Nuclear structure & decays

1 The Nuclear Periodic Table

Nuclei are made out of protons and neutrons. We label nuclei by their **atomic number** Z which is the number of protons they contain, by their neutron number N and by their **mass number** $A = Z + N$.

The symbol used to identify a nucleus is ${}^{\mathrm{A}}_{\mathrm{Z}}\mathrm{X}_{\mathrm{N}}$ where X is the name of the chemical element. For example ${}^{14}_{6}\mathrm{C}_{8}$. Since the element name specifies the atomic number Z and $A = N + Z$, we can fully specify the nucleus by just the symbol for the chemical and the mass number,

$$ {}^{\mathrm{A}}\mathrm{X} \quad \mathrm{e}.g. \quad {}^{14}\mathrm{C} $$

A table of the nuclides can be found in Figure 1. The long-lived nuclides like along the **valley of stability** which is at $N \approx Z$ for light nuclei but has $N > Z$ for heavier nuclei.

2 Nuclear size

The size of various nuclei can be determined by a variety of methods including:

- Shifts in the energy-levels of atomic electrons from the change of their Coulomb potential caused by the finite size of the nucleus.
- Muonic equivalents of the above. Muons are about 200 times heavier than electrons, so their "Bohr raduis" is about 200 times smaller. One observes the series of x-rays from the atomic (muonic) transitions
- Scattering experiments involving electromagnetically or strongly interacting probes (electrons, protons).

3 Binding energy

The total energy of a bound body is smaller than the mass of it's constituents because of the negative potential energy (**binding energy**, $B(N, Z)$) from mutually attractive (strong nuclear) forces. For nuclei this binding energy is typically of order 8 MeV per nucleon.

