B2
Symmetry and Relativity
Lecture 24

Outline

- Gauge (“internal”) transformations
- Global gauge symmetry
  - Klein-Gordon and Dirac equations
  - How to add “new physics”
- Local gauge symmetry

Gauge transformations

- Previous example: 4-vector potential

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \chi(x)$$

Klein-Gordon field equation

- From energy-momentum relationship

$$\begin{aligned} 0 &= P^\mu P_\mu + m^2 \\ &= -\partial^\mu \partial_\mu \phi + m^2 \phi \end{aligned} \qquad P_\mu \to i\partial_\mu$$

- Describes scalar (spin-0) fields
- Consider Lagrangian with complex field

Complex fields

Example: Klein-Gordon

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi^*)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^*\phi$$

$$\phi = u + iv$$

Decomposition into real-valued fields

$$0 = \delta f = \frac{\partial f}{\partial \phi}(\delta u + i\delta v) + \frac{\partial f}{\partial \phi^*}(\delta u - i\delta v)$$

Find extremum

$$0 = \frac{\partial f}{\partial \phi} + \frac{\partial f}{\partial \phi^*}$$

$$0 = \frac{\partial f}{\partial \phi} - \frac{\partial f}{\partial \phi^*}$$

Treat as two independent fields

$$0 = \frac{\partial f}{\partial \phi}$$

$$0 = \frac{\partial f}{\partial \phi^*}$$

Klein-Gordon equation

See Section 11.1.1 of old lecture notes

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi^*)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^*\phi$$

- Can treat φ and φ* as two independent fields

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{1}{2}m^2\phi^*$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\frac{1}{2}g^{\mu\nu}\partial_\nu \phi^*$$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = -\frac{1}{2}m^2\phi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = -\frac{1}{2}g^{\mu\nu}\partial_\nu \phi$$

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi}$$

$$0 = -\frac{1}{2}g^{\mu\nu}\partial_\mu \partial_\nu \phi^* + \frac{1}{2}m^2\phi^*$$

$$0 = -\partial_\mu \partial^\mu \phi^* + m^2\phi^*$$

$$0 = -\partial_\mu \partial^\mu \phi + m^2\phi$$

Two fields, same mass

A global gauge transformation

- Unitary transformation of complex fields
  - Doesn't change the Lagrangian → shouldn't change any physics
  - "Gauge transformation of the first kind" (Pauli)

$$\phi' = e^{i\lambda}\phi \qquad \delta\phi = i\lambda\phi$$

$$\phi^{*\prime} = e^{-i\lambda}\phi^* \qquad \delta\phi^* = -i\lambda\phi^*$$

- Noether general invariance:

$$\begin{aligned}
\delta\mathcal{L}[\phi, \phi^*, \phi', \phi^{*\prime}] &= \mathcal{L}[\phi + \delta\phi, \phi^* + \delta\phi^*] - \mathcal{L}[\phi, \phi^*] \\
&= -\frac{1}{2}(\partial_\mu(\phi^* - i\lambda\phi^*))(\partial^\mu(\phi + i\lambda\phi)) - \frac{1}{2}m^2(\phi^* - i\lambda\phi^*)(\phi + i\lambda\phi) \\
&= \mathcal{L}\cdot(1+\lambda^2) - \mathcal{L} \\
&= O(\lambda^2)\mathcal{L}
\end{aligned}$$

$$\Longrightarrow \quad K^\mu = O(\lambda^2)$$

A global gauge transformation

- Noether current:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\delta\phi^* - K^\mu$$

$$= -\frac{1}{2}(\partial^\mu \phi^*)(i\lambda\phi) - \frac{1}{2}(\partial^\mu \phi)(-i\lambda\phi^*) - O(\lambda^2)$$

$$= -\frac{i\lambda}{2}[(\partial^\mu \phi^*)\phi - \phi^*(\partial^\mu \phi)]$$

- More familiar form:

$$J_\mu = i\left(\frac{\partial \phi^*}{\partial x^\mu}\phi - \phi^*\frac{\partial \phi}{\partial x^\mu}\right)$$

