Complex Numbers Revision (exam questions)

1. (2015)

1. Draw an Argand diagram, with labelled values, showing the locus of points that satisfy $|z| = \arg(z)$. [2]

2. Find the values of the complex number z that satisfy $z = \mathrm{i}^{\mathrm{i}}$. [3]

2. (2016)

1. Sketch on an Argand diagram the locus of complex numbers z such that
(i) $\sin(|z|) = 0$, for $|z| < 10$
(ii) $\mathrm{Re}\{\log z\} = 2$ [4]

2. Find the real and imaginary parts of $\sin(2+3i)$, expressing each part as a product of hyperbolic and trigonometric functions. [4]

3.

(a) Obtain and carefully sketch the locus in the complex plane defined by $\mathrm{Re}(z^{-1}) = 1$. On the same diagram, sketch the locus defined by $\mathrm{Im}(z^{-1}) = 1$. At what angle do these loci intersect one another? Show that the unit circle centred at the origin of the complex plane, touches both loci but crosses neither of them. [6]

4.

1. (a) Use de Moivre's theorem to derive an expression for $\cos 3\theta$ in terms of $\cos\theta$ and $\sin\theta$, where θ is a real variable.

(b) Find all the roots of the equation $z^3 = 1$, where z is a complex variable.

(c) Show that $\dfrac{1+\mathrm{e}^{-i\theta}}{1-\mathrm{e}^{-i\theta}} = -i\cot(\theta/2)$, where θ is a real variable. [7]

5. (2018)

7.

(a) The polynomial $f(z)$ where z is a complex variable is defined as

$$f(z) = z^4 - 2z^3 + 8z^2 - 8z + 16.$$

Show that the equation $f(z) = 0$ has two purely imaginary roots. Hence or otherwise find all the roots of the equation and show that the sum and product of the roots take the expected values. [7]

(b) Use de Moivre's theorem to show that

$$\tan 4\theta = \frac{4(u - u^3)}{1 + u^4 - 6u^2},$$

where $u = \tan\theta$. Use this expression to show that $\tan 22.5° = \sqrt{3 - \sqrt{8}}$. [6]

(c) Explain why the equation of a plane in $\mathbb{R}^3$ may be written in the form $\mathbf{r} \cdot \mathbf{n} = d$, where $\mathbf{r}$ is the position vector of points on the plane. Use a sketch to define $\mathbf{n}$ and d.

Find the equation of the line of intersection of the two planes $2x + 6y - 3z = 10$ and $5x + 2y - z = 12$. Express the line equation in both the cartesian and vector forms. [7]

6. (2018)

1. Let $z = x + iy$ be a complex variable with real x, y. Show that

(a) $\tanh^{-1} z = \frac{1}{2} \ln \frac{1+z}{1-z}$

(b) $|\sin(z)| \geq |\sin(x)|$. [5]

7. (2019)

9. (a) Show that

$$64 \sin^7(\theta) = 35 \sin(\theta) - 21 \sin(3\theta) + 7 \sin(5\theta) - \sin(7\theta).$$

[6]

(b) Show that

$$\cos^{-1}(z) = -i \ln\left(z \pm i\sqrt{1 - z^2}\right),$$

and justify all steps in your working. [5]

(c) For the polynomial equation

$$z^5 - 5z^4 + 11z^3 - 3z^2 - 12z + 8 = 0,$$

find:

(i) the sum of the roots;

(ii) the product of the roots;

(iii) the values of all the roots.

(HINT: One of the roots is $(2 + 2i)$.) [9]

8. (2019)

1. Write the following in the form $a + i\,b$ where a and b are real:

(a) $(2e^{i\pi/4})^2$, (b) $\dfrac{1+i}{1-i}$, (c) $\ln(\sqrt{3}+i)$, (d) $(i^i)^i$. [4]

2. Solve

$$\det\begin{pmatrix} z & -i \\ -i & z^3 \end{pmatrix} = 0\,,$$

and show your solutions on an Argand diagram. [4]

9. (2021)

2. Solve the equation $2\sin ix - 3i\cos ix = 3i$, demonstrating your method.

10. (2022)

9. (a) Find, in the form $r\exp(i\theta)$, all values of complex number z which satisfy the following equations. Plot these solutions in the Argand diagram.

(i) $z^5 = 1 + \sqrt{3}i$

(ii) $z = \left(\sqrt{2}(1+i)\right)^i$

(iii) $\mathrm{Re}\left\{\dfrac{z-1}{z+1}\right\} = 0$

[12]

(b) By considering the binomial expansion of $(1 + e^{i\theta})^N$, or otherwise, show that

$$\sum_{n=0}^{N}\binom{N}{n}\sin(n\theta+\phi) = 2^N\cos^N\left(\frac{\theta}{2}\right)\sin\left(\frac{N\theta}{2}+\phi\right)$$

where $\binom{N}{n}$ represents the binomial coefficient $\left(\dfrac{N!}{(N-n)!n!}\right)$. [8]

11. (2022)

3. Find all roots of $\cos(z) = i$ in the form $z = x + iy$ where x and y are real.

12. (2023)

9. (a) (i) Prove that the sum and products of the roots, x_i, of the polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ satisfy $\sum_{i=1}^{n} x_i = -a_{n-1}/a_n$ and, $\prod_{i=1}^{n} x_i = (-1)^n a_0/a_n$. [2]

Hence find the sum and products of the roots of $P = x^3 - 6x^2 + 11x - 6$. Show that $x = 1$ is a root. By writing $P = (x-1)Q$, where Q is a quadratic polynomial, find the other two roots. [3]

(ii) Show that the equation $(z+1)^n - e^{2\mathrm{i}n\theta}(z-1)^n = 0$ has roots $z = -\mathrm{i}\cot(\theta + m\pi/n)$, where m and n are integers. [3]

(iii) Using the results from (i) and (ii) above, show that

$$\prod_{r=1}^{n} \cot\left(\theta + \frac{m\pi}{n}\right) = \begin{cases} (-1)^{n/2} & \text{for } n \text{ even} \\ (-1)^{(n-1)/2}\cot n\theta & \text{for } n \text{ odd.} \end{cases}$$ [7]

(b) Draw the locus of the points $z = r\exp(\mathrm{i}\theta)$ with $r = |\sin(2\theta - \pi/3)|$ in an Argand diagram. For which angles θ is $|z|$ maximal? [5]

13. (2023)

2. Express $\sin\left\{i\ln(iz \pm \sqrt{1-z^2})\right\}$ in its simplest form.