Todo list

The Oxford Nuclear and Particle Physics Basics

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Preface

This book has grown out of a third year undergraduate lecture course taught at the University of Oxford. That course is designed to introduce all physics students to the main concepts of both nuclear and particle physics, without recourse to the mathematical framework developed at the advanced undergraduate or graduate level. This book aims to provide a corresponding introductory text. It is intended to be accessible, and to be useful to all physics undergraduates whether experimentally or theoretically inclined.

Subatomic physics has a particularly close interplay between experimental results and the theories which describe them. The book uses the examples of nuclear and particle physics to show both how models – of various levels of sophistication – are developed to describe physical phenomena, and how those models can themselves be tested in future experiments. Examples include the semi-empirical mass formula for nuclei, the quark model of hadrons, the Standard Model of particle physics.

Non-relativisitic quantum scattering theory is developed via the Lippmann-Schwinger equation, from which the propagator and interaction terms arise naturally. Feynman diagrams for relativistic scattering are introduced by analogy, and used in tree-level calculations of example experimental observables.

Applications of nuclear physics are dealt with in a dedicated chapter, which outlines the important reactions both for power generation in earth, and for stars like the sun. The physics of the accelerators and detectors – which form the toolset of the experimental physicist – are also given a dedicated chapter. We do not shy away from areas of active research, such as in neutrino or Higgs boson physics, where recent progress have being made.

Each chapter ends with with a summary of the main key results. Additional information, and a list of futher reading can be found after each chapter summary. At the end of the book we provide problems for each chapter. They are of varying difficulty, and are intended both to test the concepts developed in that chapter, and to provoke further thought and discussion.

By its nature, subatomic physics deals with particles which are very small and very fast-moving, so we have to deal both with quantum mechanics and special relativity. The book assumes its readers already to have some familiarity with special relativity at the level of performing Lorentz transformations and calculating Lorentz invariants. In terms of quantum mechanics, the book expects the reader to have met the bra-ket notation of Dirac. They should also have encountered angular momentum in quantum mechanics, perturbation theory, and the Fermi Golden Rule for calculating transition rates. However no group theory is assumed, nor is it expected that students will have yet delved into the deeper waters of quantum field theory.

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Nuclear physics and decays

2.1 Structure of matter and energy scales

Subatomic physics deals with objects of the size of the atomic nucleus and smaller. We cannot see subatomic particles directly, but we may obtain knowledge of their structures by observing the effect of projectiles that are scattered from them. When a particle of any kind is scattered from a target, the typical resolution achieved is order the de Broglie wavelength,

$$\lambda = \frac{h}{p} \tag{2.1}$$

where h is Planck's constant, and p is the momentum of the projectile. Therefore, if we wish to resolve small distances, smaller than the size of the atom, we will need to do so with high-momentum projectiles. Smaller objects also tend to have larger binding energies holding them together, so require larger energies to excite their internal components. Some typical sizes of objects are given below, together with the momementum of the projectile required to resolve their size, and typical binding energies in electron-volt (eV) units.

keV	10^3 eV
MeV	10^6 eV
GeV	10^9 eV
TeV	10^{12} eV

Object	Size	$p = \frac{h}{\lambda}$	Binding energy
Atom	10^{-10} m	$10\,\text{keV}/c$	$\sim$ eV
Nucleus	$\sim 10^{-15}$ m	$1\,\text{GeV}/c$	$\sim$ MeV
Quark	$< 10^{-19}$ m	$>$ TeV$/c$	$>$ TeV

We can see that small objects also tend to have high binding energies, and hence projectiles of large energy will be required in order to excite them or break them up. The momenta are indicated in units of eV$/c$ where c is the speed of light. These units make it easy to compare the momentum of the projetile to its corresponding energy $E = pc$ for the case of a massless projectile such as a photon. The most convenient unit for describing the size of nuclei is the femtometer 10^{-15} m.[1] No

[1] The unit of 10^{-15} m or femtometer is sometimes called the 'fermi' reflecting the many seminal contributions of the Italian physicist Enrico Fermi to the field of nuclear physics.

sub-structure has yet been found for quarks even when using very high energy (TeV) projectile.

2.2 The Nuclear Periodic Table and Binding Energy

Nuclei are found to be made out of two constituents: protons and neutrons. We label nuclei by their **atomic number** Z which is the number of protons they contain, by their neutron number N, and by their **mass number** $A = Z + N$.