- 2 fields, equal mass, opposite charge

Charge conservation

- Noether current conservation:

$$0 = \partial_\mu J^\mu = \frac{\partial J^0}{\partial t} + \nabla \cdot \mathbf{J}$$

$$\int_V d^3x \nabla \cdot \mathbf{J} = \int_S \mathbf{J} \cdot d\mathbf{a} = 0$$

- Integrate over all space

$$\int_V d^3x \frac{\partial J^0}{\partial t} = \frac{d}{dt} \int_V d^3x J^0 = \frac{dQ}{dt}$$

- Charge is conserved!

$$\frac{dQ}{dt} = 0$$

Dirac field equation

- Equation of motion of spin-1/2 particles

$$0 = (i\gamma^\mu \partial_\mu - m)\psi$$

4x4 “Dirac” matrices — 4-component state

- For our purposes, form of state and matrices not important – keep in mind “spinor indices”

$$0 = (i[\gamma^\mu]^a{}_b \partial_\mu - \delta^a_b m)\psi^b$$

Dirac field Lagrangian

- Lagrangian density to obtain Dirac equation

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

$$= \bar{\psi}_a (i[\gamma^\mu]^a{}_b \partial_\mu - \delta^a_b m)\psi^b$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = i\gamma^\mu \partial_\mu \psi - m\psi$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = -m\bar{\psi}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\bar{\psi}\gamma^\mu$$

Dirac current

- Global gauge transformation

$$\delta\psi = i\lambda\psi$$
$$\delta\bar{\psi} = -i\lambda\bar{\psi}$$

$$\begin{aligned} J^\mu &= \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\delta\bar{\psi}\right) - O(\lambda^2) \\ &= (i\bar{\psi}\gamma^\mu)(i\lambda\psi) \\ &= -\lambda(\bar{\psi}\gamma^\mu\psi) \end{aligned}$$

- 2 fields, equal mass, opposite charge
- Conserved current: $J^\mu = \bar{\psi}\gamma^\mu\psi$

Global gauge invariance

- We've applied a global U(1) phase transformation to complex fields
  - Klein-Gordon equation (bosons)
  - Dirac equation (spin-1/2)

➔ The Noether currents reflect particles which have the same mass, but opposite charge

  ➢ Anti-particles

Adding new physics

- We've seen how we "add" physics to a Lagrangian (density) by simply adding terms
- Common restrictions for "new physics":
  - Local: depend only on one spacetime point
  - Real-valued action: complex-valued actions tend to result in disappearing matter
  - Lagrangian depends on no higher than 2nd derivatives: higher orders tend to violate causality
  - Action reflects other symmetries, e.g., Lorentz invariance
- Lagrangians tend to be scalar invariants

Lagrangians in the wild

- Recipe for form-invariant equations of motion:
  - Form-invariant scalar Lagrangian (density)
  - Plug into form-invariant Euler-Lagrange

A lot of particle physics papers start off by specifying such a Lagrangian

2. Toy Model

2.1. Real Scalar Triplet

The possibility of extending the SM with a real $SU(2)_W$ triplet scalar has been extensively studied [21–30] since such extensions generally lead to suppressed contributions to electroweak precision observables (EWPO). The scalar Lagrangian for a toy model including all possible gauge invariant combinations of a Higgs doublet, H, and an $SU(2)_W$ triplet, Σ, given by

$$H = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \qquad \Sigma = \frac{1}{2}\begin{pmatrix} \eta^0 & \sqrt{2}\eta^+ \\ \sqrt{2}\eta^- & -\eta^0 \end{pmatrix}, \tag{1}$$

can be written as

Kinetic energies

$$\mathcal{L}_{\text{scalar}} = (D_\mu H)^\dagger (D^\mu H) + Tr(D_\mu \Sigma)^\dagger (D^\mu \Sigma) - V(H,\Sigma), \tag{2}$$

where

$$V(H,\Sigma) = -\mu^2 H^\dagger H + \lambda_0 (H^\dagger H)^2 + \frac{1}{2} M_\Sigma^2 Tr[\Sigma^2] + \frac{b_4}{4} Tr[\Sigma^2]^2 + a_1 H^\dagger \Sigma H + \frac{a_2}{2} H^\dagger H Tr[\Sigma^2], \tag{3}$$

is the scalar potential [28, 29], and the covariant derivatives are the standard $SU(2)_W \times U(1)_Y$, as in the SM.

