3 Lectures on Complex Numbers

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last modified April 11, 2024

Preamble

The following three lectures were designed for the Oxford Undergraduate course and are given during the first year BA course. These notes are not particularly original. I would like to thank Michael Barnes, James Binney, John Magorrian and Julia Yeomans for their lecture notes; I based these notes on theirs. I would greatly appreciate it if you could email me at `pedro.ferreira@physics.ox.ac.uk` with any corrections.

1 Complex Numbers

Over time you have learnt that there are different types of numbers. You started off with counting:

$$1, 2, 3, \cdot$$

and learnt about the *Natural* numbers, $\mathbb{N}$.

You rapidly figured out that the natural number don't quite do it; if you subtract two arbitrary natural numbers, you might end up with a number that doesn't fall into that set. You then learnt about *Integer* numbers, $\mathbb{Z}$:

$$\cdots, -2, -1, 0, 1, 2, 3, \cdot$$

With the integer numbers you can add and subtract any of them and you will end up with an integer.

Something happens if you are given an equation like

$$m \times x = n,$$

where m and n are integers, and asked to solve for x. The solution

$$x = \frac{n}{m}$$

is not an integer. And so, we define the *Rational* numbers, $\mathbb{Q}$, in which elements are of the form

$$\frac{n}{m}$$

for any pair of integers, n and m. We can multiply and divide (as well as add and subtract) rational numbers and we always get a rational number.

Things get trickier when we try to solve an equation like

$$x^2 = 2$$

Let us check if we can solve it with a rational number, i.e. let us assume that $x = n/m$ in which n and m have no common factors. If we plug it in we have

$$\frac{n^2}{m^2} = 2 \to n^2 = 2m^2$$

and so n^2 is even which means n is even. If n is even, then we can write it as $n = 2q$ where q is an integer. Replacing it back into the equation we are trying to solve we have

$$(2q)^2 = 2m^2 \to m^2 = 2q^2$$

which means that m^2 is even and thus m is even. But that contradicts our starting assumption, that n and m have no common factors. So x cannot be a rational number. We denote the two solutions by

$$x = \pm\sqrt{2}$$

These solutions are *Irrational* numbers, $\mathbb{I}$. The set of irrational numbers (and there are many more irrational numbers than rational numbers) combined with the rational numbers form the set of *Real* numbers, $\mathbb{R}$.

It would seem that we have finished filling out the number landscape but not quite. Consider now the following equation:

$$z^2 = -1$$

It is very similar to the equation we solved above: a second order, polynomial equation and thus should have two roots. But we clearly can't solve it with the usual reals so need to introduce a new type of number, an *imaginary* number, i, such that

$$z = \pm \mathrm{i}$$

Now, i is distinctly different from a real number. I.e. we can't multiply, divide, add or subtract real numbers to give us an imaginary number. So we need to include imaginary numbers with our real numbers to form *Complex* numbers, $\mathbb{C}$.

Is there any physical situation where we might need such a number? You will often come across a Simple Harmonic Oscillator (SHO) throughout your degree. It characterizes a multitude of different physical systems, like a simple pendulum undergoing small oscillation, a mass connected to a spring, an electrical circuit, a standing wave, etc. It is described in terms of a differential equation of the form

$$\frac{d^2Q}{dt^2} + \omega^2 Q = 0$$

where $\omega > 0$. We can solve it with oscillatory functions

$$Q \propto \cos(\omega t), \sin(\omega t)$$

Let us try solving it with a function of the form

$$Q = Q_0 e^{\alpha t}$$

Subsituting it in and factorizing, we end up with an equation of the form

$$(\alpha^2 + \omega^2) Q_0 e^{\alpha t} = 0$$

which can be solved by

$$\alpha^2 = \sqrt{-\omega^2}$$

Using our new found knowledge, we have that

$$\alpha = \pm \mathrm{i}\omega$$

and the solutions are of the form

$$Q \propto e^{\mathrm{i}\omega t}, e^{-\mathrm{i}\omega t}$$

We will come back to these complex exponentials in a bit.