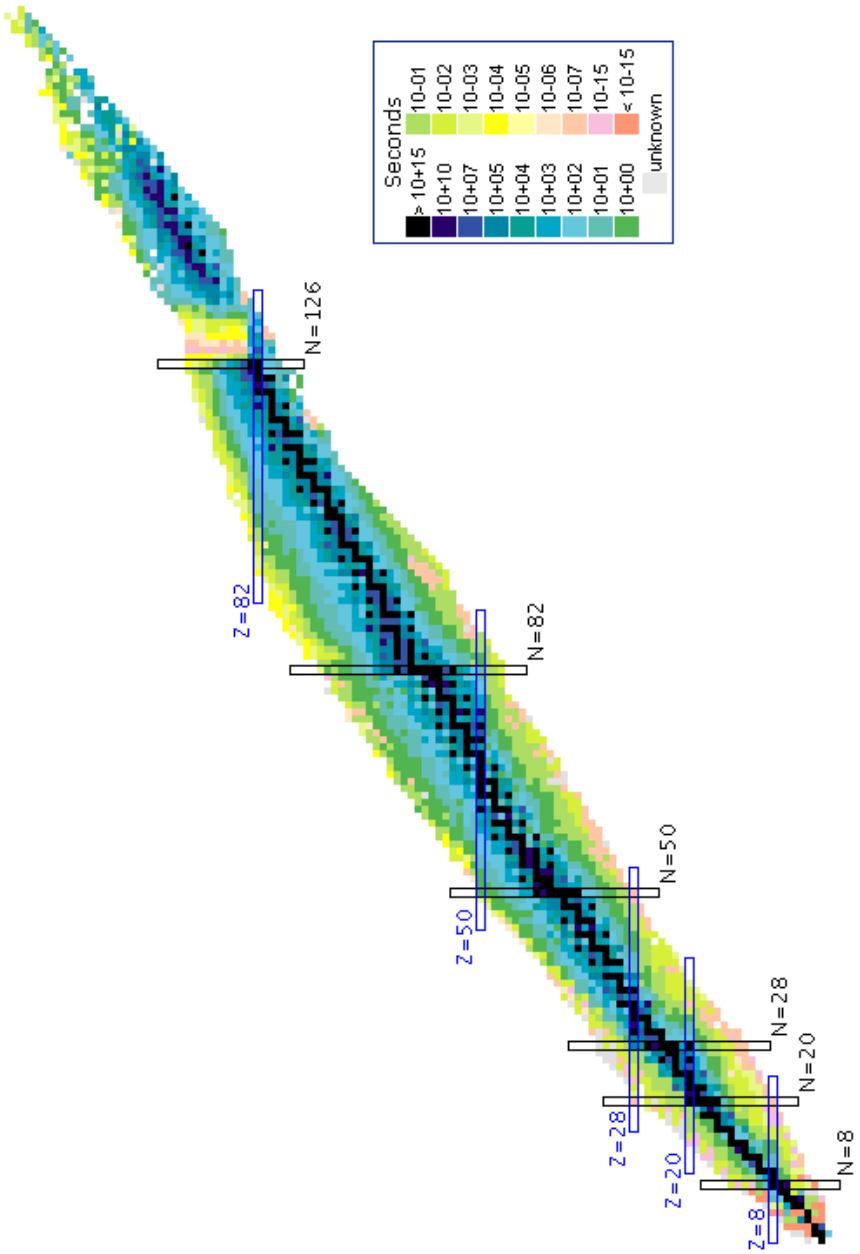


Figure 1: Table of the nuclides as a function of the number of neutrons (N) and the number of protons (Z). Darker colours represent longer-lived nuclides. The 'Magic Numbers' represent particularly stable nuclei which have closed shells. From http://www.nndc.bnl.gov/nudat2/.

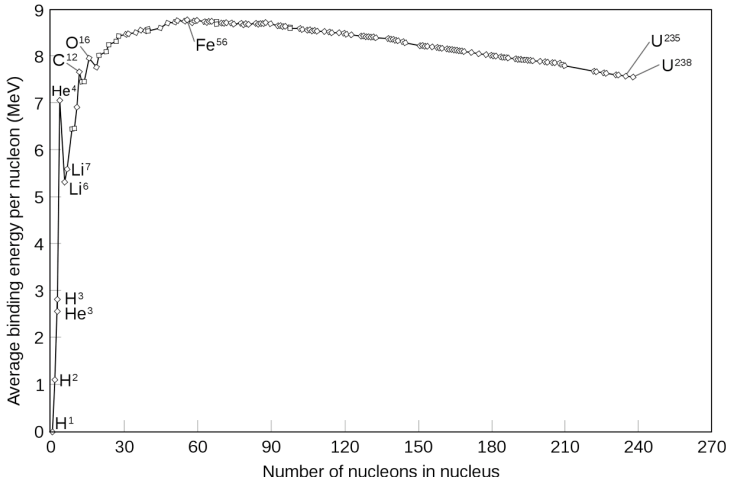


Figure 2: Binding energy per nucleon for some common nuclei. Data taken from [1]. Plot from `http://en.wikipedia.org/wiki/Binding_energy`.

The dynamics of the nucleus cannot be solved using perturbation theory. Perturbation theory is fine for atomic structure because the fine structure is of order $\alpha_{\rm EM}^2$ smaller than the gross structure. The nucleus is held together by the **strong nuclear force** and the strong coupling constant is large $\alpha_s \sim 1$, so unlike the atomic case the "fine structure" is of the same order as the "gross structure". Other methods will have to be used.

Let's build a model of the nucleus as a set of nucleons (protons and neutrons) held together by a strong short-range attractive force.

We expect that the mass of the atom will be less than the sum of the mass of its consituents by the binding energy B:

$$M(^{\rm A}_{\rm Z}{\rm X}) = Z\,M({\rm H}) + (A-Z)M(n) - B(N,Z).$$

Note that by using atomic masses on both sides of the equation we have included the masses of the electrons in this definition. The binding energy B is dominated by the nuclear binding energy (though in principle there will be a very small contribution from atomic binding energy).

The binding energy per nucleon B/A is shown for some common nuclei in Figure 2. It can be seen that the most stable nuclei are found around $^{56}{\rm Fe}$. Different behaviours can be seen in different regions. There is a broad flattish plateau for the central region $30 < A < 200$ for which $B/A \approx 8\,{\rm MeV}$. For A below about 30 the B/A is smaller than the plateau and is spiky. There is a systematic drop in B/A for large A, particularly for $A > 200$.

