B2
Symmetry and Relativity
Lecture 7

SO(3) Lie algebra

- Lie algebra → another basis → representation space

Casimir operator

$$L^2 = L_1^2 + L_2^2 + L_3^2$$

$$L_\pm = L_1 \pm i L_2$$

$$L^2 |\ell m\rangle = \ell(\ell+1)|\ell m\rangle$$

New eigenvectors labelled with l,m

$$L_3 |\ell m\rangle = m|\ell m\rangle$$

$$L_\pm |\ell m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}|\ell, m\pm 1\rangle$$

$$\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$$

$$m = -\ell, -\ell+1, \ldots, \ell-1, \ell$$

Labels for new basis / representation space

Representation space

- Element of representation space: write amplitudes of each |*lm*> in a column

$$|\psi\rangle = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} a_{\ell m} |\ell m\rangle$$

$$\Psi = \begin{pmatrix} a_{00} \\ a_{11} \\ a_{10} \\ a_{1-1} \\ a_{22} \\ a_{21} \\ a_{20} \\ a_{2-1} \\ a_{2-2} \\ \vdots \end{pmatrix}$$

Invariant subspaces

- We went from a 3-vector to one with infinite dimensions – how is that simpler?
- Rotation matrix representation → block diagonal

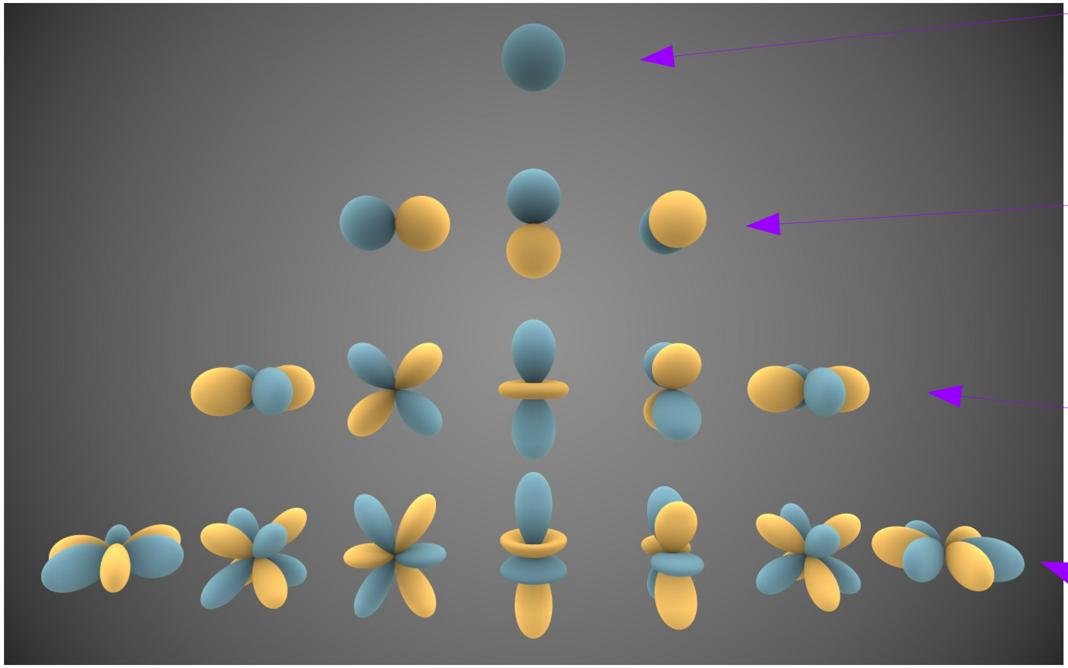
Invariant subspaces

$$\begin{pmatrix} D^{00}_{00} & & & & & & & \\ & D^{11}_{11} & D^{10}_{11} & D^{1-1}_{11} & & & & \\ & D^{11}_{10} & D^{10}_{10} & D^{1-1}_{10} & & & & \\ & D^{11}_{1-1} & D^{10}_{1-1} & D^{1-1}_{1-1} & & & & \\ & & & & D^{22}_{22} & D^{21}_{22} & \cdots \\ & & & & D^{22}_{21} & D^{21}_{21} & \cdots \\ & & & & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_{00} \\ a_{11} \\ a_{10} \\ a_{1-1} \\ a_{22} \\ a_{21} \\ \vdots \end{pmatrix}$$

Reduced representations of the rotation group

Spherical harmonics basis

- Invariant subspaces of rotations ("rotate in the same way")



L=0: invariant wrt rotations (trivial representation)

L=1: need full 2π rotation; representation isomorphic to full 3D rotations

L=2: same after π rotation

L=3: same after 2π/3 rotation

But wait, we forgot something...