The scalar potential can be minimized along the directions of the neutral components of both H and Σ, leading to two conditions:

Higgs mass and self-interactions

New particle mass and self-interactions

Higgs+new particle interactions

EC Leskow, TAW Martin, A de la Puente, arXiv:1409.3579v2, 2 Oct 2014

Lagrangians in the wild

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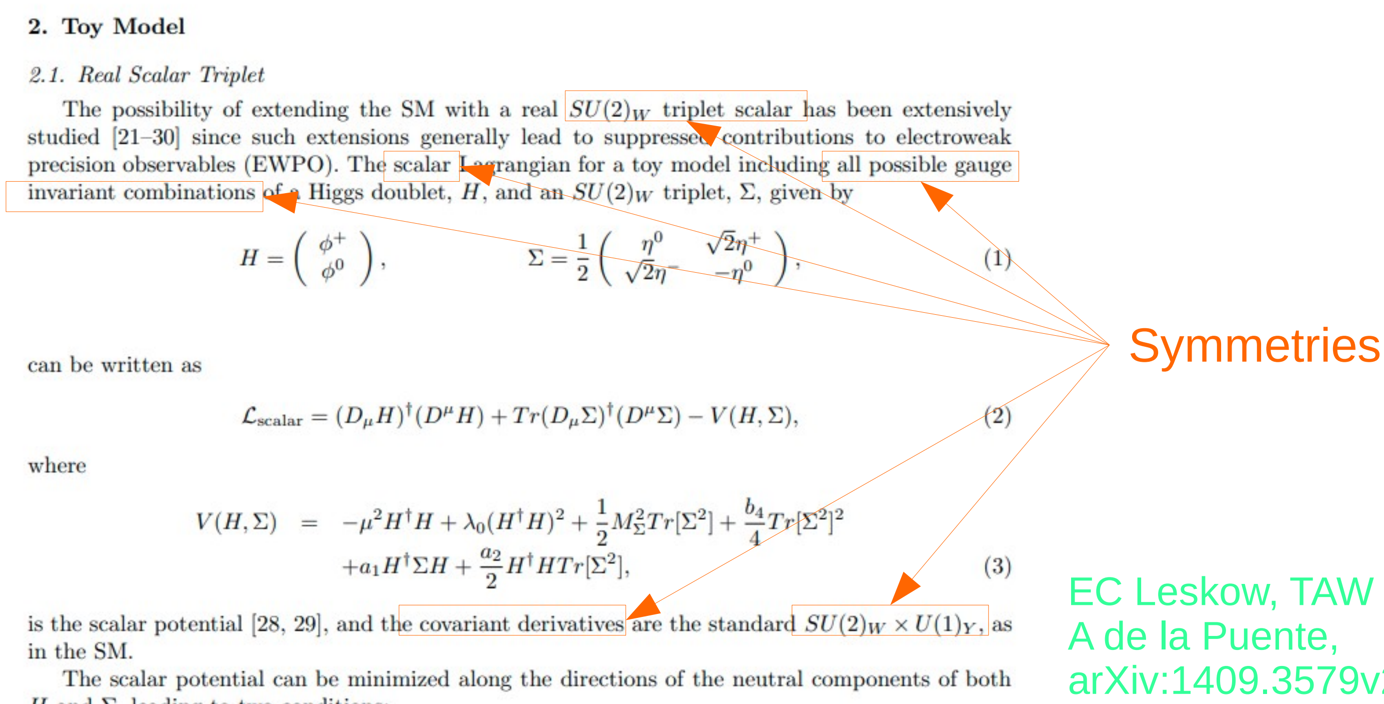

