Mobius transformation to map the upper half plane onto the circle, in such a way that the positive x-axis is mapped onto the top half of the section and the negative x-axis is mapped onto the bottom half of the cylinder. The mapping from the plane (with coordinates $w = u+\mathrm{i}v$) to the cylinder (with coordinates z) is done with

$$f(z) = e^{\mathrm{i}\theta_0}\frac{w - w_0}{w - \bar{w}_0} \tag{115}$$

where the point w_0 will be mapped onto the origin while $-\infty$ and $+\infty$ will be mapped onto the point $e^{i\theta_0}$. From what we decided above, we have $e^{i\theta_0} = -1$ and we can map $w = i$ onto $z = 0$. This leads us to

$$z = \frac{i - w}{i + w} \tag{116}$$

Now, consider the function

$$\frac{1}{\pi}\log w = \frac{1}{\pi}\log r + \frac{i}{\pi}\phi \tag{117}$$

We can see that the imaginary part takes exactly the right values on the boundary: it is 0 along the positive u axis and 1 along the negative u axis. We thus have that

$$\Phi = \frac{1}{\pi}\phi = \frac{1}{\pi}\arctan\left(\frac{v}{u}\right) \tag{118}$$

and thus

$$f = -\frac{i}{\pi}\log w \tag{119}$$

The inverse transform is

$$w = i\frac{1 - z}{1 + z} \tag{120}$$

so we can express u and v in terms of x and y to find

$$\Phi = \frac{1}{\pi}\arctan\left(\frac{1 - x^2 - y^2}{2y}\right) \quad (0 \leq \arctan t \leq \pi) \tag{121}$$

The equipotential curves, $\Phi = c$ in the circular region are arcs of circles

$$x^2 + (y + \tan\pi c)^2 = \sec^2\pi c \tag{122}$$

while the flux lines are arcs of circles with centres on the y-axis.

5 Singularities.

We have focused on analytic functions, i.e. functions $f(z)$ where df/dz is well defined. We have already seen quite a few of them; functions for which the derivative is single-valued and defined everywhere (except, maybe, at $z = \infty$) are called *entire* or *integral* functions. Examples are polynomials, e^z, $\sin z$ and $\cos z$.

Consider now functions which are analytic almost everywhere except at a few points, where the derivative is infinite. A simple, notable example is

$$f(z) = \frac{1}{z} \tag{123}$$

The derivative is infinite (and in fact takes on different signs depending on which direction you take to calculate it). One says that f has a *pole* at $z = 0$. We can also consider

$$f(z) = \frac{1}{z-a} \tag{124}$$

Now the singularity, or pole is at $z = a$.

We could have considered a more general function

$$f(z) = \frac{1}{(z-a)^n} \tag{125}$$

where n is an integer greater than 0. Again, we have that $z = a$ is a pole but we can now classify it: it is a *pole of order* n.

We can have functions with a pole that has more than one order. For example

$$f(z) = \frac{(z+1)^2}{(z-1)^2} \tag{126}$$

can be expanded to give

$$f(z) = \frac{4}{(z-1)^2} + \frac{4}{(z-1)} + 1 \tag{127}$$

In this case $z = 1$ is a pole of order 1 *and* 2.

We can also have functions with more than one pole. For example

$$f(z) = \frac{1}{1-z^2} \tag{128}$$

can be separated into

$$f(z) = \frac{1}{2}\left(\frac{1}{1-z} + \frac{1}{1+z}\right) \tag{129}$$

has poles of order 1 at $z = -1$ and $z = 1$. Or, for example

$$f(z) = \frac{1}{\sin z} \tag{130}$$

has an infinite number of poles at

$$z = n\pi \quad \text{with} \quad n \in \mathbb{Z} \tag{131}$$

all of which are of order 1.

We saw above that we could complete the Argand plane by including $z = \infty$. This means we can consider it on par with other points. So for example, if we consider the function

$$f(z) = z^4 \tag{132}$$

it clearly blows up at $z = \infty$. If we define $\omega = 1/z$, we have that the function $f(\omega)$

$$f(\omega) = \frac{1}{\omega^4} \tag{133}$$

has a pole of order 4 at $\omega = 0$, in other words at $z = \infty$.

There are two further types of singularities which we must consider. The first one is very benign and is dubbed a *removable* singularity. An example you are familiar with is

$$f(z) = \frac{\sin z}{z} \tag{134}$$

which has a singularity at $z = 0$ but, as you know has a finite limit, $\lim_{z \to 0} f(z) = 1$. This means, the function can be rendered completely analytic by setting $f(0) = 1$. In other words it just *looks* singular at $z = 0$ but isn't really.

A singularity which is neither removable or a pole is known as an *essential* singularity. A particularly notorious example is

$$f(z) = e^{\frac{1}{z}} \tag{135}$$

To see how severe it is, consider the following series expansion expression of f around $z = \infty$:

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \tag{136}$$

As you can see, $z = 0$ is a pole of arbitrary order.

Finally, let us return to poles of order 1. Let us consider the example of

$$f(z) = \frac{r}{z - a} \tag{137}$$

We have the $f(z)$ has a pole of order 1 at $z = a$ and a *residue* r. We will use the notation

$$\mathrm{Res}(f, a) = r \tag{138}$$

We can look at a slightly more complicated example:

$$f(z) = \frac{z^2 + 2z + 1}{z - 1} \tag{139}$$

If we re-express the numerator as a polynomial in $z - 1$ we have that

$$f(z) = \frac{4 + 4(z-1) + (z-1)^2}{z - 1} = \frac{4}{z - 1} + 4 + z - 1 \tag{140}$$

We then have

$$\mathrm{Res}(f, 1) = 4 \tag{141}$$

Or, if we revisit the examples we considered above, we have that

$$f(z) = \frac{(z+1)^2}{(z-1)^2} \tag{142}$$

and so

$$\mathrm{Res}(f, 1) = 4 \tag{143}$$

while

$$f(z) = \frac{1}{1-z^2} \tag{144}$$

has two poles of order 1, such that

$$\begin{aligned} \mathrm{Res}(f,1) &= -\frac{1}{2} \\ \mathrm{Res}(f,-1) &= \frac{1}{2} \end{aligned} \tag{145}$$

Finally, reconsider

$$f(z) = \frac{1}{\sin z} \tag{146}$$

As we saw, it has poles at $z = n\pi$. If we Taylor expand around such a pole we have that

$$\sin z = \sin n\pi + (z - n\pi)\cos n\pi + O(2) \simeq (-1)^n (z - n\pi)[1 + c(z - n\pi)^2] \tag{147}$$

where c is a constant. We then have that

$$f(z) = \frac{1}{(-1)^n (z-n\pi)[1 + c(z-n\pi)^2]} \simeq \frac{(-1)^n}{z - n\pi} \times [1 - c(z - n\pi)^2]. \tag{148}$$

We thus have

$$\mathrm{Res}(f, n\pi) = (-1)^n. \tag{149}$$

6 Branch points and branch cuts

Until now we have focused on single valued functions. Let us now look at multi-valued functions; we start with a particularly simple one

$$f(z) = z^{\frac{1}{2}} \tag{150}$$

We previously saw that, if $z = w^2$ there are two roots which we can express as $f = +|w|$ and $f = -|w|$ or as

$$f = |w| e^{ik\pi} \quad \text{with} \quad k = 0, 1 \tag{151}$$

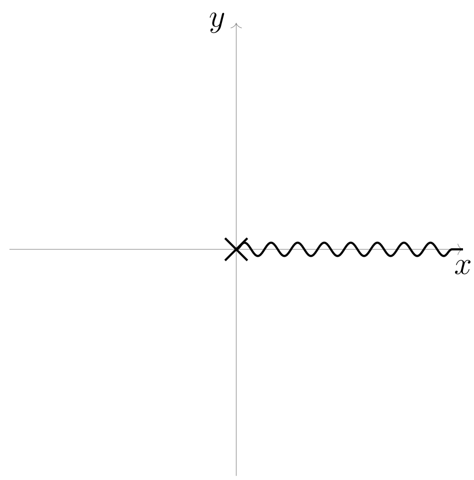
Let us now consider what happens to $f(z)$ as we move z around a circle with unit radius. We can express $z = e^{i\theta}$ and as we take θ from 0 to 2π, z will start and end at 1 while $f(z) = e^{i\theta/2}$ will start at 1 and end at -1. In other words, $f(z)$ will not return to its initial value. Now, this will be true for *any* closed loop that encompasses 0 and we call it a *branch* point.