We need to figure out a bit more on how to combine real and imaginary numbers. What happens if we now take the 4^{th} root of -1, i.e. what happens if we try and solve

$$z^4 = -1?$$

If we follow what we did above, we have now that

$$z^2 = \pm \mathrm{i}.$$

Consider one of those sets of roots

$$z^2 = \mathrm{i}.$$

The solution to this equation is (as expected) two roots of the form

$$\begin{aligned} z_1 &= \frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}} \\ z_2 &= -\frac{1}{\sqrt{2}} - \mathrm{i}\frac{1}{\sqrt{2}}. \end{aligned}$$

Look at the way I have written each one of these solution: I have added a real part and imaginary part. Let us check that they in fact solve the equation:

$$\begin{aligned} z_1^2 = z_1 \times z_1 &= \left(\frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}\right) \times \left(\frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \mathrm{i}\frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}} \times \mathrm{i}\frac{1}{\sqrt{2}} \\ &= \frac{1}{2} + \mathrm{i}\frac{1}{2} + \mathrm{i}\frac{1}{2} - \frac{1}{2} = \mathrm{i}, \end{aligned}$$

where I have used the distributive property of multiplication and, for the last term, the fact that $\mathrm{i} \times \mathrm{i} = -1$ (i.e. the whole reason why i was invented). In the same way, we can find the other two roots by solving

$$z^2 = -\mathrm{i}$$

to find

$$\begin{aligned} z_3 &= \frac{1}{\sqrt{2}} - \mathrm{i}\frac{1}{\sqrt{2}} \\ z_4 &= -\frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}} \end{aligned}$$

i.e. four roots in total, as expected.

Again, look at the form these solutions have. They are the sum of a real and imaginary number. One way to think about these numbers is that they have two dimensions, a real one and an imaginary one. And so, if we wanted to write down a general, *complex* number, we would write it as

$$z = x + \mathrm{i}y$$

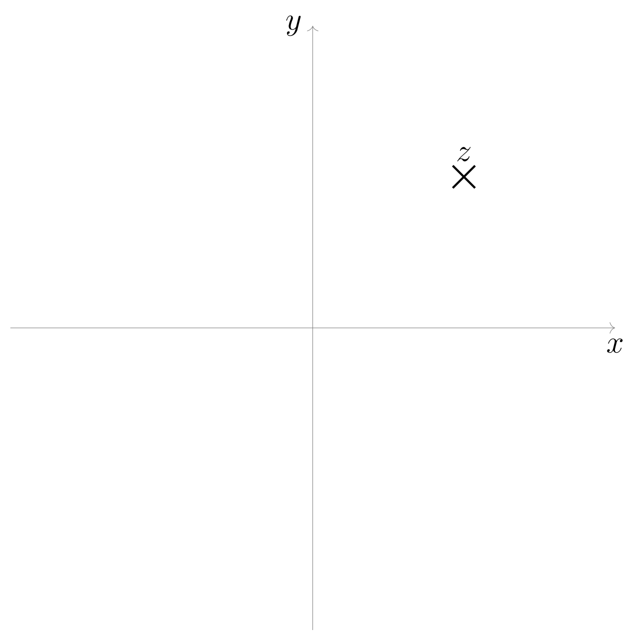
We have that the *real* part of z is x, i.e.

$$\mathrm{Re}(z) = x$$

and the *imaginary* part of z is y, i.e.

$$\mathrm{Im}(z) = y$$

If we want to find some graphical way of representing them, we would need to use a plane, the *Argand* plane. So, for example, a general complex number looks like this:



2 Other Representations of Complex Numbers

Given that we can represent a complex number, z, on a plane, i.e. as a vector, we can use another convenient representation. Note that a 2-D vector can be represented in terms of a *modulus* (or length), r and *argument* (or angle), θ, such that

$$z = r\sin\theta + \mathrm{i}r\cos\theta = r(\sin\theta + \mathrm{i}\cos\theta)$$

Note that we then have

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \theta &= \tan^{-1}\frac{y}{x} \end{aligned}$$

It is useful to define some additional notation. The modulus of z, r, can be written as $|z|$ or mod(z) while the argument of z, θ can be written as arg(z). Note that the θ of a given z is not unique - we can obtain the same z by adding $2\pi n$ (where n is an arbitrary integer) to θ. We thus (sometimes but not always!) define arg(z) to be the *principal value* of θ which lies between $-\pi$ and π.

