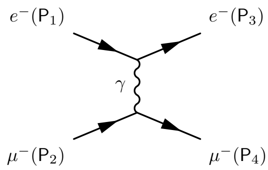
Feynman diagrams

1 Aim of the game

We are now getting ambitious and want to calculate the probabilities for *relativistic* scattering processes. To do so we need to find out the Lorentz-invariant scattering **amplitude** $\mathcal{M}_{fi}$ which takes us from an initial state $|\Psi_i\rangle$ containing some particles with well defined momenta to a final state $|\Psi_f\rangle$ containing (often different) particles also with well defined momenta.

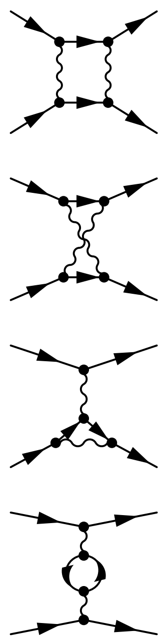


Simplest Feynman diagram for electron–muon elastic scattering.

We make use of a graphical technique popularised by Richard Feynman[1]. Each graph – known as a **Feynman Diagram** – represents a contribution to $\mathcal{M}_{fi}$. This means that each diagram actually represents a **complex number** (more generally function of the external momenta). The diagrams give a pictorial way to represent the contributions to the amplitude.

Diagrams consist of **lines** representing particles and **vertices** where particles are created or annihilated. I will place the incoming state on the LHS and the outgoing state on the RHS, so that we have an incoming electron with four-momentum P_1 and an incoming muon with four-momentum P_2. The electron and muon exchange a photon (the electromagnetic force carrier) and in the final state we have an electron with momentum P_3 and a muon with momentum P_4.

Since the diagrams represent transitions between well-defined states in 4-momentum they already include the contributions from **all possible paths in both time and space** through which the intermediate particles may have passed. So for the diagram at the top of the page it is *not* meaningful to ask "did the muon *emit* the photon or *absorb* it?" – both processes (photon going 'up' and 'down') are already summed in the diagram.



Some of the more complicated diagrams for electron–muon scattering.

2 Rules for calculating diagrams

It turns out that are simple rules for calculating the complex number represented by each diagram. These are called the **Feynman rules**. With a bit more math under our belts we could derive these rules from the Lagrangian density for any quantum field theory, but in this course we will simply quote the rules relevant for the Standard Model.

[1]American physicist (1918-1988).

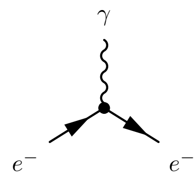
2.1 Vertices

Vertices are places where particles are created or annihilated. In the case of the electromagetic interaction there is only **one** basic vertex which couples a photon to a charged particle with strength proportional to its charge.

To calculate the contribution to $\mathcal{M}_{fi}$, for each vertex we associate a **vertex factor**. For interactions of photons with electrons the vertex factor is of size $\boxed{-g_e}$ where g_e is a dimensionless charge or **coupling constant**.[2] The coupling constant is a number which represents the strength of the interaction between the particle and the force carrier at that vertex. For the electromagnetic force the coupling strength must be proportional to the electric charge of the particle. So for the electromagnetic vertex we need a dimensionless quantity proportional to the charge. Recall that for the electromagnetic fine structure constant:

$$\alpha_{\mathsf{EM}} \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}.$$

is dimensionless.



The electromagnetic vertex. The vertex factor is $-g_e$.

It turns out to be convenient to choose g_e such that

$$\alpha_{\mathsf{EM}} = \frac{g_e^2}{4\pi}.$$

In other words the coupling constant g_e is a dimensionless measure of the electronic charge e. The size of the coupling between the photon and the electron is

$$-g_e = -\sqrt{4\pi\alpha_{\mathsf{EM}}}.$$

The electromagnetic vertex factor for any other charged particle f with charge $\mathcal{Q}_f$ times that of the proton is then

$$\boxed{g_e\mathcal{Q}_f}$$

So, for example, the electromagnetic vertex factor for an electron is of size $-g_e$ while for the up quark it is of size $+\frac{2}{3}g_e$.

2.2 Propagators

For each internal line – that is each **virtual particle** – we associate a **propagator factor**. The propagator tells us about the contribution to the amplitude from a particle travelling through space and time (integrated over all space and time). For a particle with no spin, the **Feynman propagator** is a factor

$$\boxed{\frac{1}{\mathsf{Q}\cdot\mathsf{Q} - m^2}}$$

[2]For students who have taken BII: Here we are simplifying the situation by ignoring the spin of the electron. If spin is included the vertex factor becomes $-g_e$ times a matrix, in fact a Dirac gamma matrix, allowing the spin direction of the electron as represented by a spinor wavefunction to change in response to interaction with the photon. In this course we ignore this complication and for the purpose of Feynman diagrams treat all spin $\frac{1}{2}$ fermions, such as electrons, muons, or quarks, as spinless.

where $\mathsf{Q} \cdot \mathsf{Q} = E^2 - \mathbf{q} \cdot \mathbf{q}$ is the four-momentum-squared of the internal virtual particle[3].