Note that, for example $z = 1$ is not a branch point. To see this, consider a small loop around it, $z = 1 + \epsilon e^{i\theta}$. If we take $f(z)$ and Taylor expand it, we have that

$$f(z) = (1 + \epsilon e^{i\theta})^{\frac{1}{2}} \simeq 1 + \frac{1}{2}\epsilon e^{i\theta} + \sum_{n=2}^{\infty} c_n e^{in\theta} \epsilon^n \tag{152}$$

which, as you can see, returns to its starting point when θ changes from 0 to 2π.

There is a way of taking into account that functions can be discontinuous, due to their multivaluedness. One way is to put in a "roadblock" to prevent us from varying around closed loops that lead to discontinuities. For example, consider our first example, $f(z) = \sqrt{z}$ which has a branch point at $z = 0$. We can put up a barrier by preventing any loop from crossing the line that start at 0 and extends out, along the x-axis to $+\infty$. Now, whenever we repeat the exercise we did above on *any* closed loop that doesn't cross that forbidden line, we will always arrive back at the original value; in other words, $f(e^{0\mathrm{i}}) = f(e^{2\pi\mathrm{i}})$. We call such a line a *branch cut*.



Note that we could have picked that line to be in any direction, as long as it starts off at $z = 0$. Furthermore, it could be curved or wiggly, it doesn't matter. As long as it prevents the loop from ending up with $f(e^{0\mathrm{i}}) \neq f(e^{2\pi\mathrm{i}})$.

Consider the slightly more complicated example of

$$f(z) = \sqrt{z^2 - 1} \tag{153}$$

We can factorize this function

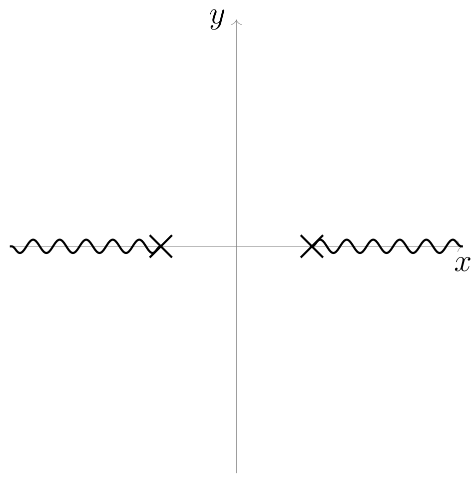
$$f(z) = \sqrt{z-1}\sqrt{z+1} \tag{154}$$

and immediately see that it has two branch points at $z = -1$ and $z = 1$. To check, let us take a closed loop around z=1. Define $z = 1 + \epsilon e^{i\theta}$. We then have that

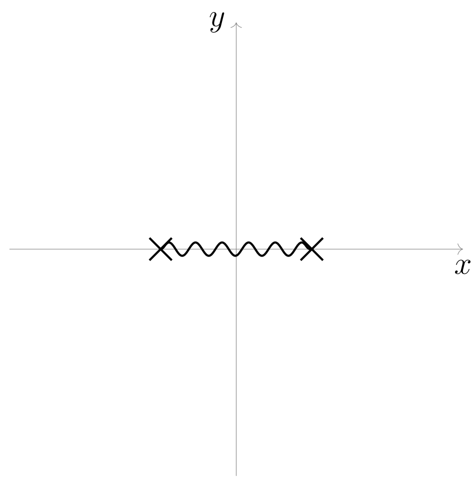
$$f(z) = \sqrt{\epsilon e^{i\theta}}\sqrt{2 + \epsilon e^{i\theta}} = \sqrt{2\epsilon}e^{\frac{i\theta}{2}}\sqrt{1 + \frac{\epsilon}{2}e^{i\theta}} \simeq \sqrt{2\epsilon}e^{\frac{i\theta}{2}}(1 + \frac{\epsilon}{4}e^{i\theta} + \cdots) \tag{155}$$

As you can see, $f(e^{0\mathrm{i}}) \neq f(e^{2\pi\mathrm{i}})$. One can show the same when $z = -1$.

Given that we encounter discontinuties when we circle either $z = 1$ or $z = -1$, we need to conjur up a branch cut. One possible is to duplicate what we did in the previous case: we draw out a line from $z = 1$ to $z = +\infty$ and one from $z = -1$ to $z = -\infty$. Any closed loop that doesn't intersect the lines is guaranteed to have $f(e^{0\mathrm{i}}) = f(e^{2\pi\mathrm{i}})$.



Another, interesting, possibility is to simply connect $z = -1$ to $z = 1$ along the x-axis.



In fact we can consider any line that connects the two branch points to be a valid branch cut. If we think of the Riemann sphere, you can even think of the two examples that we just considered as almost identical: the first one passes through the north pole while the second one passes through the south pole.

A question you can then ask is what happens to the function along a curve that surrounds the line. Consider the curve $z = (1+\epsilon)e^{i\theta}$. We have that

$$f(z) = \sqrt{(1+\epsilon)^2 e^{2i\theta} - 1} = (1+\epsilon)e^{i\theta}\sqrt{1 - \frac{e^{-2i\theta}}{(1+\epsilon)^2}} \simeq (1+\epsilon)e^{i\theta}\left(1 - \frac{e^{-2i\theta}}{2(1+\epsilon)^2} + \cdots\right) \tag{156}$$

which satisfies $f(e^{0\mathrm{i}}) = f(e^{2\pi\mathrm{i}})$.

A useful concept in discussing branch cuts and points is the *Riemann Sheet* or *surface*. Let us first consider the function

$$f(z) = \log z = \log r + \mathrm{i}\phi \tag{157}$$

We know that if we go around a closed loop that containts the origin, $f(z)$ will increase by 2π and thus the function will be multivalued. The way around this is to unfold the complex plane into an infinite number of sheets as follows: take the complex plane and cut it along the $x > 0$ axis. Then take an exact copy and glue the $y > 0$ edge of one to the $y < 0$ edge of the other. If you now go around a loop and reach $\phi = 2\pi$ you will immediately climb onto the next sheet. Now, repeat this procedure so that you systematically glue sheets whenever you come to an edge. In this way, you will have constructed what looks like an infinite multi-story car-park, a gigantic flaring cork screw. This will be the Riemann surface tied to this function so that it is single valued.

Another example which isn't infinite but has a strange topology is the Riemann surface tied to

$$f(z) = z^{1/2} = r^{1/2}e^{\mathrm{i}\phi/2} \tag{158}$$

Now, we start off in the same way by cutting the complex plane along the $x > 0$ axis, we glue the $y < 0$ of one to the $y > 0$ edge of another one. But now, we take the $y < 0$ edge of the second one and glue it back onto the $y > 0$ of the first one. So, in a way which is impossible to vizualize, the surface wraps back in on itself. Again, on this surface, the function is single-valued.

7 Cauchy's theorem

We will now look at integrals on the Argand plane. Naively you might want to write, as in the real case,

$$\int_{z_1}^{z_2} f(z)dz \tag{159}$$

but this is clearly not enough; we need to specify the path or *contour* that takes us from z_1 to z_2. And, as in the case of line integrals in vector calculus, the result will depend on the choice of path. So, from now on we write

$$\int_C f(z)dz \tag{160}$$

for a contour integral on a contour C. A closed contour is denoted by

$$\oint_C f(z)dz \tag{161}$$

It is instructive to look at the guts of what an integral looks like. We have that

$$\int_C f(z)dz = \int_C [u(x,y) + \mathrm{i}v(x,y)][dx + idy] = \int_C (udx - vdy) + \mathrm{i}\int_C (vdx + udy) \tag{162}$$

It looks like we have to do a bunch of real integrals. In fact, and in general it is much simpler.