Let us consider a few examples. If we start with 1, we have that

$$\begin{aligned} \mathrm{mod}(1) &= 1 \\ \arg(1) &= 0 \end{aligned}$$

Now consider the new number we just introduced, i,

$$\begin{aligned} \mathrm{mod}(\mathrm{i}) &= 1 \\ \arg(\mathrm{i}) &= \frac{\pi}{2} \end{aligned}$$

and for $-\mathrm{i}$ we have

$$\begin{aligned} \mathrm{mod}(-\mathrm{i}) &= 1 \\ \arg(-\mathrm{i}) &= -\frac{\pi}{2} \end{aligned}$$

Finally, let us look at the four roots we found above for

$$z^4 + 1 = 0$$

We then have

$$\begin{aligned} \mathrm{mod}(z_1) &= 1 \\ \arg(z_1) &= \frac{\pi}{4} \\ \mathrm{mod}(z_2) &= 1 \\ \arg(z_2) &= -\frac{3\pi}{4} \\ \mathrm{mod}(z_3) &= 1 \\ \arg(z_3) &= -\frac{\pi}{4} \\ \mathrm{mod}(z_4) &= 1 \\ \arg(z_4) &= \frac{3\pi}{4} \end{aligned}$$

It is useful to define the complex conjugate of z

$$z^* = z - \mathrm{i}y$$

Another way of writing the complex conjugate is $\bar{z}$. A compact way of expressing (and calculating) the modulus is

$$r^2 = |z| = \sqrt{z \times z^*}$$

Given z and z^* we can determine x and y through

$$\begin{aligned} x &= \frac{1}{2}(z + z^*) \\ x &= \frac{1}{2\mathrm{i}}(z - z^*) \end{aligned}$$

We can use the polar representation of a complex number to find an even more remarkable form. To do so, take the Taylor expansion of $\sin\theta$ and $\cos\theta$ and combine them:

$$\begin{aligned} z &= r(\sin\theta + \mathrm{i}\cos\theta) = r\left(\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!} + \mathrm{i}\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!}\right) \\ &= r\left(1 + \mathrm{i}\theta - \frac{\theta^2}{2!} - \mathrm{i}\frac{\theta^3}{3!} + \cdots\right) = r\left[1 + (\mathrm{i}\theta) + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \cdots\right] \\ &= r\sum_{n=0}^{\infty}\frac{(\mathrm{i}\theta)^n}{n!} = re^{\mathrm{i}\theta} \end{aligned}$$

which is known as Euler's equation.

Let us revisit the examples we had above. We can now write

$$\begin{aligned} \mathrm{i} &= e^{\mathrm{i}\pi/2} \\ \mathrm{i} &= e^{-\mathrm{i}\pi/2} = e^{\mathrm{i}\pi/2} \\ z_1 &= e^{\mathrm{i}\pi/4} \\ z_2 &= e^{-\mathrm{i}3\pi/4} = e^{\mathrm{i}5\pi/4} \\ z_3 &= e^{-\mathrm{i}\pi/4} = e^{\mathrm{i}7\pi/4} \\ z_4 &= e^{\mathrm{i}3\pi/4} \end{aligned}$$

It is interesting to note that we can order the four z_i which are solutions to the quartic equation, as

$$z_i = \left\{e^{\mathrm{i}\pi/4}, e^{\mathrm{i}3\pi/4}, e^{\mathrm{i}5\pi/4}, e^{\mathrm{i}7\pi/4}\right\}$$

The exponential representation of a complex number make certain operations very easy, as we shall see further on. For now, note that the conjugate of a complex number

$$z = re^{\mathrm{i}\theta}$$

is simply

$$z^* = re^{-\mathrm{i}\theta}$$

You can see this by simply re-expressing it in the polar representation:

$$z^* = re^{-\mathrm{i}\theta} = r\left[\cos(-\theta) + \mathrm{i}\sin(-\theta)\right] = r(\cos\theta - \mathrm{i}\sin\theta) = x - \mathrm{i}y$$

Let us look at a few examples where we can use the exponential representation. We can try and express

$$\mathrm{i}^{-2\mathrm{i}}$$

and

$$\ln(1 - \mathrm{i}\sqrt{3})$$

in the form $x + y\mathrm{i}$. In the first case we have

$$\mathrm{i}^{-2\mathrm{i}} = \left[e^{\mathrm{i}(\frac{\pi}{2}+2\pi n)}\right]^{-2\mathrm{i}} = e^{(-2\mathrm{i})\times\mathrm{i}(\frac{\pi}{2}+2\pi n)} = e^{\pi+4\pi n}$$

where $n \in \mathbb{Z}$. In the second case we have

$$\ln(1 - \mathrm{i}\sqrt{3}) = \ln 2e^{-\mathrm{i}\frac{\pi}{3}+\mathrm{i}2n\pi} = \ln 2 + \mathrm{i}\left(2n\pi - \frac{\pi}{3}\right)$$

again, with $n \in \mathbb{Z}$.