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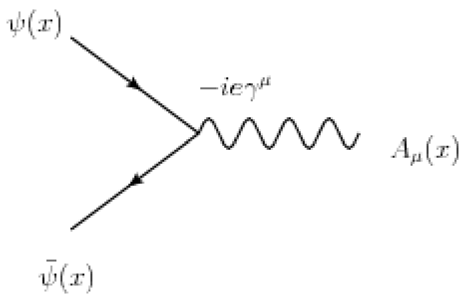
Adding matter-EM interactions

- Contract two 4-vector fields:

$$J^\mu = \bar{\psi}\gamma^\mu\psi \qquad A_\mu$$

Strength of coupling

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + e\bar{\psi}\gamma^\mu\psi A_\mu$$
$$= \bar{\psi}(\gamma^\mu(i\partial_\mu + eA_\mu) - m)\psi$$



$$P_\mu + eA_\mu$$

Canonical momentum with EM field

But now have a gauge invariance problem

Dirac current coupling

- Link gauge transformation with local phase of state

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \chi$$
$$\psi(x) \rightarrow \psi'(x) = e^{iq\alpha(x)}\psi(x)$$

U(1) gauge transformation

- Transform Dirac equation:

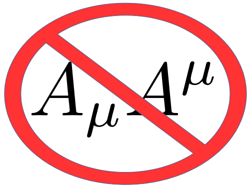
$$\begin{aligned}(i\gamma^\mu \partial_\mu + q\gamma^\mu A'_\mu - m)\psi' &= -q\gamma^\mu(\partial_\mu \alpha)e^{iq\alpha}\psi + ie^{iq\alpha}\gamma^\mu(\partial_\mu \psi) - me^{iq\alpha}\psi \\ &\quad + q\gamma^\mu A_\mu e^{iq\alpha}\psi + q\gamma^\mu(\partial_\mu \chi)e^{iq\alpha}\psi \\ &= e^{iq\alpha}(i\gamma^\mu \partial_\mu + q\gamma^\mu A_\mu - m)\psi \\ &\quad + q\gamma^\mu(\partial_\mu \chi - \partial_\mu \alpha)e^{iq\alpha}\psi\end{aligned}$$

✔ Gauge invariance restored!

Set $\chi=\alpha$

Local gauge invariance

- Turn argument around:
  Local U(1) gauge symmetry → require gauge field
  - Transformation of field hides gauge transformation
  - "Let there be U(1) gauge symmetry, and there was light"
  - Gauge field must be massless



- Can this be extended?
  - SU(n): state → complex n-tuplet
  - Evaluate change due to gauge transformation
  - Remove effect using a gauge field
  - Add gauge-invariant kinetic energy terms

Local gauge invariance

- n-tuplet:

$$\psi \rightarrow \psi' = e^{i\alpha_j T_j}\psi$$

nxn Hermitian matrices

$$\partial_\mu \psi \rightarrow \partial_\mu \psi' = e^{i\alpha_j T_j}[\partial_\mu \psi + iT_j(\partial_\mu \alpha_j)\psi]$$

Hide gauge transform with new fields (for each generator)

$$G^j_\mu \rightarrow {G^j_\mu}' = G^j_\mu - \frac{1}{g}\partial_\mu \alpha_j$$

New fields need kinetic energy terms

$$\mathcal{L} = \cdots - \frac{1}{4} G^j_{\mu\nu} G^{\mu\nu}_j$$

$$G^j_{\mu\nu} = \partial_\mu G^j_\nu - \partial_\nu G^j_\mu$$

Problem with n>1

- Higher-rank unitary groups are non-Abelian
  - Exponentiation more complicated
  - Gauge field transformation more complicated
- Example: SU(3)

$$\psi(x) \rightarrow \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix}$$

SU(3)

- Generators (Gell-Mann matrices)

$$\lambda_1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & \\ 0 & -1 & \\ & & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ 1 & & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & & -i \\ & 0 & \\ i & & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

SU(3)