Consider a few examples. Let us integrate $f(z) = z$ around a circle with a $|z| = 1$. We have that $z = e^{\mathrm{i}\theta}$ so $dz = e^{\mathrm{i}\theta}\mathrm{i}d\theta$ and so

$$\oint_C f(z)dz = \mathrm{i}\int_0^{2\pi} e^{2\mathrm{i}\theta}d\theta = \frac{1}{2}\, e^{2\mathrm{i}\theta}\Big|_0^{2\pi} = 0 \tag{163}$$

Of course, we could have done the integral directly

$$\oint_C z dz = \frac{1}{2} z^2 \Big|_1^{e^{2\pi \mathrm{i}}} = 0 \tag{164}$$

A slightly more interesting example is $f(z) = 1/z$ around a circle with *arbitrary* radius. We have $z = re^{\mathrm{i}\theta}$ so $dz = re^{\mathrm{i}\theta}\mathrm{i}d\theta$ and then

$$\oint_C f(z)dz = \mathrm{i}\int_0^{2\pi} \frac{re^{\mathrm{i}\theta}}{re^{\mathrm{i}\theta}} = 2\pi\mathrm{i} \tag{165}$$

Again, we could have done this directly

$$\oint_C \frac{dz}{z} = \log z\Big|_r^{re^{2\pi\mathrm{i}}} = 2\pi\mathrm{i} \tag{166}$$

A final example is $f(z) = z^{-n}$ with n an integer not equal to 1, around a circle with radius r

$$\oint_C f(z)dz = \mathrm{i}r^{1-n}\int_0^{2\pi} e^{(1-n)\mathrm{i}\theta}d\theta = \frac{r^{1-n}}{1-n}\, e^{(1-n)\mathrm{i}\theta}\Big|_0^{2\pi} = 0 \tag{167}$$

We will now look at the first of a set of powerful theorems that greatly simplify complex integrals.
Cauchy's Theorem states that: if $f(z)$ is an analytic function in a region R enclosed by a contour C, then

$$\oint_C f(z)dz = 0 \tag{168}$$

To prove it, we use our knowledge of vector calculus. We saw that

$$\oint_C f(z)dz = \oint_C (udx - vdy) + \mathrm{i}\oint_C (vdx + udy) \tag{169}$$

We can now use Green's theorem (which is the planar version of Stoke's theorem) to convert the two contour integrals into surface integrals. To do so, recall that Green's theorem states that

$$\oint_C [A(x,y)dx + B(x,y)dy] = \int_S \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dxdy \tag{170}$$

where S is the area contained inside C. Note that we can get this from Stoke's theorem by considering the vector function $\vec{F} = (A, B, 0)$ and the orthogonal vector to the surface S given by $\hat{k}$ (i.e. the unit vector in the z direction); Stoke's theorem is then just

$$\oint \vec{F}\cdot d\vec{s} = \int_S (\vec{\nabla}\times\vec{F})\cdot\hat{k}dA \tag{171}$$

Applying this to our integral we have

$$\oint_C f(z)dz = \int_S \left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) dxdy + \mathrm{i}\int_S \left(-\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right) dxdy \tag{172}$$

But the integrands of the surface integrals look familiar; if you go back to an earlier lecture, you will see that they take the form of the Cauchy-Riemann conditions for analytic functions and therefore are 0.

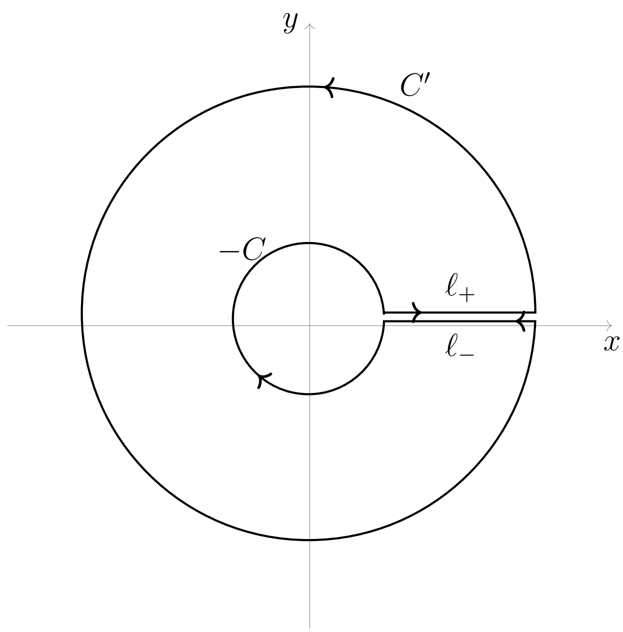
Cauchy's theorem has a number of consequences. One obvious one is that if $f(z)$ is an entire (or integral) function, then

$$\oint_C f(z)dz = 0 \tag{173}$$

on *any* closed loop. Another very useful consequence is that we can deform contours into one another if the function is analytic in the region between them. In other words, if we have two contours, C and C' and the function is analytic in the region between them, then

$$\oint_C f(z)dz = \oint_{C'} f(z)dz \tag{174}$$

We can see this if we break the contour Γ into $\Gamma = C \cup \ell_- \cup C' \cup \ell_+$:



If $f(z)$ is analytic inside Γ we have that

$$\oint_\Gamma f(z)dz = \int_{-C} f(z)dz + \int_{\ell_-} f(z)dz - \int_{C'} f(z)dz + \int_{\ell_+} f(z)dz = 0 \tag{175}$$

(note the minus sign in front of C because of the direction). But we can make ℓ_+ and ℓ_- lie arbitrarily close to each other so

$$\int_{\ell_-} f(z)dz + \int_{\ell_+} f(z)dz = 0 \tag{176}$$

and therefore

$$\int_C f(z)dz = \int_{C'} f(z)dz \tag{177}$$

Let us now put our new found knowledge to good use. Consider the integral

$$\int_{-\infty}^{+\infty} e^{-x^2} \cos(2kx) dx \tag{178}$$

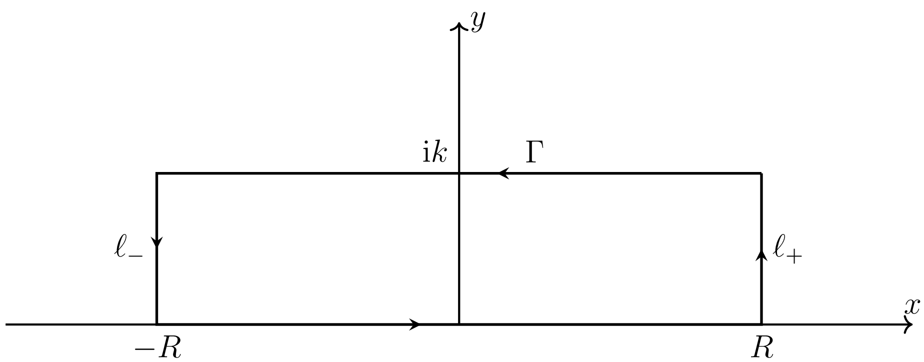
This is nothing more than the real part of the integral

$$\int_{-\infty}^{+\infty} e^{-x^2} e^{-2\mathrm{i}kx} dx \tag{179}$$

You will be familiar with it: it is the Fourier Transform of the Gaussian, $f(2k)$. We can complete the square to get

$$f(2k) = e^{-k^2} \int_{-\infty}^{\infty} e^{-(x+\mathrm{i}k)^2} dx \tag{180}$$

How do we do this integral? Normally we take a change of variables $\tilde{x} = x + \mathrm{i}k$, wave our hands, and then proceed to do the integral on the real line. But we can see if this is correct by considering a contour as follows:



The integral around that contour is 0 because the function is analytic. As you can see there are 4 parts which we can write out as

$$\oint_\Gamma e^{-z^2} dz = \int_{-R}^{R} e^{-x^2} dx + \int_0^k e^{-(R+iy)^2} dy + \int_R^{-R} e^{-(x+ik)^2} dx + \int_k^0 e^{-(-R+iy)^2} dy \tag{181}$$