The symbol used to identify a nucleus is

$$^{A}_{Z}\mathrm{X}_{\mathrm{N}}$$

where X is the name of the chemical element. For example the Carbon-14 nucleus, which contains 8 neutrons and 6 protons is denoted $^{14}_{6}\mathrm{C}_{8}$. Since the element's name specifies the number of electrons, and hence the atomic number Z, and since $A = N + Z$, we can fully specify the nucleus by just the symbol for the chemical and the mass number,

$$^{A}\mathrm{X} \quad \text{e.g.} \quad ^{14}\mathrm{C}.$$

Isotopes	Same Z
Isotones	Same N
Isobars	Same A

Notation for related nuclei

Most nuclei are spherical in shape. The nuclear radius r can be measured in scattering experiments, and follows the general rule

$$\boxed{r = r_0 A^{1/3}} \tag{2.2}$$

where the constant r_0 is the characteristic nuclear size and is about 1.2×10^{-15} m. The fact that r is proportional to $A^{1/3}$ indicates that the volume of the nucleus $V \propto r^3$ is proportional to the mass number A. Each proton or neutron is therefore making an equal contribution to the overall nuclear volume.

2.2.1 Binding energy and the semi-empirical mass formula

The mass $m(A, Z)$ of the nucleus containing Z protons and $A - Z$ neutrons should be given by the mass of its consituents, less the mass associated with the binding energy. The mass-energy is therefore

$$m(A, Z)c^2 = Zm_p c^2 + (A - Z)m_n c^2 - B(A, Z), \tag{2.3}$$

where $m_p \approx 938.3\,\mathrm{MeV}/c^2$ and $m_n \approx 939.6\,\mathrm{MeV}$ are the masses of the proton and neutron respectively. In nuclear physics it is convenient to measure energies in units of MeV and masses in units of MeV/c^2. Using these units makes it easy for us to convert from mass to mass-energy and vice versa. By assuming such units, we can omit the factors of c^2 in what follows. [2]

We can build up a functional form for the binding energy $B(A, Z)$ by considering the forces between the nuclear constituents. To find the full quantum mechanical

[2] For more on 'natural units' see appendix 2.A.

Atomic or nuclear masses?

In nature almost all nuclei exist within atoms, so measuring atomic masses is generally easier than measuring nuclear masses. The mass of the nucleus is, to an excellent approximation, equal to the mass of the atom after subtraction of the mass Zm_e of the Z atomic electrons. There is a small reduction of the atomic mass from the binding energy $B_{\rm atomic}$ of the electrons to the nucleus. This atomic binding energy is typical of order keV or smaller, about a thousand times smaller than nuclear binding energies $B(A,Z)$, and so the atomic binding energy can safely be ignored in most contexts, so

$$\begin{aligned} m_{\rm atom} &= m_{\rm nucleus} + Zm_e - B_{\rm atomic} \\ &\approx m_{\rm nucleus} + Zm_e. \end{aligned}$$

The mass of the electron, is about half an MeV, of the same order as the nuclear minding energies, so certainly can't be ignored. In examples it's easy to forget to check whether atomic or nuclear masses are being considered – so take care, particularly in β decay calculations where electrons (and positrons) are created and annihilated.

ground state for all of the protons and neutrons would be a very difficult problem. However we can understand a great deal about nuclear behaviour by building up a model of the mass which encapsulates its key features. This we will do over the rest of the section, building up towards the **semi-empirical mass formula** of equation (2.5). The 'semi-empirical' means that the model is built partly partly by demanding agreement with data, and partly from our understanding of the underlying theory.

Firstly, we will need an attractive force in order to hold the nucleus together against the mutual electrostatic replusion of its constituent protons. That force must be very strong, since the Coulomb electrostatic repulsion between a pair of protons, each of charge e and separated by distance $d \approx 1\,\text{fm}$, is

$$\begin{aligned} F &= \frac{e^2}{4\pi\epsilon_0\, d^2} \\ &\approx 230\,\text{N} \end{aligned}$$

which is macroscopic – comparable to the weight of a child.