Other than for $A < 30$ a reasonable fit to the binding energy curve $B(N,Z)$ can be found by using the binding energy model of the **semi-empirical mass formula** (SEMF). The mass of the atom containing Z protons and N neutrons in its nucleus

is given by

$$m(N,Z) = Zm_H + Nm_n - B(N,Z)$$

where for the SEMF the binding energy is modelled by the function

$$\boxed{B(N,Z) = \alpha A - \beta A^{\frac{2}{3}} - \gamma\frac{(N-Z)^2}{A} - \epsilon\frac{Z^2}{A^{\frac{1}{3}}} + \delta(N,Z)}$$

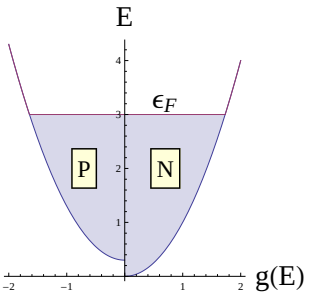
where $A = N + Z$.

The SEMF does not come from a full quantum mechanical solution of the Hamiltonian (nor from the effective field theory needed to describe it properly), but instead from modelling of just its most important features.

Let's assume that the interaction which is binding the nucleons together generates a force only between nearest neighbours, and that this force is equal for protons and neutrons. This will lead to a **volume term** in B which scales proportional to the number of nucleons, since most nucleons have the same number of neighbours. This is the α term above.

There will be a correction to the nearest-neighbour interactions coming from those nucleons at the boundary of the nucleus which have fewer neighbours. This **surface term** is expected to be proportional to $A^{\frac{2}{3}}$ since the surface area will scale according to r^2 and so $A \propto r^3$. It is the β term above.

With the two terms considered so far there is nothing to prevent arbitrarily large nucleons forming. These are not observed in nature, so we must be missing something. The something that we are missing is the **Coulomb repulsion** term. This electrostatic term will provide negative contribution to B (from the repulsive force between protons) and will scale as Z^2 because every proton feels the potential from all of the other protons (not just nearest neighbours). It will be inversely proportional to the size of the nucleus, since the potential energy of a uniform sphere of charge is proportional to Q^2/r. This is the ϵ term above.



The density of states for protons and neutrons.

$$\rho(E) = \frac{1}{e^{(E-\mu)/k_BT} + 1}$$

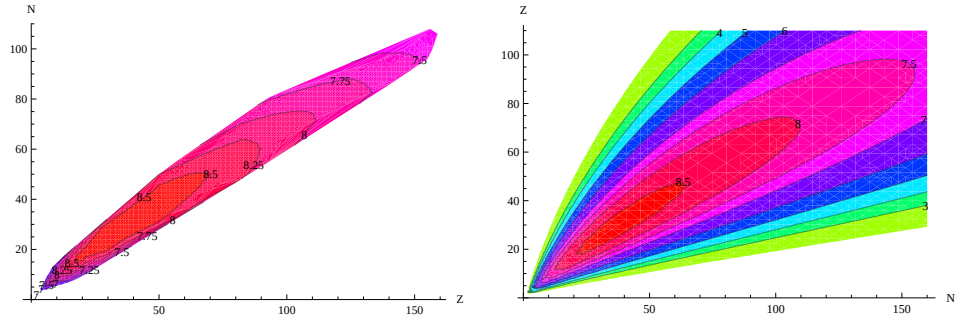
The Fermi-Dirac function.

There are two further terms in the SEMF, both of which are quantum mechanical in origin.

Firstly there is an **asymmetry term**. The origin of this term is as follows. As identical fermions no proton may exist in the same state as any other proton, nor may any neutron occupy the same state as any other neutron. However a proton and a neutron may exist in the same space $\times$ spin state since these are non-identical particles. The distribution of filled states is therefore different for the protons and neutrons. At large mass number the Coloumb term would tend to favour larger N and smaller Z. However if there are more neutrons than protons they must (on average) be placed in higher energy levels, since all of the lower-energy states are already filled.