When $R \to \infty$ we get rid of the integrals associated to paths ℓ_- and ℓ_+; flipping the order of integration in the third integral gives us

$$\int_{-\infty}^{\infty} e^{-(x+ik)^2} dx = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \tag{182}$$

Taking the real part, we find the integral we desired:

$$\int_{-\infty}^{+\infty} e^{-x^2} \cos(2kx) dx = \sqrt{\pi} e^{-k^2} \tag{183}$$

8 Cauchy's integral formula

We can use Cauchy's theorem to find a useful relationship between integrals of $f(z)$ and its derivatives. Consider a contour C inside which $f(z)$ is analytic. Now consider the following integral

$$\oint_C \frac{f(z)}{z-a} dz \tag{184}$$

We have that the integrand is analytic everywhere inside C except for, maybe, at a. Using Cauchy's theorem, we can deform the contour to a new one, γ, so that it becomes a circle of radius ϵ around a, $z_\gamma = a + \epsilon e^{\mathrm{i}\theta}$

We then have

$$\oint_C \frac{f(z)}{z-a} dz = \oint_\gamma \frac{f(a+\epsilon e^{\mathrm{i}\theta})}{\epsilon e^{\mathrm{i}\theta}} \mathrm{i}\epsilon e^{\mathrm{i}\theta} d\theta = \mathrm{i} \int_\gamma f(a+\epsilon e^{\mathrm{i}\theta}) d\theta \tag{185}$$

We can now take the limit $\epsilon \to 0$ to get

$$\oint_C \frac{f(z)}{z-a} dz = \lim_{\epsilon \to 0} \mathrm{i} \oint_\gamma f(a+\epsilon e^{\mathrm{i}\theta}) d\theta = 2\pi \mathrm{i} f(a) \tag{186}$$

This is a remarkable result: it means we know the value of the f at any point inside the two dimensional surface contained within the contour, if we know the value of f on the one-dimensional contour. Very much like a 2-dimensional version of a hologram.

Not only that, we can use the Cauchy integral formula to determine the value of any derivative of f. Indeed, we have that

$$f^{(n)}(a) = \frac{d^n}{da^n} f(a) = \frac{n!}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \tag{187}$$

We can now use this expression to explicitely construct the Taylor expansion around a:

$$\begin{aligned} f(a+\epsilon) \qquad &= \frac{1}{2\pi\mathrm{i}} \oint_C \frac{f(z)}{z-a-\epsilon} dz = \frac{1}{2\pi\mathrm{i}} \oint_C \frac{f(z)}{(z-a)\left(1-\frac{\epsilon}{z-a}\right)} dz \\ &= \frac{1}{2\pi\mathrm{i}} \oint_C \frac{f(z)}{(z-a)} dz \sum_{n=0}^{\infty} \frac{\epsilon^n}{(z-a)^n} = \sum_{n=0}^{\infty} f^{(n)}(a) \frac{\epsilon^n}{n!} \end{aligned} \tag{188}$$

9 The Laurent expansion

Let us now consider a function which has a pole of order p at $z = a$. Recall that a pole can have multiple orders so we assume that p is the maximum of these. We then know that $(z-a)^p f(z)$ is analytic at $z = a$ and that, locally we can Taylor expand it. But this means we have that

$$f(z) = \frac{1}{(z-a)^p} \sum_{n=0}^{\infty} b_n (z-a)^n \tag{189}$$

We can now relabel the coefficients of the series expansion as

$$f(z) = \sum_{n=-p}^{\infty} a_n (z-a)^n \tag{190}$$

Note that we now have terms with inverse powers of $(z-a)$. Furthermore, we can see that the residue of the function is given by one of these terms, a_{-1}.

We call the *principal* part of the Laurent series of $f(z)$, the sum of the terms with negative exponents. In the example we just saw above, the principle part of f is

$$\mathcal{P}[f,a] = \sum_{n=-p}^{-1} a_n (z-a)^n \tag{191}$$

10 Cauchy's residue theorem

Before we tackle the main topic of this section, let us do some preparatory work. As we have seen, we can expand a function around a pole of order p using the Laurent expansion, i.e. in terms of monomials of the form $(z-a)^n$ where n can take positive and negative values.

Consider now the following integral

$$\oint_C (z-a)^n dz \tag{192}$$

where C is a contour that surrounds $z = a$. Let us take the contour to be a circle with an infinitesimal radius $z = a + \epsilon e^{i\theta}$. We then have

$$\oint_C (z-a)^n dz = i\int_0^{2\pi} \epsilon^{n+1} e^{i(n+1)\theta} d\theta = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases} \tag{193}$$

We can now use this fact to calculate such an integral for an $f(z)$ which has a Laurent expansion:

$$\oint_C f(z)dz = \sum_{n=-p}^{\infty} a_n \oint_C (z-a)^n dz = 2\pi i a_{-1} \tag{194}$$

In other words, the contour integral for a function $f(z)$ around a loop that contains a pole is simply given by the residue of the pole (upto a factor of $2\pi i$). This is known as the *Cauchy Residue Theorem*.

We can generalize this expression to the case where we the contour encompasses multiple (N) poles, w_k $(k = 1, \cdots, N)$ each one with a residue, $\text{Res}(f, w_k)$. The integral is then given by

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{N} \text{Res}(f, w_k) \tag{195}$$

Let us now look at some examples where the Cauchy Residue Theorem can be used to great effect. To begin with, let us apply the residue theorem in the most straightforward case. Consider the following integral

$$\oint_C \frac{z+1}{z(z^2+1)} dz \tag{196}$$

where C is the circle centered at $z = 0$ with radius 2. We can see that the integrand has 3 poles at $z_0 = 0$, $z_\pm = \pm i$. To find the residues of each of these poles we find the corresponding Laurent series. Let us do this in two ways. First of all, let us use partial fraction to factorize the integrand:

$$\begin{aligned}\frac{z+1}{z(z^2+1)} &= \frac{z+1}{z}\frac{1}{(z+\mathrm{i})(z-\mathrm{i})} = \frac{\mathrm{i}}{2}\frac{z+1}{z}\left(\frac{1}{z+\mathrm{i}} - \frac{1}{z-\mathrm{i}}\right) \\ &= \frac{1}{2}(z+1)\left(\frac{2}{z} - \frac{1}{z+\mathrm{i}} - \frac{1}{z-\mathrm{i}}\right) \qquad (197)\end{aligned}$$

As we can see, we can immediately read off the residues from this expression: 1, $-(1+\mathrm{i})/2$ and $-(1-\mathrm{i})/2$.

We can do this another, more general way. Consider each one of the poles in turn:

- Expanding the integrand around z_0 we have

$$\frac{z+1}{z(z^2+1)} \simeq \frac{1}{z}(1+z)(1-z^2+\cdots) \qquad (198)$$

so

$$\mathrm{Res}(f, z_0) = 1 \qquad (199)$$

- Expanding around z_+ we have

$$\frac{z+1}{z(z^2+1)} = \frac{1+i+(z-i)}{[i+(z-i)](z-i)[2i+(z-i)]} \qquad (200)$$

For ease of notation, let us define $\xi = z - i$, we then have

$$\frac{1+i+\xi}{[i+\xi]\xi[2i+\xi]} \simeq \frac{1}{\xi}[(1+i)+\xi]\frac{1}{i}[1-\frac{\xi}{i}+\cdots]\frac{1}{2i}[1-\frac{\xi}{2i}+\cdots] \qquad (201)$$

so

$$\mathrm{Res}(f, z_+) = -\frac{1+i}{2} \qquad (202)$$

- Expanding around z_- we have

$$\frac{z+1}{z(z^2+1)} = \frac{1-i+(z+i)}{[-i+(z+i)](z+i)[-2i+(z+i)]} \qquad (203)$$