What form should that nucleon-nucleon attractive force take? We can get clues about the force by looking at the binding energy per nucleon B/A is shown for some common nuclei, shown in Figure 2.1. For nuclei this binding energy is typically of order 8 MeV per nucleon. It can be seen that the most stable nuclei are found for values of A around 60. Different behaviours can be seen in different regions. There is a broad flattish plateau for the central region $30 < A < 200$ for which $B/A \approx 8\,\text{MeV}$. For A below about 30 the binding energy per nucleon is smaller than the plateau value and is spiky. There is a systematic drop in B/A for large A, particularly for $A > 200$.

To obtain a value of B/A that is rather flat, we cannot permit strong attractions between each of the constituent nucleons and every one of the others. If every nucleon felt an attraction to each of the others, then the binding energy would be

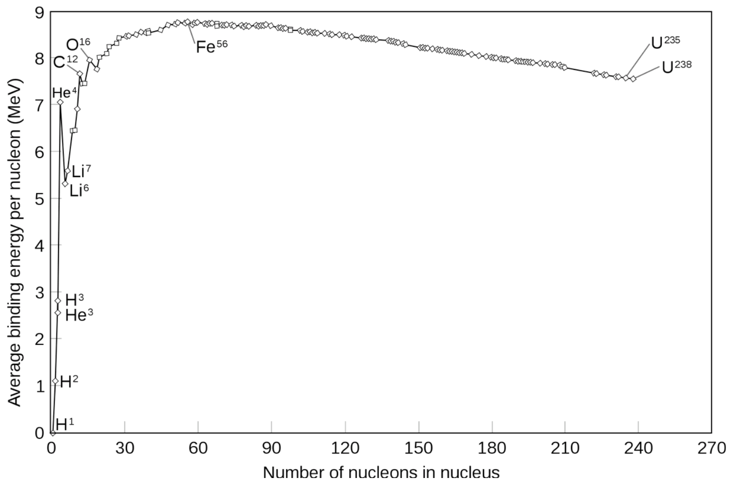


Figure 2.1: Binding energy per nucleon (B/A) as a function of A for some common nuclei. Data taken from [1]. Plot from [source].

expected to grow as approximately $B \propto A(A-1) \sim A^2$, and hence B/A would be approximately proportional to A. Such a linear growth in B/A is ruled out by the data (Figure 2.1).

To obtain the flat B/A found in nature we must assume that the strongly attractive force acts only between **nearest neighbour** nucleons. In this way, each nucleon binds to the same number of nearest neighbours, independently of the size of the nucleus, and hence the binding energy per nucleon is the same regardless of the nuclear size,

$$B \approx \alpha A$$

where α is a constant with units of energy. The use of nearest-neighbour interactions indicates that the force must either be short-range, or screened from long-range interactions by the effects of the nucleons in between.

In modelling a nearest-neighbour force we ought to make a correction for the fact that those nucleons on the surface have fewer neighbours. To correct for the reduced number of binding opportunities on the surface we reduce the binding energy by an amount proportional to the surface area, leading to the corrected formula

$$B \approx \alpha A - \beta A^{\frac{2}{3}}. \tag{2.4}$$

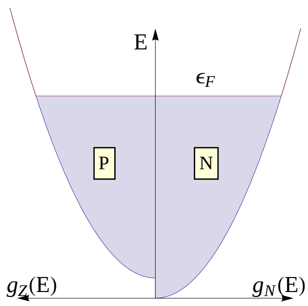
The new contribution is negative since it reduces the binding energy. We have made use of the observation (2.2) that since the volume of the nucleus scales as $r^3 \propto A$, the surface area scales as $r^2 \propto A^{2/3}$.

These two terms (2.4) in this first approximation to the binding energy are known as the **volume term** and the **surface term** respectively. Together they form what is known as the **liquid drop model**, since a similar result would be found for a drop of fluid with nearest neighbour interactions and a surface tension parameterised by β. The liquid drop model is consistent with the observation that each nucleon requires the same volume of space, in agreement with equation (2.2).

So far, so good. However there is nothing in this liquid drop model to prevent the growth of arbitrarily large nuclei. Such large nuclei are not observed in nature, so we must be missing something. The obvious candidate is the **Coulomb repulsion**, which interacts over long distances, and so will tend to push larger nuclei apart. This electrostatic repulsion between protons will reduce the binding energy by an amount proportional to $Z(Z-1) \approx Z^2$ because every proton feels the repulsion from all of the other protons (not just nearest neighbours). The binding energy will be reduced by the electrostatic binding energy which can be parameterised by

$$\epsilon \frac{Z^2}{A^{\frac{1}{3}}}.$$

Here ϵ is a another constant with dimensions of energy, which we will calculate a value for in the examples. The Coulomb repulsion energy is inversely proportional to the radius of the nucleus, and hence to $A^{\frac{1}{3}}$, since the potential energy of a uniform sphere of charge Q is proportional to Q^2/r.