We can express trigonometric functions in terms of complex numbers. Consider the case where $|z| = 1$. We now have that

$$\begin{aligned} z &= e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta \\ z^* &= e^{-\mathrm{i}\theta} = \cos\theta - \mathrm{i}\sin\theta \end{aligned}$$

We can invert this system of equations to find that

$$\begin{aligned} \cos\theta &= \frac{e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}}{2} \\ \sin\theta &= \frac{e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}}{2\mathrm{i}} \end{aligned}$$

We can revisit the solution to the SHO we had before. We have that the general solutions is given by

$$Q = Q_1\cos(\omega t) + Q_2\sin(\omega t)$$

But now replace the sin and cos by complex exponentials and you have

$$\begin{aligned} Q &= Q_1\left(\frac{e^{\mathrm{i}\omega t} + e^{-\mathrm{i}\omega t}}{2}\right) + Q_2\left(\frac{e^{\mathrm{i}\omega t} - e^{-\mathrm{i}\omega t}}{2\mathrm{i}}\right) \\ &= \frac{1}{2}(Q_1 - \mathrm{i}Q_2)e^{\mathrm{i}\omega t} + \frac{1}{2}(Q_1 + \mathrm{i}Q_2)e^{-\mathrm{i}\omega t} = Q_+e^{\mathrm{i}\omega t} + Q_-e^{-\mathrm{i}\omega t} \end{aligned}$$

3 Operations with Complex Numbers

We have already done some operations with complex numbers but let us look in more detail. Consider the main four:

- Addition:

$$z_1 + z_2 = x_1 + \mathrm{i}y_1 + x_2 + \mathrm{i}y_2 = (x_1 + x_2) + \mathrm{i}(y_1 + y_2)$$

- Subtraction:

$$z_1 - z_2 = x_1 + \mathrm{i}y_1 - x_2 - \mathrm{i}y_2 = (x_1 - x_2) + \mathrm{i}(y_1 - y_2)$$

- Multiplication:

$$z_1 z_2 = (x_1 + \mathrm{i}y_1)(x_2 + \mathrm{i}y_2) = (x_1x_2 - y_1y_2) + \mathrm{i}(x_1y_2 + x_2y_1)$$

  Note that if we represent these numbers as complex exponentials, the product looks much simpler

$$z_1 z_2 = (r_1 e^{\mathrm{i}\theta_1})(r_2 e^{\mathrm{i}\theta_2}) = r_1 r_2 e^{\mathrm{i}(\theta_1+\theta_2)}$$

  We can immediately infer that

$$\begin{aligned} |z_1 z_2| &= |z_1||z_2| \\ \arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) \end{aligned}$$

- Division:

$$\frac{z_1}{z_2} = \frac{(x_1 + \mathrm{i}y_1)}{(x_2 + \mathrm{i}y_2)} = \frac{(x_1 + \mathrm{i}y_1)(x_2 - \mathrm{i}y_2)}{(x_2 + \mathrm{i}y_2)(x_2 - \mathrm{i}y_2)} = \frac{(x_1x_2 + y_1y_2) + \mathrm{i}(x_1y_2 - x_2y_1)}{x_2^2 + y_2^2}$$

  Again, if we represent these numbers as complex exponentials, division looks much simpler

$$\frac{z_1}{z_2} = \frac{r_1 e^{\mathrm{i}\theta_1}}{r_2 e^{\mathrm{i}\theta_2}} = \frac{r_1}{r_2} e^{\mathrm{i}(\theta_1-\theta_2)}$$

  We can immediately infer that

$$\begin{aligned} \left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|} \\ \arg\left(\frac{z_1}{z_2}\right) &= \arg(z_1) - \arg(z_2) \end{aligned}$$

I have introduced complex numbers as roots of a number. Generally, if we want to find roots of a complex number, it makes sense to write them as a complex exponential. Consider a general equation of the form

$$z^n = w$$

Now express w as a complex exponential

$$w = re^{\mathrm{i}(\theta+2\pi k)}$$

with $k \in \mathbb{Z}$. Note that I have included an extra factor of $e^{\mathrm{i}2k\pi}$, which is just 1. Now take the n-th root of w to find

$$z = r^{\frac{1}{n}} e^{\mathrm{i}\frac{\theta+2\pi k}{n}}$$

As you can see there are a number of roots as you would expect. In fact you expect n roots which you can obtain, simply running $k = 0, 1, \cdots, n-1$. You will note that the roots lie on a circle in the complex plane with radius $r^{1/n}$. Furthermore the roots are equidistant in angle along the circle.