For ease of notation, let us define $\xi = z + i$, we then have

$$\frac{1-i+\xi}{[-i+\xi]\xi[-2i+\xi]} \simeq \frac{1}{\xi}[(1-i)+\xi]\frac{1}{-i}[1+\frac{\xi}{i}]\frac{1}{-2i}][1+\frac{\xi}{2i}] \qquad (204)$$

so

$$\mathrm{Res}(f, z_-) = -\frac{1-i}{2} \qquad (205)$$

Applying the residue theorem we have that

$$I = 2\pi i \left[1 - \frac{1+i}{2} - \frac{1-i}{2}\right] = 0 \tag{206}$$

We can use the residue theorem to evaluate real integrals. Let us consider one of the simplest examples:

$$I = \int_0^{2\pi} \frac{d\theta}{2 - \cos\theta} \tag{207}$$

Recall that, with $z = e^{i\theta}$ we have

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \tag{208}$$

and therefore

$$I = \oint_C \frac{1}{2 - \frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz} = \oint_C \frac{2i}{z^2 - 4z + 1} dz \tag{209}$$

where C is the unit circle. The integrand has two poles, $z_\pm = 2 \pm \sqrt{3}$, one of which lies in the contour. The residue of the function at z_- is

$$\text{Res}(f, z_-) = -\frac{i}{\sqrt{3}} \tag{210}$$

So

$$I = \frac{2\pi}{\sqrt{3}} \tag{211}$$

We can generalize this calculation to

$$I = \int_0^{2\pi} \frac{d\theta}{a\cos\theta + b} d\theta \tag{212}$$

where $a, b > 0$. We are going to convert this real integral into a contour integral by choosing $z = e^{i\theta}$ which corresponds to a circular contour of radius 1.We then have

$$I = \oint_C \frac{dz}{iz} \frac{1}{\frac{a}{2}\left(z + \frac{1}{z}\right) + b} = -\frac{2i}{a} \oint_C \frac{dz}{z^2 + 2\frac{b}{a}z + 1} = -\frac{2i}{a} \oint_C \frac{dz}{(z - z_+)(z - z_-)} \tag{213}$$

where

$$z_\pm = -\frac{b}{a} \pm \sqrt{\left(\frac{b}{a}\right)^2 - 1} \tag{214}$$

There a few different cases to consider.

- If $b/a > 1$, the roots are real; note that $z_+ z_- = 1$ so z_+ (z_-) lies inside (outside) the contour. Thus, applying the residue theorem, we have

$$I = -\frac{2i}{a}\frac{2\pi i}{z_+ - z_-} = \frac{2\pi}{\sqrt{b^2 - a^2}}. \tag{215}$$

- If $b/a = 1$, then $z_\pm = -1$ and there is a double pole on the contour so $I = \infty$.
- If $b/a < 1$ then

$$z_\pm = -\frac{b}{a} \pm i\sqrt{1 - \left(\frac{b}{a}\right)^2} \tag{216}$$

  and, the two poles lie on the contour, again leading to $I = \infty$.

More generally, we can undertake trigonometric integrals of the form

$$I = \int_0^{2\pi} F(\sin\theta, \cos\theta) d\theta \tag{217}$$

by simply transforming them to

$$I = \oint_C F\left[\frac{1}{2i}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right]\frac{dz}{iz} \tag{218}$$

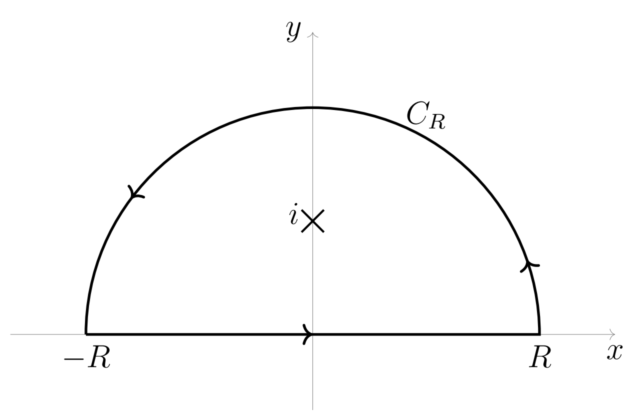
We can also do improper integrals, i.e. real integrals with limits at ∞. Consider the following integral

$$I = \int_{-\infty}^{+\infty} \frac{dx}{1 + x^2} \tag{219}$$

If we convert the integrand into a complex function, we can factorize the denominator to find two poles $z_\pm = \pm i$. Let us now consider the following contour integral

$$\oint_C \frac{dz}{1 + z^2} \tag{220}$$

where the contour C is the semi-circle with radius R, limited by the $x - axis$.



We have that, one of the poles, $z = i$ lies inside the contour and the residue of the function at that pole is

$$\text{Res}(f, z_+) = \frac{1}{2i} \tag{221}$$

Now we can split the contour up into two parts. The first one is along the real axis and the second follows the semi-circle with radius R:

$$\oint_C \frac{dz}{1+z^2} = \int_{-R}^{R} \frac{dx}{1+x^2} + \int_0^{\pi} \frac{\mathrm{i}Re^{\mathrm{i}\theta}d\theta}{1+R^2e^{2\mathrm{i}\theta}} = \pi \tag{222}$$

Now, note that, as we make R large, the integrand in the semi-circle is supressed by a $1/R$. If we take the limit $R \to \infty$ we get

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi \tag{223}$$

Let us make this slightly more complicated and consider

$$I = \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2} \tag{224}$$

We can now play the same trick as above

$$I = \int_C \frac{dz}{(1+z^2)^2} = \int_C \frac{dz}{(z+i)^2(z-i)^2} \tag{225}$$

and we can see that there is a double pole at $z = z_+ \equiv i$. We now need to find the Laurent series around z_+ and to do so, we need expand the other factor in the denominator:

$$\frac{1}{(z+i)^2} = \frac{1}{(z-i+2i)^2} = \frac{1}{(2i)^2}\frac{1}{\left(1+\frac{z-i}{2i}\right)^2} = -\frac{1}{4}\left(1 - 2\frac{z-i}{2i} + \mathcal{O}(z-i)^2\right) \tag{226}$$

We can now see that the residue of the integrand is

$$\text{Res}(f, z_+) = \frac{1}{4i} \tag{227}$$

Now, as above, we can break up the integral into

$$I = \int_C \frac{dz}{(1+z^2)^2} = \int_{-R}^{R} \frac{dx}{(1+x^2)^2} + \int_0^{\pi} \frac{\mathrm{i}Re^{\mathrm{i}\theta}d\theta}{(1+R^2e^{2\mathrm{i}\theta})^2} \tag{228}$$

The second part is supressed by $1/R^3$. In the limit where $R \to \infty$ we have that

$$I = \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2} \tag{229}$$

A slightly more complicated example is

$$I = \int_{-\infty}^{+\infty} \frac{\cos ax}{1+x^2} dx \tag{230}$$

with $a > 0$. It helps to consider

$$\tilde{I} = \int_{-\infty}^{+\infty} \frac{e^{iax}}{1+x^2} dx \tag{231}$$

This seem slightly more complicated except for the fact that y is positive and so

$$e^{iaz} = e^{\mathrm{i}a(x+\mathrm{i}y)} = e^{-ay} e^{\mathrm{i}ax} < e^{\mathrm{i}ax} \leq 1 \tag{232}$$

So we have that

$$\tilde{I} = \oint_C \frac{e^{iaz}}{1+z^2} dz = \int_{-R}^{+R} \frac{e^{iax}}{1+x^2} dx + \int_0^{\pi} e^{\mathrm{i}aRe^{\mathrm{i}\theta}} \frac{\mathrm{i}Re^{\mathrm{i}\theta} d\theta}{(1+R^2 e^{2\mathrm{i}\theta})} \tag{233}$$

but

$$\int_0^{\pi} e^{\mathrm{i}aRe^{\mathrm{i}\theta}} \frac{\mathrm{i}Re^{\mathrm{i}\theta} d\theta}{(1+R^2 e^{2\mathrm{i}\theta})} < \int_0^{\pi} \frac{\mathrm{i}Re^{\mathrm{i}\theta} d\theta}{(1+R^2 e^{2\mathrm{i}\theta})} \tag{234}$$

which goes to zero as $R \to \infty$. The residue at z_+ is now

$$\mathrm{Res}(f, z_+) = \frac{1}{2\mathrm{i}} e^{-a} \tag{235}$$

and so

$$I = \mathrm{Re}(\tilde{I}) = \int_{-\infty}^{+\infty} \frac{\cos ax}{1+x^2} dx = \pi e^{-a} \tag{236}$$