We can work out the size of this term by calculating the number of states available. Neutrons and protons are both fermions, and so obey Fermi-Dirac statistics.

At temperatures small compared to the chemical potential ($k_T \ll \mu$) the Fermi-



(a) **(a)**

Figure 3: Diagram showing binding energys as a function of proton and neutron number for **(a)** data and **(b)** the Semi-Empirical Mass formula. Data taken from [1].

Volume	Surface	Asymmetry	Coulomb	Pairing
α	β	γ	ϵ	δ
15.835	18.33	23.2	0.71	$11.2/\sqrt{A}$

Figure 4: Typical values of the SEMF parameters (in MeV). From Bowler.

Dirac distribution tends to a step function – all levels are filled up to some energy level, known as the **Fermi Level** ϵ_F, and all levels above this are vacant.

In the problem set we show that this leads to a term in the SEMF of the form $\gamma\frac{(N-Z)^2}{A}$.[1]

Finally there is a **pairing term** which accounts for the observation that nuclei with either (a) even numbers of protons (Z even) or (b) with even numbers of neutrons (N even) tend to be especially stable. If *both* Z and N are even numbers then the nucleus tends to be even more tightly bound.

The pairing term in the SEMF that accounts for these effects is set to zero for odd-A nuclei. Even A nuclei have two possibilities. If both Z and N are *even* then the nucleus is more tightly bound and have an extra binding contribution, so B is increased by δ. If both Z and N are *odd* then the nucleus is less tightly bound and so B is decreased by δ.

4 Shell Model [non-examinable]

The SEMF provides a reasonable description of the binding energies of the nuclei for $A > 30$ but only the overall structure, not the details.

[1] In fact our calculation only accounts for *part* of that term.

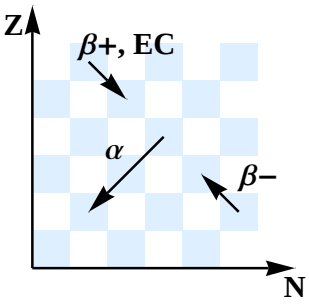
Differences at small A (e.g. the tightly bound isotopes ${}^{4}_{2}\mathrm{He}$ and ${}^{16}_{8}\mathrm{O}$) are already obvious in Figure 2. Figure 5 shows the difference between the true binding energy (per nucleon) and the SEMF prediction. Islands of stability are clearly visible near the special values of $N = 50, 82, 126$.

The complete special values of either N or Z (or both) are:

$$\{2, 8, 20, 28, 50, 82, 126\}.$$

These are known as the **magic numbers**. They correspond to closed configurations of nuclear shells. Evidence for this shell structure can be found in the binding energies, excitation energies, abundances, spins, and magnetic moments. The model gives further insight into a variety of nuclear properties, but is beyond the scope of this course.

5 Decays



The changes in (Z, A) induced by various decays.

For all nuclear reactions we define the Q value to be amount of energy 'released' by the decay

$$\boxed{Q \equiv \sum M_i - \sum M_f}$$

The first sum is over the masses of the initial particles in the decay (including their binding energies), and the second sum is over the masses of the final-state particles (including binding energies).

5.1 Alpha Decays

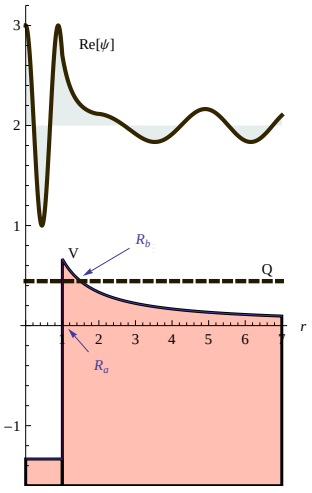
Alpha decays occur when (usually heavy) nuclei eject a helium nucleus:

$$\boxed{{}^{\mathrm{A}}_{\mathrm{Z}}\mathrm{X} \rightarrow {}^{\mathrm{A}-4}_{\mathrm{Z}-2}\mathrm{Y} + {}^{4}_{2}\mathrm{He}.}$$