- Lie algebra $$[T_a, T_b] = i f_{abc} T_c \qquad a, b, c = 1 \cdots 8$$

- Structure constants f completely antisymmetric in indices

$$f_{123} = 1 \qquad f_{458} = f_{678} = \frac{\sqrt{3}}{2}$$

$$f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}$$

Problem with n>1

- Gauge field transformation more complicated

$$G^j_\mu \rightarrow {G^j_\mu}' = G^j_\mu - \frac{1}{g}\partial_\mu \alpha_j - f_{jmn}\alpha_m G^n_\mu$$

$$D_\mu = \partial_\mu + i g_s T_j G^j_\mu$$

$$G^j_{\mu\nu} = \partial_\mu G^j_\nu - \partial_\nu G^j_\mu - g_s f_{jmn} G^m_\mu G^n_\nu$$

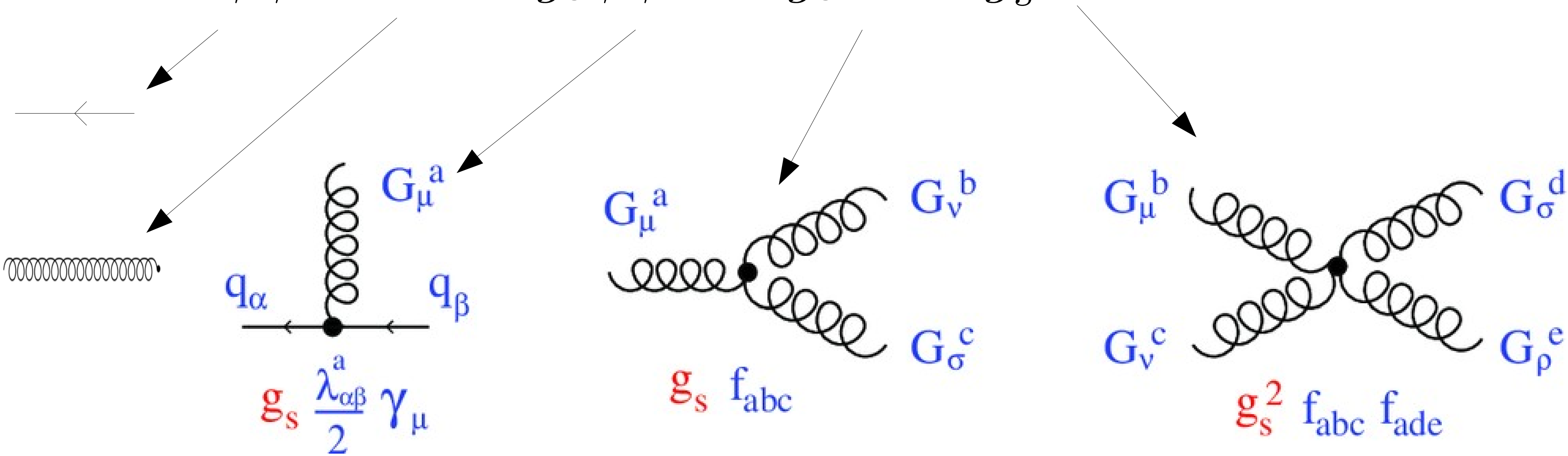
New stuff with structure constants

Problem with n>1

- QCD is the land of heroic calculation

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi - g_s(\bar{\psi}\gamma^{\mu}T_a\psi)G^a_{\mu} - \frac{1}{4}G^a_{\mu\nu}G_a^{\mu\nu}$$

$$\sim \bar{\psi}\psi + GG + g_s\bar{\psi}\psi G + g_s G^3 + g_s^2 G^4$$



SU(2)

- Weak force has 3 gauge bosons
  - SU(2) has 3 generators (Pauli spin matrices)

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- You've seen this algebra – it's for the rotation of a spin-1/2 field
  - Note: it's actually isospin – not connected to rotational symmetry
  - Introduce doublet field representation

$$\psi(x) \rightarrow \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$$

- You'd think this would be simple

SU(2)

$$W^a_\mu \rightarrow W^{a\prime}_\mu = W^a_\mu - \frac{1}{g_W}\partial_\mu \alpha_a - \epsilon_{abc}\alpha_b W^{\mu c}$$

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g_W \epsilon^{abc} W^\mu_b W^c_\nu$$

- Gauge invariance → only massless W,Z
  - Experimental reality alert: 80/90 GeV
- Also a chirality problem
  - Deep dive into Dirac, parity, etc.