The density of states $g(E)$ for protons and neutrons as a function of energy E.

Two further terms are required to give a good match between our model and the data. Both of them are quantum mechanical in origin.

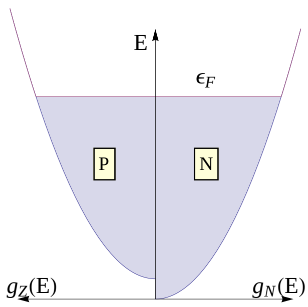
Firstly there is an **asymmetry term**. The origin of this term is as follows. Since protons are identical fermions, the Pauli exclusion principal states that no two of them may exist in the same state. Nor may any neutron occupy the same state as any other neutron. However it is possible for a proton and a neutron to exist in the same state since the two particles are not identical. The allowed states are therefore distinct, and are separately filled for the protons compared to the neutrons.

$$p(E_i) = \frac{1}{e^{(E_i - \mu)/k_B T} + 1}$$

The Fermi-Dirac function gives the probability $p(E_i)$ of filling a state with energy E_i for a system at temperature T and with chemical potential μ. k_B is the Boltzmann constant.

We can work out the size of the asymmetry effect by calculating the number of states available. Neutrons and protons are both fermions, and so obey Fermi-Dirac statistics. The temperatures we are interested in are small compared to the chemical potential ($k_B T \ll \mu$). Under these circumstances the Fermi-Dirac distribution tends towards a step function — all levels are filled up to some energy level, known as the **Fermi Energy** ϵ_F, with all states with energy above ϵ_F left vacant.

At large mass number A the Coloumb repulsion term would tend to favour larger N and smaller Z, since neutrons do not suffer from the Coulomb repulsion as protons do. However this energic advantage of neutrons over protons will be partially cancelled out by the fact that the additional neutrons must (on average) be placed in higher energy levels than additional protons, since all of the lower-energy neutron states will already be filled.



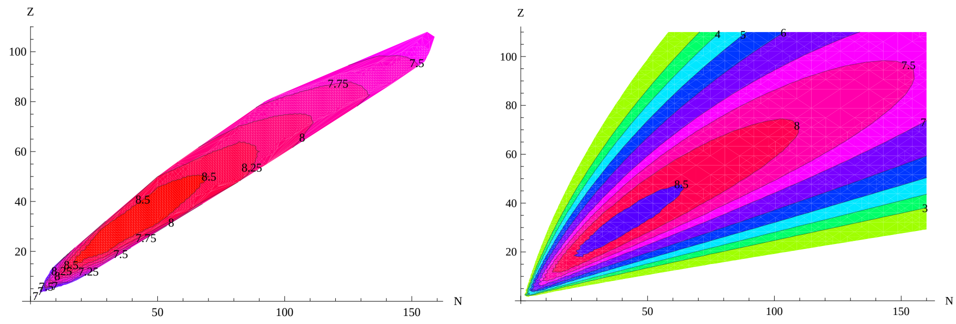
The density of states $g(E)$ for protons and neutrons as a function of energy E.

If we approximate the states available as being a continuum, which is not a bad approximation provided that both Z and N are larger than about 10, then the density of available states is found to be proportional to $E^{\frac{1}{2}}$. In the examples we show that this leads to an energy equation of the form

$$\gamma \frac{(N-Z)^2}{A}.$$

This **asymmetry term** reduces the binding energy, doing so most when the difference between the number of protons and of neutrons is largest.

Finally there is a **pairing term** which accounts for the observation that nuclei with either even numbers of protons (Z even) or with even numbers of neutrons (N



(a) **(a)**

Figure 2.2: Diagram showing binding energies as a function of proton and neutron number for **(a)** data [1] and **(b)** the Semi-Empirical Mass Formula.

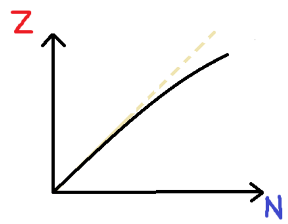
even) tend to be more stable than those with odd numbers. The pairing term is zero for odd-A nuclei. Even-A nuclei have two possibilities. If both Z and N are even then the nucleus is more tightly bound and have an extra binding contribution, so B is increased by δ. If both Z and N are odd then the nucleus is less tightly bound and so B is decreased by δ.