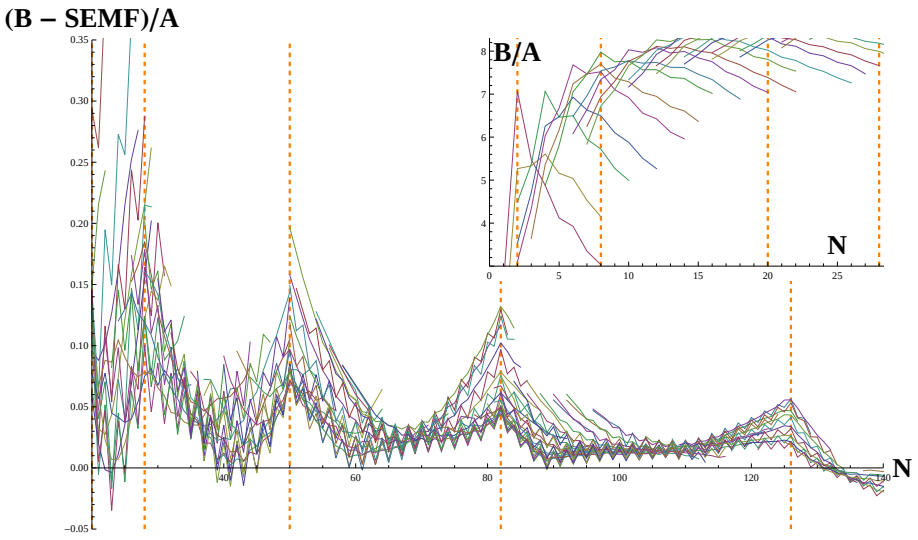
The change of mass number is $\Delta A = 4$ for α decays. Alpha decays of heavy nuclei are often associated with sequential *chains* of decays. When recall that $\Delta A = 0$ for β and γ decays we see that for any nucleus starting with mass number A, if it only decays via alpha, beta or gamma decay processes, all other nuclei in that chain must have some other set of mass numbers $A' = A - 4m$, where m is an integer indicating the number of sqeuential alpha decays that have occured.

There are therefore *four* non-overlapping decay chains for the heavy elements, corresponding to $A = n$, $A = n + 1$, $A = n + 2$, and $A = n + 3$ respectively, where n is an integer. So for example a chain initiated by ${}^{238}\mathrm{U}$ will decay via nuclides with mass numbers separated by four units $A = \{238,\ 234,\ 230,\ \ldots,\ 4n + 2, \ldots\}$, in the series of decays:

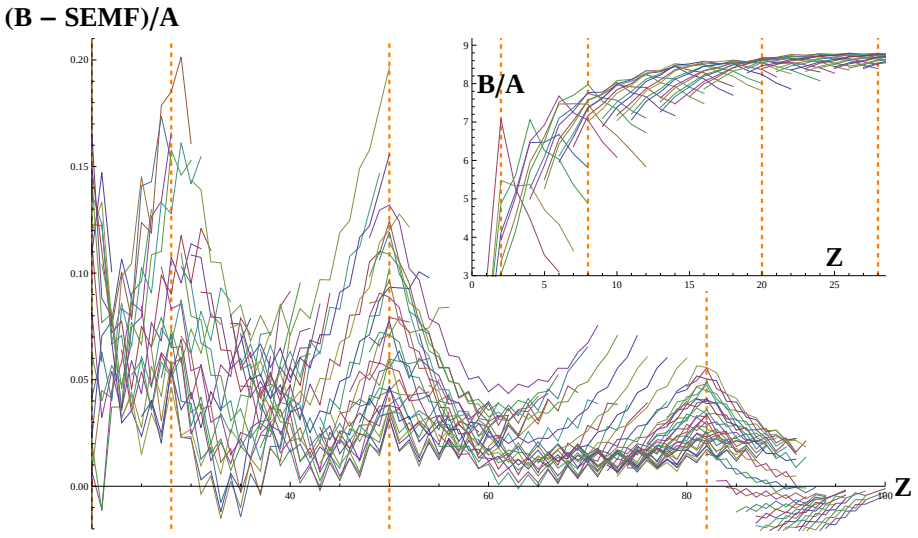
$${}^{238}\mathrm{U} \xrightarrow{\alpha} {}^{234}\mathrm{Th} \xrightarrow{\beta} {}^{234}\mathrm{Pa} \xrightarrow{\beta} {}^{234}\mathrm{U} \xrightarrow{\alpha} {}^{230}\mathrm{Th} \xrightarrow{\alpha} {}^{226}\mathrm{Th} \quad \text{etc.}$$