➢ Lagrangian may have manifest symmetry, but symmetry could be broken in ground state (from CMP)

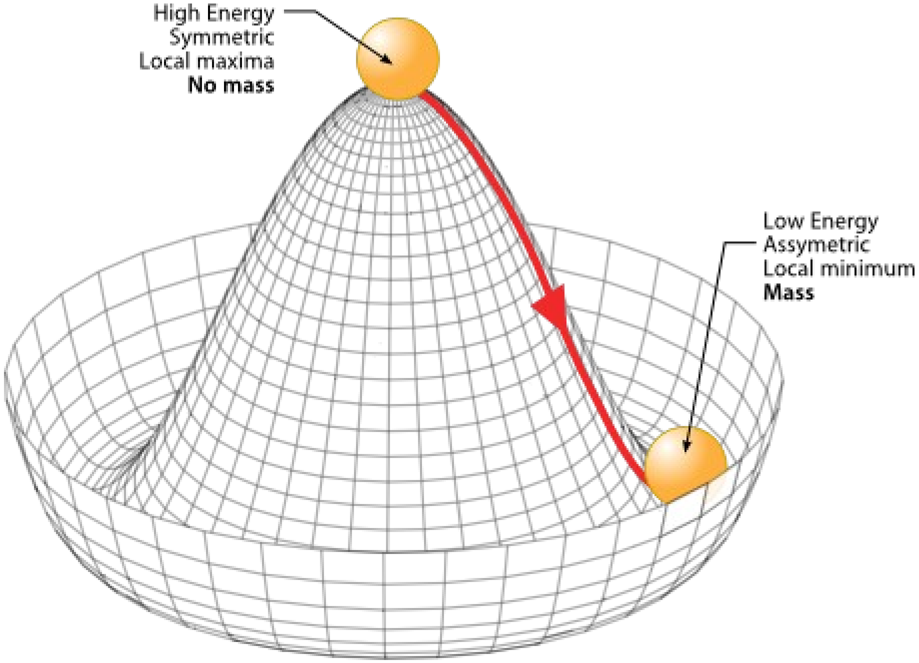
Spontaneous symmetry breaking

- Postulate a complex doublet field

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \psi_2(x) \end{pmatrix}$$

- Add potential

$$V(\phi) = \mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2$$



Standard Model Lagrangian

$$\mathcal{L} = -\frac{1}{4} W^a_{\mu\nu} W^{\mu\nu}_a - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} G^b_{\mu\nu} G_{\mu\nu b}$$

$$+ \bar{\psi} \gamma^\mu \left[i\partial_\mu - \frac{g_W}{2} \tau_a W^a_\mu - \frac{g'_W}{2} Y B_\mu - g_s T_b G^b_\mu \right] \psi$$

$$+ \left| \left(i\partial_\mu - \frac{g_W}{2} \tau_a W^a_\mu - \frac{g'_W}{2} Y B_\mu \right) \phi \right|^2 - V(\phi)$$

$$- G_1 \bar{\psi} \phi \psi + \cdots$$

W,Z,γ,g self-energy, kinetic energy (Note: Zγ mixing)

Fermion interactions with W,Z,γ,g

W,Z,γ interactions with Higgs

Symmetry breaking potential

Fermion masses via coupling to Higgs field

Recent lecture notes from Corfu Summer Institute 2021: JI Illana and AJ Cano, “Quantum field theory and the structure of the Standard Model”, https://arxiv.org/abs/2211.14636

Conclusion

- SM Lagrangian: manifestly invariant with respect to both gauge and Lorentz transformations
  - Pretty successful theory
  - Elegant structure
  - At least 26 free parameters

- That's it for B2 (+)
- Congratulations for sticking with it
- Feedback →