$$L^2 = L_1^2 + L_2^2 + L_3^2$$

$$L_\pm = L_1 \pm i L_2$$

$$L^2 |\ell m\rangle = \ell(\ell+1)|\ell m\rangle$$

$$L_3 |\ell m\rangle = m|\ell m\rangle$$

$$L_\pm |\ell m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}|\ell, m\pm 1\rangle$$

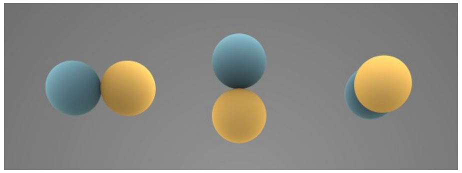
$$\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$$

$$m = -\ell, -\ell+1, \ldots, \ell-1, \ell$$

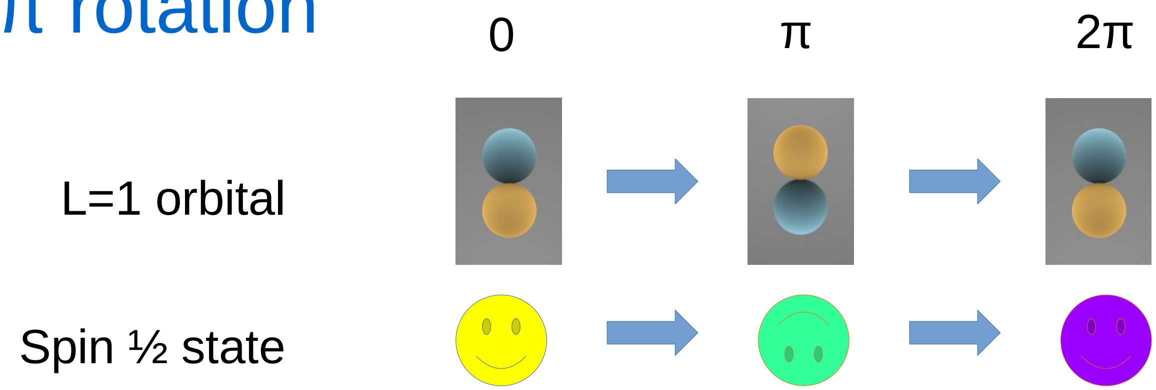
We've seen these in action ($\ell = 0, 1, 2$)

What about these? ($\ell = \frac{1}{2}, \frac{3}{2}$)

“Intrinsic” angular momentum

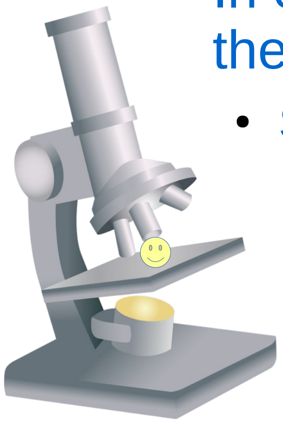


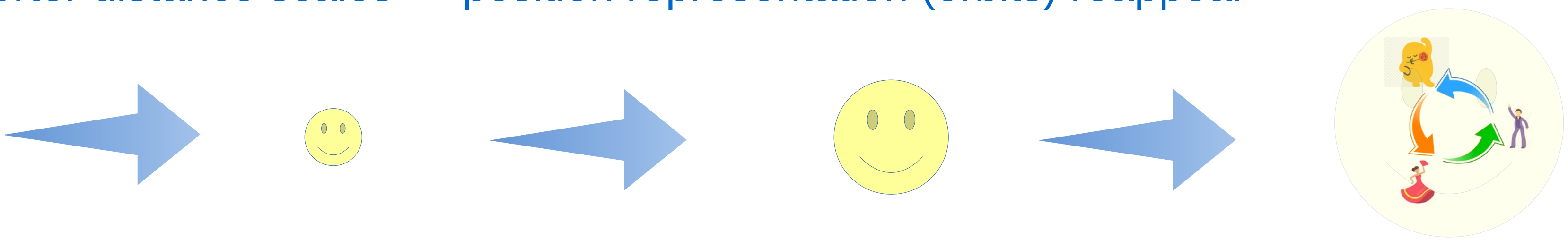
- Consider an electron in an atomic orbital
- To rotate in space, we have to rotate the orbital
  - But we also have to “rotate the electron”
  - In spite of lack of a position representation, the spin state still rotates, just with 4π rule
- Example: 2π rotation



“Intrinsic” angular momentum

- Total angular momentum of an electron orbital is J=L+S, not just L
  - You’ll see this atomic physics, possibly in B4 as well
  - Can also see this with Noether current if you apply infinitesimal rotation to the Dirac Lagrangian
  - In spite of being “intrinsic” and without position representation, electron spin is actually a property connected with spatial rotations
  - In composite particles, the intrinsic spin is the total angular momentum of the constituents
    - Shorter distance scales → position representation (orbits) reappear