These intermediate particles are called **virtual particles**. They do **not** satisfy the usual relativistic energy-momentum constraint, i.e.

$$\boxed{\mathsf{Q} \cdot \mathsf{Q} \neq m^2}$$

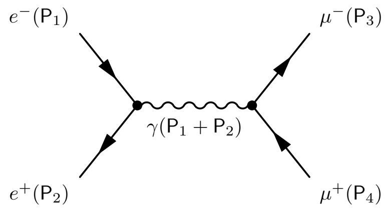
for these short-lived virtual particles. Such particles are sometimes said to be **off their mass-shell**.

If this inequality worries you, it might help you if you consider that their energy and momentum cannot be measured without hitting something against them. So you will never "see" off-mass-shell particles, you will only see the effect they have on other objects they interact with.

External particles in Feynman diagrams *always* satisfy the usual relativistic energy-momentum constraint $E^2 - p^2 = m^2$, and for these particles we **do not** include a propagator factor. The external lines are included in the diagram purely to show which kinds of particles are in the initial and final states.

2.2.1 Propagator example

Let's have a look at the annihilation process $e^+e^- \to \mu^+\mu^-$ via a virtual photon. For the moment we will ignore the spin of all the particles, so that we can concentrate on the vertex factors and propagators.



We can calculate the photon's energy-momentum four-vector from that of the electron and the positron. Four momentum is conserved **at each vertex** so the photon four-vector is $\mathsf{Q}_\gamma = \mathsf{P}_{e^+} + \mathsf{P}_{e^-}$. Let's calculate the virtual photon momentum components in the zero momentum frame:

$$\mathsf{P}_1 = (E, \mathbf{p}), \quad \mathsf{P}_2 = (E, -\mathbf{p}). \tag{1}$$

Conserving energy and momentum at the first vertex, the energy-momentum vector of the internal photon is

$$\mathsf{Q}_\gamma = (2E, \mathbf{0}).$$

[3] This propagator is the relativistic equivalent of the non-relativistic version of the Lippmann-Schwinger propagator $(E - H + i\epsilon)^{-1}$ that we found in non-relativistic scattering theory. Why are the forms different? Non-relativistic propagators are Greens functions for integration over all space. Relativistic propagators by contrast are Greens functions for integrations over both space *and* time.

So this *virtual* photon has more energy than momentum.

The propagator factor for the photon in this example is then

$$\frac{1}{(2E)^2 - m_\gamma^2} = \frac{1}{4E^2}.$$

The contribution to $\mathcal{M}_{fi}$ from this diagram is obtained my multiplying this propagator by two vectex factors each of size g_e. The modulus-squared of the matrix element is then

$$|\mathcal{M}_{fi}|^2 = \left|\frac{g_e^2}{4E_e^2}\right|^2.$$

If we pop this $|\mathcal{M}_{fi}|^2$ into Fermi's Golden Rule with the appropriate density of states

$$\frac{dN}{dp_\mu} = \frac{p_\mu^2 d\Omega}{(2\pi)^3},$$

and divide by an incoming flux factor $2v_e$, then we should get the differential cross section

$$d\sigma = \frac{1}{v_e}\ 2\pi\ |\mathcal{M}_{fi}|^2\ \frac{p_\mu^2}{(2\pi)^3}\frac{dp_\mu}{d(E_f)}\ d\Omega.$$

A little care is necessary in evaluating this. Momentum conservation means that only *one* of the outgoing particles is free to contribute to the density of states. The muon energy in the ZMF is $E_\mu = \frac{1}{2}E_0$, so

$$\frac{dp}{dE_0} = \frac{1}{2}\frac{dp_\mu}{dE_\mu},$$

where p_μ and E_μ are the momentum and energy of one of the outgoing muons. Since those muons are real they satisfy

$$p_\mu^2 + m_\mu^2 = E_\mu^2$$

and so taking a derivative $p_\mu\, dp_\mu = E_\mu\, dE_\mu$. Sticking this into the F.G.R. we'll want

$$\frac{dp_\mu}{dE_0} = \frac{1}{2}\frac{dp_\mu}{dE_\mu} = \frac{1}{2}\frac{E_\mu}{p_\mu} = \frac{1}{2}\frac{1}{v_\mu}.$$

We then integrate over all possible outgoing angles to gain a factor of 4π and note that $g^2/4\pi = \alpha$, and that $\frac{p_\mu}{E_\mu} = v_\mu$. Gathering all the parts together we find we have a total cross-section for $e^+ + e^- \to \mu^+ + \mu^-$ of [4]

$$\sigma = \pi\frac{\alpha^2}{s}\left|\frac{v_\mu}{v_e}\right|$$

where $s = (2E)^2$ is the center-of-mass energy squared.