Real part of $\langle r|\Psi\rangle$ (as calculated in the WKBJ approximation) and $V(r)$ for a thin Coulomb potential barrier.



(a)



(b)

Figure 5: Difference between the measured binding energy (per nucleon) and the SEMF prediction. **(a)** The x-axis shows the number of neutrons in the nucleus; curves show isoto**p**es (same Z). **(b)** The x-axis shows the number of protons in the nucleus; curves show isoto**n**es (same N). In both cases the inset shows the binding energy per nucleon for the low-A nuclei. The magic numbers $\{2, 8, 20, 28, 50, 82, 126\}$ are marked with dashed lines.

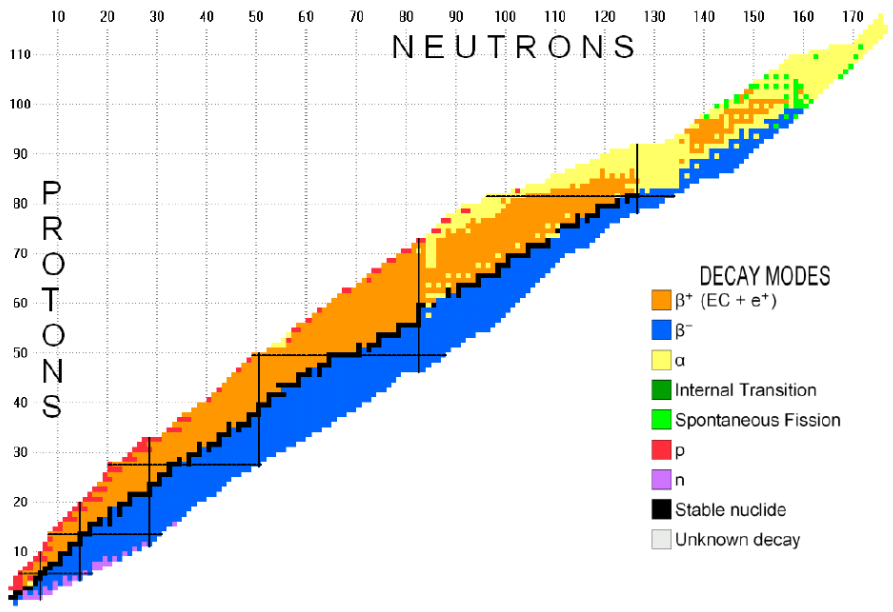


Figure 6: Decay modes of the nuclei. From [1].

The other three types of chain can, for example, be initiated from decays of the following isotopes: $^{232}\mathrm{Th}$ (leading to nuclides in the $A = 4n$ series); $^{237}\mathrm{Np}$ (leading to nuclides in the $A = 4n + 1$ series); $^{239}\mathrm{Pu}$ (leading to nuclides in the $A = 4n + 3$ series).

5.1.1 Rate calculation for α decays

We can model the α decay as a two-step process, in which α particles within the nucleus have a large number of collisions with the edge of the nuclear potential, but a small probability of tunnelling through the Coulomb barrier to the classically allowed region.

If the Q value of the decay is positive, then the decay is energetically favourable, but it may still be suppressed by a large tunnelling factor. Let us try to model the probability of tunnelling through the barrier, assuming that this large exponential factor will dominate the calculation of the rate of decay.

