Notes on Rates and Cross Sections

Definitions

Beam flux:

$$\Phi = \rho v$$

where ρ is the number density of the beam and v is the velocity.

Cross section:

$$\sigma = \frac{wV}{v}$$

where w is the rate, V is the normalization volume, and v is the relative velocity of the reaction particles (more strictly speaking, the incoming flux of the two individual particles).

Reaction rate is $\Phi\sigma_{total}$.

For particles incident on a target, reverse the roles such that the incident particles are the "target" and the target is the "beam". Then the reaction rate is $\rho_{target}v_{beam}\sigma_{total}$, and the mean time before an interaction is $\tau = (\rho_{target}v_{beam}\sigma_{total})^{-1}$. The mean free path is therefore $\ell = v_{beam}\tau = (\rho_{target}\sigma_{total})^{-1}$.

Rate and width:

$$w_i = \frac{\Gamma_i}{\hbar}$$

Total width and lifetime:

$$\Gamma = \sum_i \Gamma_i = \tau^{-1}$$

Fermi's Golden Rule

$$w = \frac{2\pi}{\hbar}|\mathcal{M}|^2\frac{dN}{dE}$$

where $\mathcal{M}$ is the matrix element $\langle f|H|i\rangle$, linking the initial and final state through an interaction Hamiltonian; and $\frac{dN}{dE}$ is the final state density. N is the number of final states with energy $\leq E$, and E is the total energy available to the final state.

Wavefunctions

In many nuclear physics applications, we approximate light particles, such as the e and ν, with plane waves (which is fine far away from the interaction).

$$\psi_{e,\nu} = \frac{1}{\sqrt{V}}e^{-i\vec{k}\cdot\vec{x}}$$

Again, V is the normalization factor such that, for instance,

$$\langle e|e\rangle = \int_V \psi_e^*\psi_e d^3x = \int_V d^3x = 1$$

The heavy particles, in this case the n and p, are considered to have largely localized wavefunctions. We don't specify their form, because all we need to know is that

$$\langle n|n\rangle \approx \int_V \psi_n^* \psi_n d^3x = 1$$

The approximation is due to the fact that in principle we'd integrate over all space in calculating the norm of the wavefunction (remember that this is different from plane waves, which don't have a well-defined integral over all space). If, however, the wavefunction is mostly localized, we just define V to envelope it. We can now integrate over V for both light and heavy wavefunctions.

The wavefunction of a multi-particle state have multiple coordinates, one for each constituent, so for example if pe is the final state, the wavefunction would be

$$|pe\rangle \approx \psi_p(\vec{x}_p)\frac{1}{\sqrt{V}}e^{-i\vec{k}_\nu\cdot\vec{x}_\nu}$$

Two coordinates are required, since two particles are really being described, and they aren't being constrained to any interrelationship (the constraint of the "contact interaction" happens through the interaction Hamiltonian, not the wavefunction). The norm of the two-particle wavefunction would then be calculated as follows:

$$\langle pe|pe\rangle = \int_V d^3x_p d^3x_e \psi_p^*(\vec{x}_p)\psi_e^*(\vec{x}_e)\psi_e(\vec{x}_e)\psi_p(\vec{x}_p)$$

Matrix element

For Fermi's theory of β decay, we postulate a contact interaction of four particles. This interaction would be characterized by a coupling strength, G_F. The contact is expressed through a delta function which joins the constituent wavefunctions together:

$$H = G_F\delta^3(\vec{x}-\vec{y})$$

where the coordinates are those of the interacting constituents.

The form of the Fermi process, without antiparticles, is $n\nu \to pe$. Note that the matrix element is the same regardless of whether it is describing the antiparticle process, $\overline{p}\overline{\nu} \to \overline{n}e^+$, or a decay such as $n \to pe\overline{\nu}$, since we take antiparticle wavefunctions to be the complex conjugates of normal particle wavefunctions.

Therefore, for the Fermi β process,

$$\begin{aligned}
\mathcal{M} &= \langle pe|H|n\nu\rangle \\
&= \int_V d^3x d^3y \psi_p^*(x)\psi_e^*(y) H \psi_n(x)\psi_\nu(y) \\
&\approx \int_V d^3x d^3y \psi_p^*(x)\psi_e^*(y) G_F \delta^3(x-y)\psi_n^(x)\psi_\nu(y) \\
&= G_F \int_V d^3x \psi_p^*(x)\psi_e^*(x)\psi_n(x)\psi_\nu(x) \\
&\approx G_F \int_V \psi_p^*(x)\frac{1}{V}e^{-i(k_\nu-k_e)\cdot x}\psi_n(x)
\end{aligned}$$

At this point, we note that for the typical low energies of nuclear β decay, the e and ν plane waves will have long wavelengths compared to the localized nuclear wavefunctions. We therefore use the approximation

$$e^{-(k_\nu - k_e)\cdot x} = 1 - i(k_\nu - k_e)\cdot x + \cdots$$

and take just the leading term (i.e., 1):

$$\mathcal{M} \approx \frac{G_F}{V}\int_V d^3x \psi_p^*(x)\psi_n(x)$$

If we then note that the proton and neutron remain essentially stationary, and that the only real difference between them is their Coulomb energy, then

$$\psi_p \approx \psi_n$$

which then implies that

$$\begin{aligned}\mathcal{M} &\approx \frac{G_F}{V}\int_V d^3x \psi_n^*(x)\psi_n(x) \\ &\approx \frac{G_F}{V}\end{aligned}$$

which is the familiar result.

Final state density

For a single free particle, we use the Fermi gas model of plane wave states, so the number of final states with momenta less than magnitude P is simply

$$N(P) = \frac{\frac{4}{3}\pi P^3}{(2\pi\hbar)^3/V}$$

where the numerator is the volume in P space, and the denominator is the volume (in P space) occupied by one state. V is the normalization volume of the plane waves.

Note that the above formula gives N as a function of P, not E as required in Fermi's Golden Rule. You will have to find the relationship between E and P—or more specifically (and usually easier), the relationship between the differential elements dE and dP.

In the approximation that the nucleons remain stationary, they do not contribute to the tally of final states (*i.e.*, they only have one state).

Beta decay

In the case of β decay, $n \to pe\overline{\nu}$, we need to have two plane waves in the final state, and this must be reflected in the final state density (the matrix element remains unchanged, of course). The total final state, however, is simply the direct product of electron and antineutrino states, so the number of states is

$$N(P_e, P_\nu) = \frac{16\pi^2 P_e^3 P_\nu^3 V^2}{9(2\pi\hbar)^6}$$

Typically, one calculates the energy spectrum of the electron, so we are really calculating

$$\frac{dw}{dP_e} = \frac{2\pi}{\hbar}|\mathcal{M}|^2 \frac{d^2N}{dP_e dE}$$

for a given (fixed) P_e. Because of this, along with the relationship

$$E = E_e + E_\nu$$

where E is the total final state energy and we are neglecting the proton energy (since it's stationary and consequently mostly constant), we find that

$$dE = dE_\nu = c dP_\nu$$

Therefore

$$\frac{d^2N}{dP_e dE} = \frac{16\pi^2 P_e^2 P_\nu^2 V^2}{(2\pi\hbar)^6 c}$$

which then gives the result

$$\frac{dw}{dP_e} = \frac{G_F^2 P_e^2 P_\nu^2}{2\pi^3 \hbar^7 c}$$