11 Jordan's Lemma

When we worked out

$$I = \int_{-\infty}^{+\infty} \frac{e^{iax}}{1+x^2} dx \tag{237}$$

we got rid of the integral on the semi-circle because, as we saw, it tends to 0 as $R \to \infty$ But there are some cases where this limit is not so clear. Take, for example

$$\int_{-\infty}^{\infty} dx \frac{xe^{\mathrm{i}ax}}{1+x^2} \tag{238}$$

and let us explicitly write out the integral on the semi-circle ("SC")

$$I_{SC} = \int_{SC} \frac{ze^{iaz} dz}{1+z^2} = \int_0^{\pi} e^{iaRe^{\mathrm{i}\theta}} \frac{R^2 e^{2\mathrm{i}\theta}}{1+R^2 e^{2\mathrm{i}\theta}} \mathrm{i}d\theta \tag{239}$$

It is not at all clear that $I_{SC} \to 0$ as $R \to \infty$. As before, we note that

$$e^{iaz} = e^{\mathrm{i}aR\cos\theta} e^{-aR\sin\theta} \tag{240}$$

and define the function

$$f_R(\theta) = e^{\mathrm{i}aR\cos\theta}e^{2i\theta}\frac{R^2}{1+R^2e^{2i\theta}} \tag{241}$$

we then have

$$I_{SC} = \int_0^\pi e^{-aR\sin\theta} f_R(\theta)\mathrm{i}d\theta \tag{242}$$

We now have

$$|f_R(\theta)|^2 = \frac{R^4}{1+2R^2\cos(2\theta)+R^4} \tag{243}$$

but

$$(R^2-1)^2 \leq 1+2R^2\cos(2\theta)+R^4 \leq (R^2+1)^2 \tag{244}$$

and so

$$f_R(\theta) < \frac{R^2}{R^2-1} \quad \text{when } R > 1 \tag{245}$$

Then we have that

$$|I_{SC}| \leq \int_0^\pi d\theta|e^{-aR\sin\theta}f_R(\theta)| \leq \frac{R^2}{R^2-1}\int_0^\pi d\theta e^{-aR\sin\theta} = \frac{2R^2}{R^2-1}\int_0^{\frac{\pi}{2}} d\theta e^{-aR\sin\theta} \tag{246}$$

where we have used the symmetry of the integrand for the last step. We can determine that the integral has an upperbound if we note the following. $\sin\theta$ is monotonic and hits a maximum at $\theta = \pi/2$. It is bounded from above by θ but more importantly, in the $[0, \pi/2]$ range, it is bounded from below by $\theta/(\pi/2)$. Which means that

$$e^{-aR\sin\theta} < e^{-aR\frac{2}{\pi}\theta} \tag{247}$$

which allows us to do the integral and find

$$|I_{SC}| \leq -\frac{2R^2}{R^2-1}\frac{\pi}{2aR}\left(e^{-aR}-1\right) < \frac{2R^2}{R^2-1}\frac{\pi}{2aR} \tag{248}$$

We can clearly see that $I_{SC} \to 0$ as $R \to \infty$. We can now finish the problem. The first terms in the Laurent expansion of the integrand are

$$\begin{aligned} \frac{ze^{\mathrm{i}az}}{(z-\mathrm{i})(z+\mathrm{i})} &= \frac{e^{\mathrm{i}a\times\mathrm{i}}}{(z-\mathrm{i})} \times \frac{(\mathrm{i}+z-\mathrm{i})e^{\mathrm{i}a(z-\mathrm{i})}}{(2\mathrm{i}+z-\mathrm{i})} \\ &\simeq \frac{e^{-a}}{z-\mathrm{i}}[\mathrm{i}+(z-\mathrm{i})+\cdots] \times \frac{1}{2\mathrm{i}}\left[1-\frac{z-\mathrm{i}}{2\mathrm{i}}+\cdots\right] \times [1+\mathrm{i}a(z-\mathrm{i})+\cdots] \end{aligned} \tag{249}$$

so the residue is

$$\mathrm{Res}(f, z_+) = \frac{e^{-a}}{2} \tag{250}$$

and thus

$$I = \mathrm{i}\pi e^{-a}. \tag{251}$$

This was a particular case of a more general Lemma. *Jordan's Lemma* can be stated as follows: consider the contour C_R such that $z = Re^{\mathrm{i}\theta}$ with $\theta \in [0, \pi]$ and the function $F(z) = e^{\mathrm{i}az} f(z)$ such that

$$|f(Re^{i\theta})| \leq L_R \tag{252}$$

then

$$\left| \int_{C_R} F(z) dz \right| \leq \frac{\pi}{a} L_R \tag{253}$$

12 Integrals through singularities

Until now we have considered real integrals in which, when we convert them to complex, contour integrals, their poles lie off the contour. Let us now consider integrals which pass through a singularity.

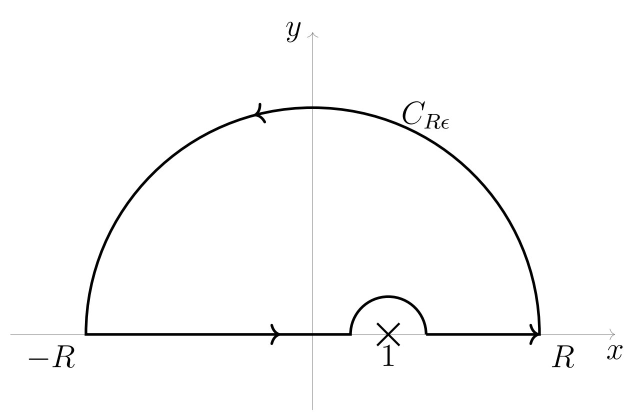
A possible example is

$$I = \int_{-\infty}^{+\infty} \frac{\cos ax}{x - 1} dx \tag{254}$$

We can rephrase this as

$$\tilde{I} = \int_{-\infty}^{+\infty} \frac{e^{\mathrm{i}ax}}{x - 1} dx \tag{255}$$

where $I = Re(\tilde{I})$. This integral looks obviously infinite: close to the singularity it behaves as $\log(x - 1)$. If we now consider the semi-circular integral, that used in previous real integrals we see that, the contour *still* passes through the singularity. So we will tweak the contour slightly and make it pass just over $x = 1$ in a small semi-circle of radius ϵ.



This new contour, $C_{R\epsilon}$ doesn't contain any poles so we have that

$$\oint_{C_{R\epsilon}} \frac{e^{\mathrm{i}az}}{z-1} dz = 0 \tag{256}$$

Let us now break down this integral into all its parts. We have that

$$\oint_{C_{R\epsilon}} \frac{e^{\mathrm{i}az}}{z-1} dz = \int_{-R}^{1-\epsilon} \frac{e^{\mathrm{i}ax}}{x-1} dx + \int_{\pi}^{0} \mathrm{i}e^{\mathrm{i}a(1+\epsilon e^{\mathrm{i}\theta})} d\theta + \int_{1+\epsilon}^{R} \frac{e^{\mathrm{i}ax}}{x-1} dx + \int_{0}^{\pi} \mathrm{i}Re^{\mathrm{i}\theta} \frac{e^{\mathrm{i}aRe^{\mathrm{i}\theta}}}{Re^{\mathrm{i}\theta}-1} d\theta \tag{257}$$

We can get rid of the last integral (through Jordan's Lemma). The sum of the first and third part (in the limit where $R \to \infty$ and $\epsilon \to 0$) gives the integral we desire. The remaining part gives us

$$\int_{\pi}^{0} \mathrm{i}e^{\mathrm{i}a(1+\epsilon e^{\mathrm{i}\theta})} d\theta = -\mathrm{i}\pi e^{\mathrm{i}a} \tag{258}$$