The time independent Schrödinger equation defines the energy eigenstate

$$E|\Psi\rangle = \left(-\frac{p^2}{2m} + V\right)|\Psi\rangle.$$

For simplicity, let's ignore the spherical geometry and treat the problem as a one-

dimensional one in the radial direction so that for a state with energy Q,

$$Q\langle x|\Psi\rangle = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\langle x|\Psi\rangle.$$

Without losing any generality we can write

$$\langle x|\Psi\rangle = \exp[\eta(x)].$$

After differentiating (5.1.1) and dividing by $\exp(\eta)$ we find

$$Q = -\frac{\hbar^2}{2m}\left[\eta'' + (\eta')^2\right] + V(x).$$

We can model the potential felt by any α particle by the function

$$V(r) = \begin{cases} \text{const} & r < R_a \\ \frac{zZ\alpha}{r} & r > R_a \end{cases}$$

where z and Z are the charges of the α-particle and the daughter nucleus respectively, and α is the electromagnetic fine structure constant.

Within the barrier the potential is smooth, so η should be a smoothly varying function of (x), and so we expect $\eta'' \ll (\eta')^2$. We therefore neglect the η'' term[2].

Neglecting the velocity in the flux terms, the tunnelling probability is given by the square of the ratio of the amplitudes at the edges of the classically forbidden regions:

$$P = \frac{|\langle R_b|\Psi\rangle|^2}{|\langle R_a|\Psi\rangle|^2} = e^{-2G}.$$

where

$$G = \eta(b) - \eta(a) = \sqrt{\frac{2m}{\hbar^2}}\int_{R_a}^{R_b} dr\,\sqrt{V(r) - Q}.$$

The limits of the integration are

$$R_a \approx r_0 A^{\frac{1}{3}},$$

the radius of the nuclear potential and

$$R_b = \frac{Z_1 Z_2 \alpha}{Q},$$

the radius at which the α particle enters the classically allowed region.

5.2 Beta decays, electron capture

There are three related nuclear decay processes which are all mediated by the **weak nuclear interaction**.

[2]This is known as the WKBJ approximation. It is a good approximation if the wave-function contain have many wave-lengths (or in the classically forbidden regions, as here, many factors of $1/e$) before the potential changes significantly.

Neutron-rich isotopes can decay through β decay:

$$ {}^{\rm A}_{\rm Z}{\rm X} \rightarrow {}^{\rm A}_{\rm Z+1}{\rm Y} + e^- + \bar{\nu}_e $$

The effect is to increase the atomic number by one, but to leave the mass number unchanged.

Isotopes which have a surplus of protons can proceed via either emission of a positron (anti-electron)

$$ {}^{\rm A}_{\rm Z}{\rm X} \rightarrow {}^{\rm A}_{\rm Z-1}{\rm Z} + e^+ + \nu_e $$

or by capture of one of the S-shell atomic electrons

$$ {}^{\rm A}_{\rm Z}{\rm X} + e^- \rightarrow {}^{\rm A}_{\rm Z-1}{\rm Z} + \nu_e $$

These interactions are mediated at the smallest scales by charged spin-1 particles known as $W^\pm$ bosons.

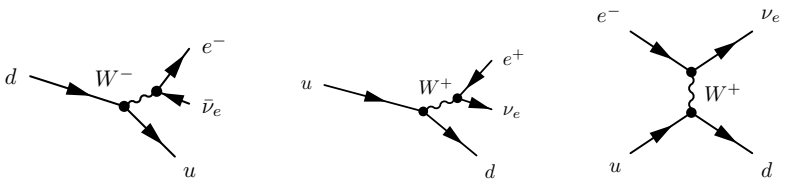
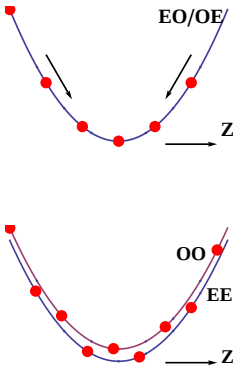


Figure 7: **Feynman diagrams** for β^- decay, β^+ decay, and electron capture. Lines on the left represent incoming particles. Lines on the right indicate outgoing particles. See the handout on that topic for explanation of the direction of the arrows.