Let us now consider a few examples. In particular, let us look at two polynomial equations we considered above. If we look at

$$z^2 = -1 = e^{\mathrm{i}(\pi+2k\pi)}$$

for $k \in \mathbb{Z}$. We have that roots will be

$$z = -1 = e^{\mathrm{i}(\pi+2k\pi)/2}$$

for $k = 0, 1$ so that

$$\{z_1, z_2\} = \left\{e^{\mathrm{i}\pi/2}, e^{\mathrm{i}3\pi/2}\right\}$$

as we saw above.

For

$$z^4 = -1 = e^{\mathrm{i}(\pi+2k\pi)}$$

for $k \in \mathbb{Z}$. We have that roots will be

$$z = -1 = e^{\mathrm{i}(\pi+2k\pi)/4}$$

for $k = 0, 1, 2, 3$ so that we now have

$$z_i = \left\{e^{\mathrm{i}\pi/4}, e^{\mathrm{i}3\pi/4}, e^{\mathrm{i}5\pi/4}, e^{\mathrm{i}7\pi/4}\right\}$$

again, as we saw above.

We can use these operations with complex numbers to tackle a ubiquitous problem in maths and physics: finding the roots of a polynomial equation. Consider an n^{th} order polynomial:

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = \sum_{i=0}^{n} a_i z^i$$

It turns out that

$$P_n(z) = 0$$

can always be solved in terms of complex numbers, i.e., we can expand $P_n(z)$ in the form

$$P_n(z) = a_n(z - z_1)(z - z_2)\cdots(z - z_n)$$

where z_i are the roots of the equation. We can use this factorized way of writing $P_n(z)$ to show that

$$\begin{aligned}\sum_{i=1}^{n} z_i &= -\frac{a_{n-1}}{a_n} \\ \prod_{i=1}^{n} z_i &= (-1)^n \frac{a_0}{a_n}\end{aligned}$$

For example, let us look at

$$z^4 + 1 = 0$$

We have that

$$\begin{aligned}a_4 &= 1 \\ a_0 &= 1\end{aligned}$$

Now let us consider each one of the relations. First we have

$$\sum_{i=1}^{4} z_i = \frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \mathrm{i}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \mathrm{i}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}} = 0$$

as advertised. Then we have (and note that I have put them in a different order)

$$\begin{aligned}\prod_{i=1}^{4} z_i &= \left(\frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}\right) \times \left(\frac{1}{\sqrt{2}} - \mathrm{i}\frac{1}{\sqrt{2}}\right) \times \left(-\frac{1}{\sqrt{2}} - \mathrm{i}\frac{1}{\sqrt{2}}\right) \times \left(-\frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}\right) \\ &= \frac{1}{2}(1+1) \times \frac{1}{2}(1+1) = 1 \times 1 = 1\end{aligned}$$

again, as advertised.

4 Curves on the complex plane.

Sometimes one can consider an equation that leads to a degenerate set of solutions. This typically arises if one just enforces one part of a complex equation of the form

$$f(z, z^*) = c.$$

For example

$$\begin{aligned}|f(z, z^*)| &= |c| \\ \arg[f(z, z^*)] &= \arg[c] \\ \mathrm{Re}[f(z, z^*)] &= \mathrm{Re}[c] \\ \mathrm{Im}[f(z, z^*)] &= \mathrm{Im}[c]\end{aligned}$$

can all lead to lines on the complex plane.

Suppose we consider the equation

$$P_1(z) = z - a = c$$

where a and c are complex numbers. It has one root

$$z = c + a$$

But now consider the equation

$$|P_1(z)| = |z - a| = |c|$$

We can write it explicitly in terms of x and y and $a = a_1 + \mathrm{i}a_2$ so that

$$|x + \mathrm{i}y - a_1 - \mathrm{i}a_2| = |(x - a_1) + \mathrm{i}(y - a_2)| = \sqrt{(x - a_1)^2 + (y - a_2)^2} = |c|$$

This is the equation for a circle centred at (a_1, a_2) and radius $|c|$.