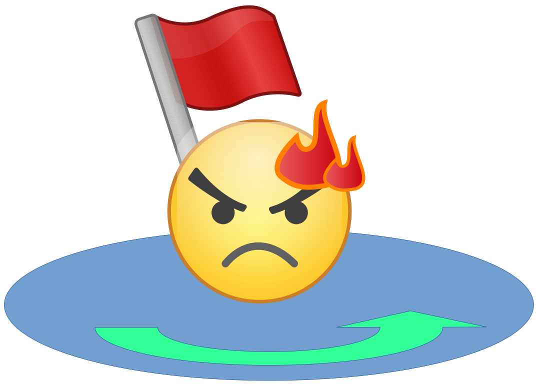




  - But it appears that fundamental fermions still have an intrinsic spin

Final caution

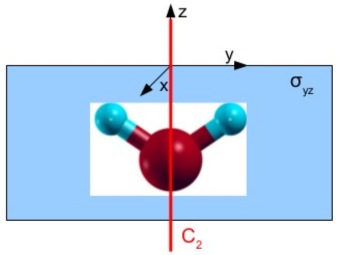
- Some quantum numbers only "look like" spin in that they have SU(2) symmetry
  - e.g., isospin: protons don't turn into neutrons when you rotate them



Other applications

- B2 emphasizes continuous (Lie) groups
  - Rotations
  - Lorentz group
- Discrete groups important in CMP, chemistry
  - Symmetries → molecular vibration modes
  - Degenerate perturbation theory
- Rubik's Cube

Symmetry group: C_{2v} (water)



Symmetry elements:

E: identity
C_2: rotation by π
σ_{yz}: mirror plane (yz)
σ_{xz}: mirror plane (xz)

Multiplication table C_{2v}:

C_{2v}	E	C_2	σ_{yz}	σ_{xz}
E	E	C_2	σ_{yz}	σ_{xz}
C_2	C_2	E	σ_{xz}	σ_{yz}
σ_{yz}	σ_{yz}	σ_{xz}	E	C_2
σ_{xz}	σ_{xz}	σ_{yz}	C_2	E

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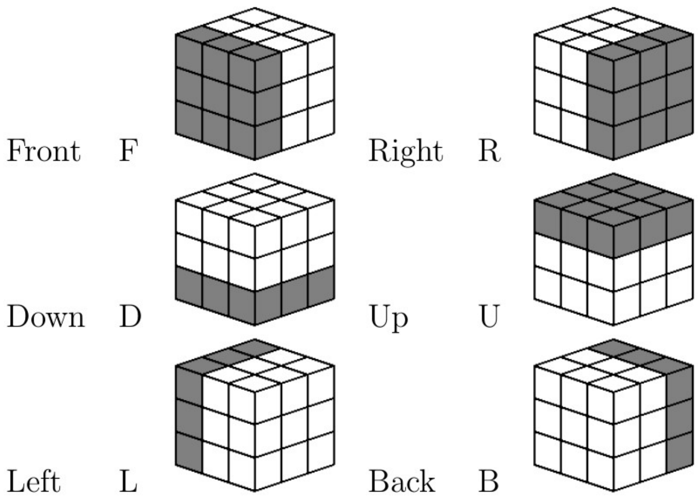
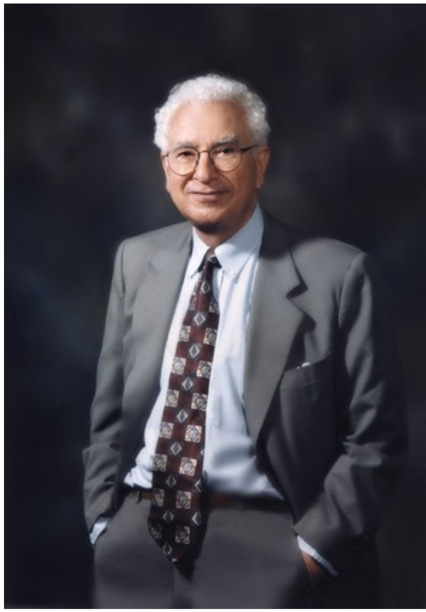
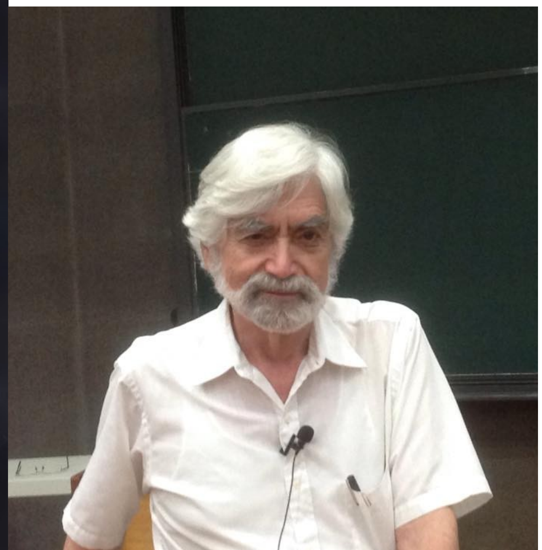
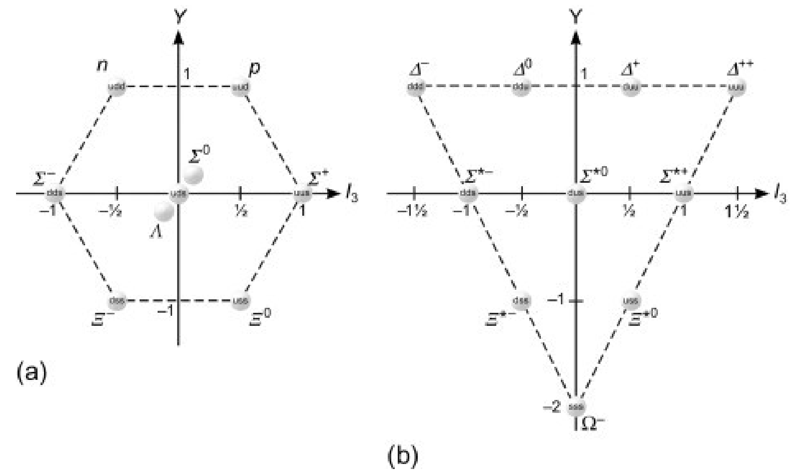