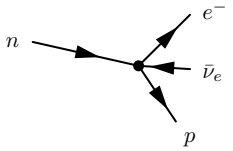


Curves of different Z for the same A for odd-A nuclei (above) and even-A nuclei (below). The even-A case has two curves separated by 2δ.

For probes with wavelength $\lambda \gg \lambda_{c,W}$, where $\lambda_{c,W}$ is the Compton wavelength, $\frac{\hbar}{m_W c}$, or similarly for probes with energies $E \ll m_W$, the small-distance behaviour of the interaction is not resolved. We do not resolve the W boson and instead we get what appears to be a four-body interaction. For example for β^- decay there is an effective interaction which appears to change a neutron into a proton in a point interaction, giving out an electron and a anti-neutrino:

$$ n \rightarrow p + e^- + \bar{\nu}_e $$

which is drawn[3]



[3]See comments above about directions of arrows.

In the **Fermi model** of β decay the amplitude of the interaction denoted by this point-like four-body vertex is assumed to be proportional to a constant – the **Fermi constant** G_F. The other assumptions are that the interaction happens at a point in space and does not depend on the spins of the incoming or outgoing particles.

The rates of decays can be calculated using **Fermi's golden rule**

$$\boxed{\Gamma = 2\pi \left|\mathcal{M}\right|^2 \frac{dN}{dE_f}}$$

with an appropriate matrix element and density of final states.

In the lectures and the exercises we define the initial nuclear wave-function $\langle \mathbf{x}|\Psi_i\rangle$, the final nuclear wave-function $\langle \mathbf{x}|\Psi_f\rangle$, and the electron and neutrino wave-functions $\langle \mathbf{x}|\Psi_e\rangle = \exp\left(\mathbf{p}_e \cdot \mathbf{x}\right)$ and $\langle \mathbf{x}|\Psi_\nu\rangle = \exp\left(\mathbf{p}_\nu \cdot \mathbf{x}\right)$ respectively. We prove the result that the rate (differential in the electron momentum) is given by:

$$\frac{d\Gamma}{dp} = \frac{2\pi}{\hbar} G_F^2 |M_{fi}|^2 \frac{1}{4\pi^4 \hbar^6 c^3} (E - E_0)^2 p^2,$$

where M_{fi} is the nuclear spatial overlap integral, $\int d^3x \langle \Psi_i|\mathbf{x}\rangle\langle \mathbf{x}|\Psi_f\rangle$, G_F is the Fermi coupling constant defined above, and E and p are the energy and momentum respectively of the electron.

5.3 Gamma decays

Gamma decays are electromagnetic transitions, and are found when excited nuclear states relax to their ground states.

We can again work out the rate using Fermi's golden rule. The final state consists of a plane wave representing the photon and a nuclear final state

$$\Psi_f \propto e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

The electric matrix element is then

$$\langle \Psi_f|M|\Psi_i\rangle = \int d^3x \, \Psi_b^* (\mathsf{A} \cdot \mathsf{J}) \, e^{-i\mathbf{k}\cdot\mathbf{x}} \Psi_a$$

where A represents the electromagnetic 4-potential and $\mathsf{J} = q\mathsf{p}/m$ is the electric 4-current operator.

Further Reading

- *"An Introduction to Nuclear Physics"*, W. N. Cottingham and D. A. Greenwood, 2001 for the basics
- *"Nuclear Physics"*, M.G. Bowler, Pergamon press, 1973

- *"Introductory Nuclear Physics"*, P.E. Hodgeson, E. Gadioli and E. Gadioli Erba, OUP, 2003
- The BNL table of the nuclides provides good reference data http://www.nndc.bnl.gov/nudat2/.

Bolwer and Hodgeson et. al. are good books which go well beyond this course.

References

[1] G. Audi, A. H. Wapstra, and C. Thibault. The 2003 atomic mass evaluation: (ii). tables, graphs and references. *Nuclear Physics A*, 729(1):337 – 676, 2003. The 2003 NUBASE and Atomic Mass Evaluations.