In the previous example we took the modulus of the equation and made it degenerate. Something similar happens of we take the argument. Specifically, if we consider

$$\arg(P_1(z)) = \arg(z - a) = \arg c = \theta_0$$

corresponds to a ray emanating from (a_1, a_2) along the direction $(\sin\theta_0, \cos\theta_0)$.

One can consider a multitude of such curves based on the same idea - finding a set of degenerate roots to an equation. For example, consider the following equation

$$|z - 1| + |z + 1| = 8$$

We can rewrite and manipulate this expression to get

$$\begin{aligned}
|z + 1| &= 8 - |z - 1| \\
\sqrt{(x + 1)^2 + y^2} &= 8 - \sqrt{(x - 1)^2 + y^2} \\
(x + 1)^2 + y^2 &= 64 - 16\sqrt{(x - 1)^2 + y^2} + (x - 1)^2 + y^2 \\
2x &= 64 - 16\sqrt{(x - 1)^2 + y^2} - 2x \\
16\sqrt{(x - 1)^2 + y^2} &= 64 - 4x \\
\sqrt{(x - 1)^2 + y^2} &= 4 - \frac{x}{4} \\
x^2 - 2x + 1 + y^2 &= 16 - 2x + \frac{x}{16} \\
\frac{15}{16}x^2 + y^2 &= 15 \\
\frac{x^2}{16} + \frac{y^2}{15} &= 1
\end{aligned}$$

In other words, we find that the curve is an ellipse.

Another example is

$$\mathrm{Im}\left(z^2\right) = \frac{z^2 - z^{*2}}{2} = 2$$

Again, working it out we have

$$\begin{aligned}(x^2+y^2)+\mathrm{i}2xy-(x^2+y^2)+\mathrm{i}2xy &= 4\mathrm{i}\\ 4xy &= 4\\ xy &= 1\end{aligned}$$

which is a hyperbola.

5 Functions of a Complex Variable

Let us revisit some functions you are familiar with. Consider first the exponential function:

$$e^z = e^{x+\mathrm{i}y} = e^x e^{\mathrm{i}y} = e^x(\cos y + \mathrm{i}\sin y)$$

The inverse function is the logarithm. Let us write the z in exponential notation

$$z = re^{\mathrm{i}\theta} = re^{\mathrm{i}(\theta+2n\pi)}$$

for $n \in \mathbb{Z}$ (recall that there are multiple ways of writing z). Now take the logarithm

$$\ln z = \ln\left[re^{\mathrm{i}(\theta+2n\pi)}\right] = \ln r + \ln\left[e^{\mathrm{i}(\theta+2n\pi)}\right] = \ln r + \mathrm{i}(\theta+2n\pi)$$

for $n \in \mathbb{Z}$. We have found that the logarithm of a complex number is multivalued, much like the root of a complex number, though in this case it can take an infinite number of values.

We have the trignometric and hyperbolic functions:

$$\begin{aligned}\cos z &= \frac{e^{\mathrm{i}z}+e^{-\mathrm{i}z}}{2} \quad \cosh z = \frac{e^z+e^{-z}}{2}\\ \sin z &= \frac{e^{\mathrm{i}z}-e^{-\mathrm{i}z}}{2\mathrm{i}} \quad \sinh z = \frac{e^z-e^{-z}}{2}\end{aligned}$$

which are related via

$$\begin{aligned}\cos \mathrm{i}z &= \cosh z \quad \cosh \mathrm{i}z = \cos z\\ \sin \mathrm{i}z &= \mathrm{i}\sinh z \quad \sinh \mathrm{i}z = \mathrm{i}\sin z\end{aligned}$$

For example, apply these relations to find the real and imaginary part of

$$\sin(2+3\mathrm{i})$$

expressing each part as a product of hyperbolic and trigonometric functions. We have that

$$\sin(2+3\mathrm{i}) = \sin(2)\cos(3\mathrm{i}) + \cos(2)\sin(3\mathrm{i}) = \mathrm{i}\left[\sin 2\cosh(3) + \cos(2)\sinh(3)\right]$$

Given that we have expressed the trigonometric functions in terms of a complex exponential, we can now find explicit inverses for them. Consider the inverse of sin. We have that

$$w = \sin^{-1} z \to z = \sin w = \frac{e^{\mathrm{i}w}-e^{-\mathrm{i}w}}{2\mathrm{i}}$$