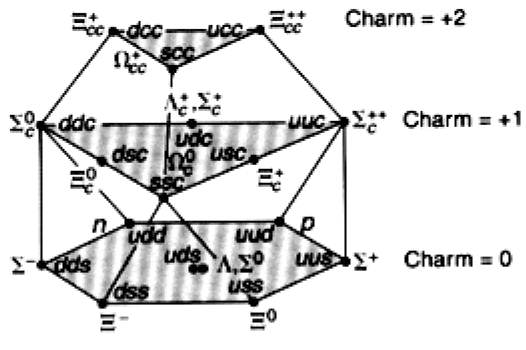
Generators	Size	Factorization
U	4	2^2
U, RR	14400	$2^6 \cdot 3^2 \cdot 5^2$
U, R	73483200	$2^6 \cdot 3^8 \cdot 5^2$
RRLL, UUDD, FFBB	8	2^3
Rl, Ud, Fb	768	$2^8 \cdot 3$
RL, UD, FB	6144	$2^{11} \cdot 3$
FF, RR	12	$2 \cdot 3^2$
FF, RR, LL	96	$2^5 \cdot 3$
FF, BB, RR, LL, UU	663552	$2^{13} \cdot 3^4$
LLUU	6	$2 \cdot 3$
LLUU, RRUU	48	$2^4 \cdot 3$
LLUU, FFUU, RRUU	82944	$2^{10} \cdot 3^4$
LLUU, FFUU, RRUU, BBUU	331776	$2^{12} \cdot 3^4$
LUlu, RUru	486	$2 \cdot 3^5$

Quark model

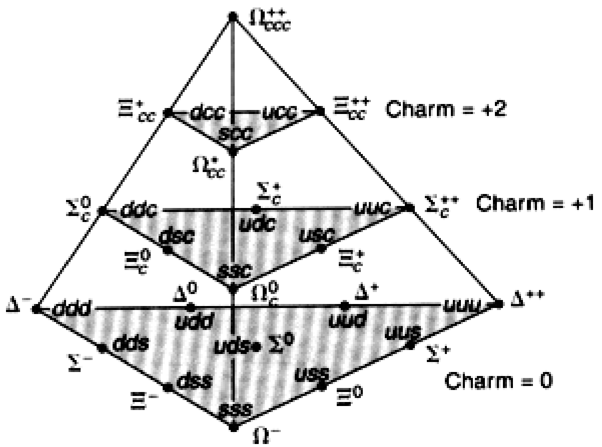



M Gell-Mann and G Zweig





$\frac{1}{2}^+$ Baryons



$\frac{3}{2}^+$ Baryons

Gell-Mann on quarks, 1967

- “Even if there are light real quarks, and the threshold comes from a very high barrier, the idea that mesons and baryons are made primarily of quarks is difficult to believe, since we know that, in the sense of dispersion theory, they are mostly, if not entirely, made up out of one another.”
- “The probability that a meson consists of a real quark pair rather than two mesons or a baryon and antibaryon must be quite small. Thus it seems to me that whether or not real quarks exist, the q and qbar we have been talking about are mathematical; in particular, I would guess that they are mathematical entities that arise when we construct representations of current algebra…”
- “Group theory is a useful technique, but it is no substitute for physics.” (H Georgi, *Lie Algebras in Particle Physics*)