A quick check of dimensions is in order. The dimensions of s are $[E]^2$, while those of σ should be $[L]^2 = [E]^{-2}$. The structure constant α and the velocities v are dimensionless so all is well.

[4] Neglecting spin and relativistic normalization and flux factor issues – see 'caveats'.

2.2.2 Second propagator example

In the previous example the virtual photon's four-momentum vector $(E, \mathbf{0})$ was time-like.

In the electron–muon scattering case $e^- + \mu^- \to e^- + \mu^-$ introduced at the start of this handout it turns out that this other virtual photon carries momentum and not energy, so the propagator is space-like.

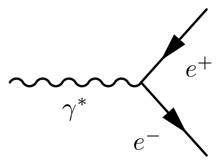
To see this, transform to in the zero-momentum frame. In the ZMF the electron is kicked out with the same energy as it came in with, so it has received no energy from the photon, and conserving energy at the vertex $E_\gamma = 0$. The direction of the electron momentum vector *has* changed so it *has* received momentum from the photon, $\mathbf{p}_\gamma \neq 0$. Therefore $E_\gamma^2 - |\mathbf{p}|_\gamma^2 < 0$ and the propagator is space-like.

2.3 Anti-particles

An anti-particle has the same mass as its corresponding particle cousin, but his charge is the opposite to that of the particle.[5]

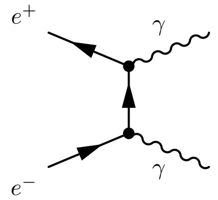
Creating anti-particles A photon travelling along in free space cannot produce an e^+ and an e^-. Let's remind ourselves why not. For a massless photon obeying the usual Lorentz invariant condition on its energy-momentum four-vector $\mathsf{Q}_\gamma \cdot \mathsf{Q}_\gamma = 0$ this reaction does *not* conserve energy and momentum. So it can't happen for a *real* photon.

But what about for *virtual* photons? The photon is couples to charge that's what our vertex is telling us. And charge is indeed conserved at the vertex – so we don't break that law. No other laws are violated, so it seems that if somehow we could shoe-horn *extra energy* into the photon while at the same time giving it *no extra momentum* (so that $\mathsf{Q}_\gamma \cdot \mathsf{Q}_\gamma > 2m_e$) – then we might be able to have that photon "decay" to an electron and positron. This situation – having a time-like photon with energy greater than its momentum – was exaction what we found for our time-like photon in §2.2.1, which was why that photon was able to 'decay' into a muon and an anti-muon.



Creation of an electron-positron pair from a (virtual) photon.

Anti-particle lines When we draw anti-particle lines (like the examples to the right) we put arrows on the particles to indicate the direction of *particle*-flow. An electron has an arrow pointing to the right as it is a *particle*. A positron which is an *anti*-particle has an arrow pointing to the left.



Electron–positron annihilation to form two photons.

[5] In fact if the particle is charged under under *more* than one force then *all* the charges have to be reversed for the anti-particle. For example an anti-quark, which has electromagnetic, strong and weak charges will have the opposite value of each of those compared to the corresponding quark.

Internal electron/positron lines — such as that in the lower diagram to the right — could have *either* a electron moving from the lower to the upper vertex *or* a positron moving from the upper to the lower vertex. We don't and can't know which. The question is not phrased as an observable so in quantum mechanics we simply ain't privy to that sort of information. Both possibilities are included in the internal electron/positron propagator.

3 Leading order diagrams

In principle to calculate $|\mathcal{M}_{fi}|$ we are supposed to draw and calculate *all* of the infinite number of possible Feynman diagrams. Then we'd have to add up all those complex numebrs and mod-squark the answer.

However the fact that the fine structure constant is small ($\alpha_{\mathsf{EM}} \ll 1$) explains we can get away with just summing the simplest diagram(s) when calculating $\mathcal{M}_{fi}$.