We can re-express this as a quadratic equation, multiplying by $e^{\mathrm{i}w}$ and $2\mathrm{i}$ and rearranging it:

$$2\mathrm{i}ze^{\mathrm{i}w} = e^{2\mathrm{i}w} - 1$$
$$e^{2\mathrm{i}w} - 2\mathrm{i}ze^{\mathrm{i}w} - 1 = 0$$

This is a quadratic equation with a solution

$$\begin{aligned} e^{\mathrm{i}w} &= \frac{2\mathrm{i}z \pm \sqrt{-4z^2+4}}{2} = \mathrm{i}z \pm \sqrt{1-z^2} \\ \mathrm{i}w &= \ln\left(\mathrm{i}z \pm \sqrt{1-z^2}\right) \end{aligned}$$

so

$$\sin^{-1} z = -\mathrm{i}\ln\left(\mathrm{i}z \pm \sqrt{1-z^2}\right)$$

with $n \in \mathbb{Z}$.

6 De Moivre's Theorem

Consider, $z = e^{\mathrm{i}\theta}$ and the following equation

$$z^n = e^{\mathrm{i}n\theta}$$

Now take each side in the polar form and we have

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$

Consider a few examples. For $n = 2$ we can recover a formula you are familiar with, the double angle formula. We have that

$$(\cos\theta + \mathrm{i}\sin\theta)^2 = \cos^2\theta + \mathrm{i}2\sin\theta\cos\theta - \sin^2\theta = \cos 2\theta + \mathrm{i}\sin 2\theta$$

Equating the real and imaginary part we have

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta \end{aligned}$$

If we want to invert these formulas, it is easier to take the Euler formula. For example

$$\cos^2\theta = \left(\frac{e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}}{2}\right)^2 = \frac{e^{2\mathrm{i}\theta} + 2 + e^{-2\mathrm{i}\theta}}{4} = \frac{1}{2}(1 + \cos 2\theta)$$

But we can, easily, go further. For $n = 3$ we can use the binomial formula to show that

$$\cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta = \cos 3\theta + \mathrm{i}\sin 3\theta$$

Which gives us some interesting equalities

$$\begin{aligned} \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ \sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta \end{aligned}$$

And, again, we can invert these expression by using the complex exponential to, for example find:

$$\begin{aligned}
\sin^3\theta &= \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^3 = \frac{e^{3i\theta}-3e^{i\theta}+3e^{-i\theta}-e^{-3i\theta}}{(2i)^3} \\
&= \frac{1}{(2i)^2}\left(\frac{e^{3i\theta}-e^{-3i\theta}}{2i} - 3\frac{e^{i\theta}-e^{-i\theta}}{2i}\right) = -\frac{1}{4}\left(\sin 3\theta - 3\sin\theta\right)
\end{aligned}$$

7 Applications of complex numbers

If you consider the following equation

$$z^n = re^{i\theta}$$

we have that the solution is

$$z = r^{1/n}e^{i(\theta+2\pi k)/n}$$

with $k = 0, 1, \cdots (n-1)$. You will note that the roots lie on a circle in the complex plane with radius $r^{1/n}$. Furthermore the roots are equidistant in angle along the circle. Now recall that

$$\sum_{k=0}^{n-1} z_k = -\frac{a_{n-1}}{a_n}$$

The RHS is 0 in this case. The LHS is

$$r^{1/n}\sum_{k=0}^{n-1} e^{i(2\pi k+\theta)/n}$$

which leads to

$$\sum_{k=0}^{n-1} e^{i(2\pi k+\theta)/n} = 0$$

The real and imaginary part of this equality are

$$\begin{aligned}
\sum_{k=0}^{n-1}\sin\frac{2\pi k+\theta}{n} &= 0 \\
\sum_{k=0}^{n-1}\cos\frac{2\pi k+\theta}{n} &= 0
\end{aligned}$$

For example, we saw that was the case when we added the roots of the equation

$$z^4 = 1$$

Consider now the following series

$$S_n = \sum_{k=1}^{n} \cos(k\theta)$$

We can use our new found knowledge on the connection between trigonometric functions and complex exponentials to work it out. We have that

$$S_n = \sum_{k=1}^{n} \mathrm{Re}(e^{\mathrm{i}k\theta}) = \mathrm{Re}(\sum_{k=1}^{n} e^{\mathrm{i}k\theta})$$