Thus we have

$$\tilde{I} = \int_{-\infty}^{+\infty} \frac{e^{\mathrm{i}ax}}{x-1} dx = \mathrm{i}\pi e^{\mathrm{i}a} \tag{259}$$

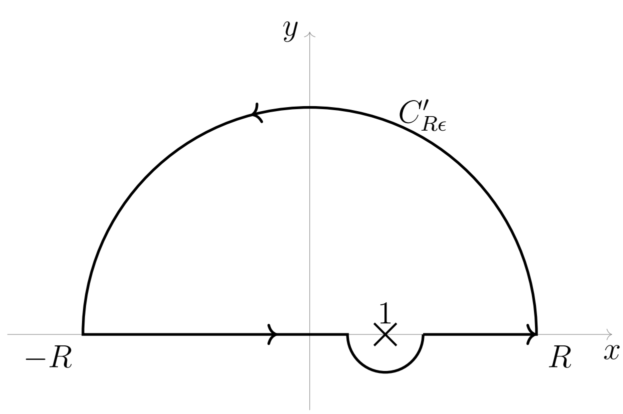
and

$$I = -\pi \sin a \tag{260}$$

Note that, for free we have also calculated

$$\int_{-\infty}^{+\infty} \frac{\sin ax}{x-1} dx = \pi \cos a \tag{261}$$

You may wonder if there is any arbitrariness in our choice of contour. In particular, in the fact that we chose the small semi-circular contour to go *above* the singularity and therefore excluding it from the interior of the contour. Let us redo the integral but now extending the contour *below* the singularity (and let us denote this contour as $C'_{R\epsilon}$).



The pole now falls inside the contour and we need to find the residue. Let us calculate the Laurent expansion

$$\frac{e^{\mathrm{i}az}}{z-1} = \frac{e^{\mathrm{i}a}}{z-1}e^{\mathrm{i}a(z-1)} \simeq \frac{e^{\mathrm{i}a}}{1-z} \times [1 + \mathrm{i}a(z-1) + \cdots] \tag{262}$$

which allows us to read off the residue:

$$\mathrm{Res}(f, 1) = e^{\mathrm{i}a} \tag{263}$$

We now have that

$$\oint_{C'_{R\epsilon}} \frac{e^{\mathrm{i}az}}{z-1}dz = 2\pi\mathrm{i}e^{\mathrm{i}a} \tag{264}$$

We can now break down the integral as above to get

$$\oint_{C'_{R\epsilon}} \frac{e^{\mathrm{i}az}}{z-1}dz = \int_{-R}^{1-\epsilon} \frac{e^{\mathrm{i}ax}}{x-1}dx + \int_{\pi}^{2\pi} \mathrm{i}e^{\mathrm{i}a(1+\epsilon e^{\mathrm{i}\theta})}d\theta + \int_{1+\epsilon}^{R} \frac{e^{\mathrm{i}ax}}{x-1}dx + \int_0^{\pi} \mathrm{i}Re^{\mathrm{i}\theta}\frac{e^{\mathrm{i}aRe^{\mathrm{i}\theta}}}{Re^{\mathrm{i}\theta}-1}d\theta \tag{265}$$

which all looks very similar except for the small semi-circle around the pole. If we focus on that integral we have

$$\int_{\pi}^{2\pi} \mathrm{i}e^{\mathrm{i}a(1-\epsilon e^{\mathrm{i}\theta})}d\theta = \mathrm{i}\pi e^{\mathrm{i}a} \tag{266}$$

and so we find

$$\tilde{I} + \pi\mathrm{i}e^{\mathrm{i}a} = 2\pi\mathrm{i}e^{\mathrm{i}a} \tag{267}$$

and so find the same answer as above.

13 Integrals of functions with branch points and cuts

We have just looked at an integral that passes through a singularity and we "fixed" it by deforming the contour to go around it. But what if, instead of a point we have a line, i.e. a branch cut. We need to play the same game and avoid it. Let us look at a concrete example. Consider the following real integral

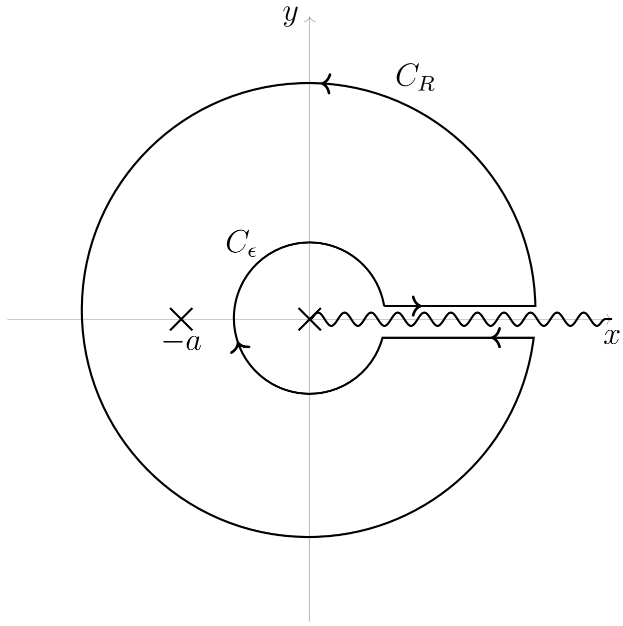
$$I = \int_0^{\infty} \frac{dx}{(x+a)^2\sqrt{x}} \tag{268}$$

with $a > 0$. The first thing we do is promote it to a complex, contour integral

$$\tilde{I} = \oint_C \frac{dz}{(z+a)^2\sqrt{z}} \tag{269}$$

The question is now: how do we pick C?

Inspecting the integrand we see that there is a branch cut arising from the $\sqrt{z}$ which we can choose to be the halfline extending along the x from 0 to ∞. As with the singularities, the contour shouldn't cross this line. Note, however, how that line is exactly the support of the real integral. We can choose the contour C to be a giant circle, with radius R, with an indent the excludes the branch cut, capped off by a tiny circle around $z = 0$, i.e. the branch point.



This contour contains a pole at $z=-a$ and so we have that

$$\tilde{I} = 2\pi \mathrm{i}\mathrm{Res}(f,-a) \tag{270}$$

To proceed, we need to calculate the residue at $-a$. The procedure is, as always, to find the appropriate terms in the Laurent expansion. We have that

$$\begin{aligned}\frac{1}{(z+a)^2}\frac{1}{\sqrt{z}} &= \frac{1}{(z+a)^2}\frac{1}{\sqrt{-a+(z+a)}} = \frac{1}{\sqrt{-a}}\frac{1}{(z+a)^2}\frac{1}{\sqrt{1-(\frac{z+a}{a})}} \\ &= -\frac{\mathrm{i}}{\sqrt{a}}\frac{1}{(z+a)^2}\left(1+\frac{1}{2}\frac{z+a}{a}+\mathcal{O}(z+a)^2\right)\end{aligned} \tag{271}$$

so we have that

$$\tilde{I} = 2\pi\mathrm{i}\times\frac{-\mathrm{i}}{\sqrt{a}}\times\frac{1}{2a} = \frac{\pi}{a^{3/2}} \tag{272}$$

Now let us unpack the contour integral. For a start, the integral around the big circle is 0; it is proportional to $1/R^{3/2}$ and goes to 0 as $R\to\infty$. We can do the integral around the small circle $z=\epsilon e^{\mathrm{i}\theta}$,

$$\int_{2\pi}^{0}\frac{\mathrm{i}\epsilon e^{\mathrm{i}\theta}d\theta}{(\epsilon e^{\mathrm{i}\theta}+a)^2\sqrt{\epsilon}e^{\mathrm{i}\theta/2}} \to -\int_0^{2\pi}\frac{\mathrm{i}\sqrt{\epsilon}e^{\mathrm{i}\theta/2}d\theta}{a^2} \to 0 \tag{273}$$