The simplest "tree level" scattering diagram has two vertices so contains two factors of e. The diagrams with the loops contain four vertices and hence four factors of e. Since $g_e^2/4\pi = \alpha_{\mathsf{EM}} \ll 1$, we can see that the more complicated diagrams with with more vertices will (all other things being equal) contribute much less to the amplitude than the simplest ones.

$$\begin{aligned} \alpha_{\mathsf{EM}} &= \frac{g_e^2}{4\pi} &\approx \frac{1}{137} \\ \alpha_{\mathsf{Weak}} &= \frac{g_W^2}{4\pi} &\approx \frac{1}{29} \\ \alpha_{\mathsf{Strong}} &= \frac{g_s^2}{4\pi} &\sim 1 \end{aligned}$$

Coupling strengths of the forces.

The other forces also have coupling constants, which have different strengths. The **strong force** is so-called because it has a fine structure constant close to 1 which is about a hundred times larger than α_{EM}. In fact the **weak force** actually has a larger coupling constant $\approx 1/29$ than the electromagnetic force $\approx 1/137$. The reason why this force *appears* weak is because the force is transmitted by very heavy particles (the W and Z bosons) so it is very short-range.

4 Key concepts

- Feynman (momentum-space) diagrams help us calculate relativistic, Lorentz-invariant scattering amplitudes.
- Vertices are associated with dimensionless **coupling constants** g with vertex factors that depend on the charge $\mathcal{Q}g$
- Internal lines are integrated over **all time and space** so include all internal time orderings.
- Intermediate/virtual/off-mass-shell particles have $\boxed{\mathsf{Q}^2 \neq m^2}$ and have propagators $\boxed{\frac{1}{\mathsf{Q}^2 - m^2}}$.
- For fermions, arrows show the sense of particle flow. Anti-particles have arrows pointing the "wrong way".

Caveats

- Sometimes you will see books define a propagator with a plus sign on the bottom line: $1/(q^2 + m^2)$. One of two things is going on. *Either* (a) q^2 is their notation for a four-vector squared, but they have defined the metric $(-, +, +, +)$ in the opposite sense to us so that $q^2 = -m^2$ is their condition for being on-mass-shell *or* (b) q^2 is acually intended to mean the three-momentum squared. A bit of context may be necessary, but regardless of the convention used the propagator should diverge in the case when the virtual particle approaches its mass-shell.

- We have not attempted to consider what the effects of spins would be. This is done in the fourth year after the introduction of the Dirac equation – the relativistic wave equation for spin-half particles. The full treatment is done in e.g. Griffiths Chs. 6 & 7.

- We have played fast and loose with phase factors (at vertices and overall phase factors). You can see that this will not be a problem so long as only one diagram is contributing to $\mathcal{M}_{fi}$, but clearly relative phases become important when adding diagrams together.

- Extra rules are needed for diagrams containing loops, because the momenta in the loops are not fully constrained. In fact one must integrate over all possible momenta for such diagrams. We will not need to consider such diagrams in this course.

- The normalization of the incoming and ougtoing states needs to be considered more carefully. The statement "*I normalize to one particle per unit volume*" is **not** Lorentz invariant. The volume of any box at rest will compress by a factor of $1/\gamma$ due to length contraction along the boost axis when we Lorentz transform it. For relativistic problems we want to normalize to a Lorentz invariant number of particles per unit volume. To achieve this we conventionally normalize to $1/(2E)$ particles per unit volume. Since $1/(2E)$ also scales like $1/\gamma$ it transforms in the same manner as V. Therefore the statement "*I normalize to $1/(2E)$ particles per unit volume*" **is** Lorentz invariant.

Terminology

$\mathcal{M}_{fi}$	Lorentz invariant amplitude for $\|\Psi_i\rangle \rightarrow \|\Psi_f\rangle$ transition
Feynman diagram ...	Graphical representation of part of the scattering amplitude
Vertex	Point where lines join together on such a graph
Constant coupling (g)	Dimensionless measure of strength of the force
Vertex factor ($\mathcal{Q}g$) ..	The contribution of the vertex to the diagram
Propagator	Factor of $1/(\mathsf{Q}\cdot\mathsf{Q}-m^2)$ associated with an internal line
Tree level / leading order	Simplest diagrams for any process with the smallest number of g factors. Contain no closed loops.

References and further reading

- *"Introduction to Elementary Particles"* D. Griffiths Chapters 6 and 7 does the full relativistic treatment, including spins, relativistic normalization and relativistic flux factor.
- *"Femptophysics"*, M.G. Bowler – contains a nice description of the connection between Feynman propagators and non-relativistic propagators.
- *"Quarks and Leptons"*, Halzen and Martin – introduction to the Dirac equation and full Feynman rules for QED including spin.
- *"QED - The Strange Theory of Light and Matter"*, Richard Feynman. Popular book with almost no maths. Even a PPE student could understand it – if you explained it slowly to him. In fact it has a lot to recommend it, not least that you can order it off Amazon for about a fiver.