Now, we have that the sum of a geometric series is

$$a(1 + r + r^2 + \cdots r^{n-1}) = a\frac{1-r^n}{1-r}$$

We can apply this now to our expression:

$$\sum_{k=1}^{n} e^{\mathrm{i}k\theta} = \sum_{k=1}^{n} \left(e^{\mathrm{i}\theta}\right)^k = e^{\mathrm{i}\theta}\frac{1-e^{\mathrm{i}n\theta}}{1-e^{\mathrm{i}\theta}} = e^{\mathrm{i}\theta}\frac{e^{\mathrm{i}n\theta/2}}{e^{\mathrm{i}\theta/2}}\frac{\left(e^{-\mathrm{i}n\theta/2}-e^{\mathrm{i}n\theta/2}\right)}{\left(e^{-\mathrm{i}\theta/2}-e^{\mathrm{i}\theta/2}\right)} = e^{\mathrm{i}(n+1)\theta/2}\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}.$$

If we now take the real part, we finally find

$$S_n = \sum_{k=0}^{n} \cos(k\theta) = \cos\frac{(n+1)\theta}{2}\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$

Consider another example and show that

$$\cos\theta + \frac{1}{3}\sin 2\theta - \frac{1}{9}\cos 3\theta - \frac{1}{27}\sin 4\theta + \frac{1}{81}\cos 5\theta + \frac{1}{243}\sin 6\theta - \cdots$$

Let us make a guess and assume that it is the real part of some series. If we do that, we can rewrite it as

$$\mathrm{Re}\left[e^{\mathrm{i}\theta} - \frac{\mathrm{i}}{3}e^{\mathrm{i}2\theta} - \frac{1}{3^2}e^{\mathrm{i}3\theta} + \frac{\mathrm{i}}{3^3}e^{\mathrm{i}4\theta}\cdots\right]$$

which can be expressed more compactly as

$$\sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n e^{\mathrm{i}(n+1)\theta}}{3^n}$$

This, again, is a geometric series so we have

$$\sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n e^{\mathrm{i}(n+1)\theta}}{3^n} = \frac{3e^{\mathrm{i}\theta}}{3+\mathrm{i}e^{\mathrm{i}\theta}}$$

Taking the real part, we then have

$$\cos\theta + \frac{1}{3}\sin 2\theta - \frac{1}{9}\cos 3\theta - \frac{1}{27}\sin 4\theta + \frac{1}{81}\cos 5\theta + \frac{1}{243}\sin 6\theta - \cdots = \frac{9\cos\theta}{10-6\sin\theta}$$

Let us now consider the following integral

$$I = \int dx e^{ax} \sin bx = \mathrm{Im}\left[\int dx e^{ax} e^{\mathrm{i}bx}\right]$$

We have that

$$\begin{aligned}
\int dx e^{(a+\mathrm{i}b)x} &= \frac{e^{(a+\mathrm{i}b)x}}{a+\mathrm{i}b} + C \\
&= \frac{(a-\mathrm{i}b)e^{(a+\mathrm{i}b)x}}{a^2+b^2} + C \\
&= \frac{e^{ax}}{a^2+b^2}(a-\mathrm{i}b)(\cos bx + \mathrm{i}\sin bx) + C \\
&= \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx + \mathrm{i}\left(a\sin bx - b\cos bx\right)\right] + C
\end{aligned}$$

If we take the Imaginary part, we now have

$$I = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right) + C'$$

Another integral that you will come across in a variety of settings is

$$I = \int_{-\infty}^{+\infty} dx e^{-x^2/2}\cos(kx) = \mathrm{Re}\left[\int_{-\infty}^{+\infty} dx e^{-x^2/2} e^{\mathrm{i}kx}\right]$$

We can complete the squares so

$$\begin{aligned}
\int_{-\infty}^{+\infty} dx e^{-x^2/2} e^{\mathrm{i}kx} &= \int_{-\infty}^{+\infty} dx e^{-(x-ik)^2/2-k^2/2} \\
&= \int_{-\infty}^{+\infty} d\tilde{x} e^{-\tilde{x}^2/2-k^2/2} \\
&= e^{-k^2/2}\int_{-\infty}^{+\infty} d\tilde{x} e^{-\tilde{x}^2/2} = \sqrt{\frac{\pi}{2}} e^{-k^2/2}
\end{aligned}$$

where I have used Gaussian integral.