We are left then with the integrals along the branch cuts:

$$\tilde{I} = \int_0^\infty \frac{dx}{(x+a)^2\sqrt{x}} + \int_\infty^0 \frac{dx}{(x+a)^2\sqrt{xe^{\mathrm{i}2\pi}}} \tag{274}$$

where the factor of $e^{\mathrm{i}2\pi}$ is explicitly inserted to flag the fact that $\sqrt{z}$ is multivalued. We can then see that

$$\tilde{I} = \int_0^\infty \frac{dx}{(x+a)^2\sqrt{x}} - \int_\infty^0 \frac{dx}{(x+a)^2\sqrt{x}} = 2\int_0^\infty \frac{dx}{(x+a)^2\sqrt{x}} = \frac{\pi}{a^{3/2}} \tag{275}$$

and we have our answer:

$$I = \frac{1}{2}\tilde{I} = \frac{\pi}{2a^{3/2}} \tag{276}$$

Let us now consider a slightly more subtle integral

$$I = \int_0^\infty \frac{x^{-\alpha}dx}{1+x} \tag{277}$$

where $0 < \alpha < 1$. This integral has a pole at $z = -1$ and a branch cut along the positive $x - axis$. We can use exactly the same contour as we used above. We first need to find the residue at $z = -1$; the Laurent expansion is

$$\frac{x^{-\alpha}}{1+x} = \frac{[-1+(z+1)]^{-\alpha}}{z+1} = (-1)^{-\alpha}[1+\alpha(z+1)+\mathcal{O}(z+1)^2] \times \frac{1}{z+1} \tag{278}$$

Recall that $-1 = e^{\mathrm{i}\pi}$ so we have that

$$\tilde{I} = \oint_C \frac{z^{-\alpha}dz}{1+z} = 2\pi \mathrm{i} e^{-\alpha\pi\mathrm{i}} \tag{279}$$

Let us split up the parts

$$\tilde{I} = \oint_C \frac{z^{-\alpha}dz}{1+z} = \int_\epsilon^\infty \frac{x^{-\alpha}dx}{1+x} + \int_0^{2\pi} \frac{\mathrm{i}Re^{\mathrm{i}\theta}(Re^{\mathrm{i}\theta})^{-\alpha}d\theta}{1+Re^{\mathrm{i}\theta}} + \int_\infty^\epsilon \frac{(e^{2\pi\mathrm{i}}x)^{-\alpha}dx}{1+x} + \int_{2\pi}^0 \frac{\mathrm{i}\epsilon e^{\mathrm{i}\theta}(\epsilon e^{\mathrm{i}\theta})^{-\alpha}d\theta}{1+\epsilon e^{\mathrm{i}\theta}} \tag{280}$$

where, again, we have introduced a factor of $e^{2\pi\mathrm{i}}$ to signal the multivaluedness of $z^{-\alpha}$. As above, the circular parts can be discarded. We are then left with

$$\tilde{I} = \int_0^\infty \frac{x^{-\alpha}dx}{1+x} + \int_\infty^0 \frac{e^{-\alpha 2\pi\mathrm{i}}x^{-\alpha}dx}{1+x} = (1-e^{-\alpha 2\pi\mathrm{i}})I \tag{281}$$

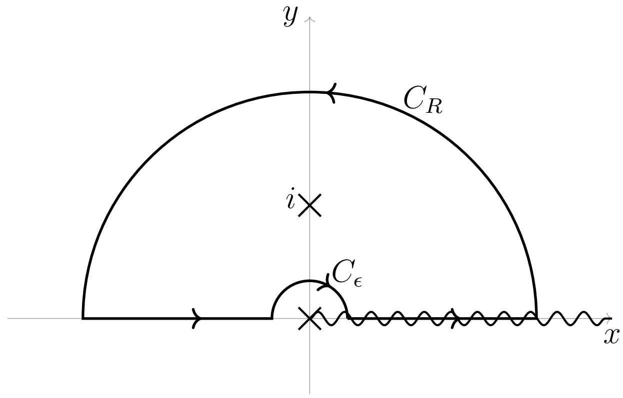
We thus have

$$I = \frac{2\pi i e^{-\alpha\pi\mathrm{i}}}{1-e^{-\alpha 2\pi\mathrm{i}}} = \pi \operatorname{cosec} \alpha\pi \tag{282}$$

Let us now consider a more complicated integral:

$$\int_0^\infty \frac{(\log x)^2 dx}{1+x^2} \tag{283}$$

The integrand has poles at $z_\pm = \pm\mathrm{i}$ and a branch cut along the positive x axis. Let us now consider a contour that consists of a semicircle above the x axis.



It encompasses z_+ so we need to find the residue there. We first factorize $1+z^2 = (z+\mathrm{i})(z-\mathrm{i})$ to get

$$\begin{aligned}\frac{[\log(\mathrm{i}+z-\mathrm{i})]^2}{(z-\mathrm{i})(2\mathrm{i}+z-\mathrm{i})} &= \frac{1}{z-\mathrm{i}} \times \frac{1}{2\mathrm{i}} \times \frac{1}{1+\frac{z-\mathrm{i}}{2\mathrm{i}}} \times \left[\log \mathrm{i} + \log\left(1+\frac{z-\mathrm{i}}{\mathrm{i}}\right)\right]^2 \\ &\simeq \frac{1}{2\mathrm{i}}\frac{1}{z-\mathrm{i}} \times \left[1 - \frac{z-\mathrm{i}}{2\mathrm{i}} + \cdots\right] \times \left[(\log \mathrm{i})^2 + 2\log \mathrm{i}\frac{z-\mathrm{i}}{\mathrm{i}} + \cdots\right] \quad (284)\end{aligned}$$

so the residue is given by

$$\mathrm{Res}(f, z_+) = \frac{(\log \mathrm{i})^2}{2\mathrm{i}} = \frac{(\mathrm{i}\pi/2)^2}{2\mathrm{i}} = \mathrm{i}\frac{\pi^2}{8} \quad (285)$$

The contour consists of four parts:

$$\begin{aligned}\oint_C \frac{(\log z)^2 dz}{1+z^2} &= \int_{-R}^{-\epsilon} \frac{(\log |x|e^{\mathrm{i}\pi})^2 dx}{1+x^2} + \int_{\pi}^{0} \mathrm{i}\epsilon e^{\mathrm{i}\theta}\frac{(\log \epsilon + \mathrm{i}\theta)^2}{1+\epsilon^2 e^{2\mathrm{i}\theta}} d\theta \\ &+ \int_{\epsilon}^{R} \frac{(\log x)^2 dx}{1+x^2} + \int_0^{\pi} \mathrm{i}Re^{\mathrm{i}\theta}\frac{(\log R + \mathrm{i}\theta)^2}{1+R^2 e^{2\mathrm{i}\theta}} d\theta \quad (286)\end{aligned}$$

Note that the integral around the large semi-circle is surpressed by powers of $(\log R)^2/R$ when $R \to \infty$ while around the small semi-circle it supressed by powers of $\epsilon(\log \epsilon)^2$ when $\epsilon \to 0$. We are then left with:

$$\begin{aligned}&\int_{-\infty}^{0} \frac{[(\log |x|)^2 + 2\mathrm{i}\pi \log |x| - \pi^2]dx}{1+x^2} + \int_0^{\infty} \frac{(\log x)^2 dx}{1+x^2} \\ &= 2\int_0^{\infty} \frac{(\log x)^2 dx}{1+x^2} + 2\mathrm{i}\pi \int_0^{\infty} \frac{\log x dx}{1+x^2} - \pi^2 \int_0^{\infty} \frac{dx}{1+x^2} \quad (287)\end{aligned}$$

We have that the last integral is simply $\pi/2$ (as we saw before). Using the residue theorem we then have (and equating the real and imaginary parts)

$$\begin{aligned}\int_0^{\infty} \frac{(\log x)^2 dx}{1+x^2} &= \frac{1}{2}\left(-\frac{\pi^3}{4} + \frac{\pi^3}{2}\right) = \frac{\pi^3}{8} \\ \int_0^{\infty} \frac{\log x dx}{1+x^2} &= 0 \quad (288)\end{aligned}$$