Putting all five terms together we obtain a formula for the binding energy,

$$B(A,Z) = \alpha A - \beta A^{\frac{2}{3}} - \gamma \frac{(A-2Z)^2}{A} - \epsilon \frac{Z^2}{A^{\frac{1}{3}}} + \delta(A,Z),$$

having eliminated N in favour of A. Substituting this into the formula defining the binding energy (2.3) we obtain the **semi-empirical mass formula** (SEMF)

$$\boxed{M(A,Z) = Zm_p + (A-Z)m_n - \alpha A + \beta A^{\frac{2}{3}} + \gamma \frac{(N-Z)^2}{A} + \epsilon \frac{Z^2}{A^{\frac{1}{3}}} - \delta(N,Z).} \tag{2.5}$$



Sketch of the shape of the valley of stability.

Other than for $A < 30$, where our approximations are less valid, the SEMF gives a rather good description of the binding energies of the observed nuclei (Figure 2.2). In particular the SEMF correctly predicts the shape of the curved **valley of stability** in the Z, N plane within which the stable nuclei are found. The relative numbers of protons and neutrons along this valley reflects a trade-off between the Coulomb and asymmetry terms. At low A the asymmetry term favours $N = Z$. At larger A the Coulomb term starts to compete with the asymmetry term, reducing the ratio of protons to neutrons.

The SEMF confirms the observation that the most energetically favourable nuclei are found close to ${}^{56}\mathrm{Fe}$. The energetically favourable location of iron goes some way to explaining why this it one of the most common elements inside the Earth.

It is energetically favourable for nuclei far from that valley to migrate towards the valley by nuclear decay. The ways in which nuclei can migrate around this space are explored in the following section.

Volume	Surface	Asymmetry	Coulomb	Pairing
α	β	γ	ϵ	δ
15.835	18.33	23.2	0.71	$11.2/\sqrt{A}$

Figure 2.3: Typical values of the SEMF parameters (in MeV). From Bowler.

2.3 Spin and parity

2.4 Decays and reactions

A table of the nuclides can be found in Figure 2.4. The stable long-lived nuclides lie along the valley of stability where the binding energy per nucleon is largest. The valley lies along $N \approx Z$ for light nuclei but has $N > Z$ for heavier nuclei. Nuclei far from that valley, and very heavy nuclei, tend to be unstable against nuclear decay.

While unstable nuclei will decay spontaneously, other reactions can be initiated by firing projectiles at a nucleus. Reactions are said to be **elastic** if the final state contains the same set of particles e.g. the elastic scattering of a photon from a nucleus, via an excited intermediate state:

$$X + \gamma \longrightarrow X^* \longrightarrow X + \gamma.$$

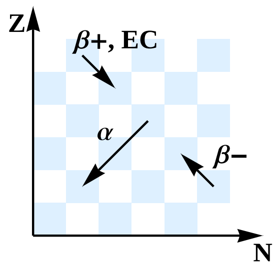
Reactions are **inelastic** if there is a change in particle content during the reaction e.g. radiative capture of a neutron

$${}^{A}\mathrm{X} + n \longrightarrow {}^{A+1}\mathrm{X}^* \longrightarrow {}^{A+1}\mathrm{X} + \gamma.$$

For all nuclear decays and reactions we define the Q value to be amount of energy 'released' by the decay,

$$\boxed{Q = \sum M_i - \sum M_f}$$

The first sum is over the masses of the initial particles in the decay (including their binding energies), while the second sum is over the masses of the final-state particles (including their binding energies). A positive Q value shows that a reaction is energetically favourable.



The changes in (Z, A) induced by various decays.

2.4.1 Alpha Decays

Alpha decays occur when (usually heavy) nuclei eject an 'α particle', that is a helium nucleus containing two protons and two neutrons. The decay is

$$\boxed{{}^{A}_{Z}\mathrm{X} \longrightarrow {}^{A-4}_{Z-2}\mathrm{Y} + {}^{4}_{2}\mathrm{He}.}$$

The change of mass number of the heavy nucleus is $\Delta A = -4$ and the change in its atomic number is -2.

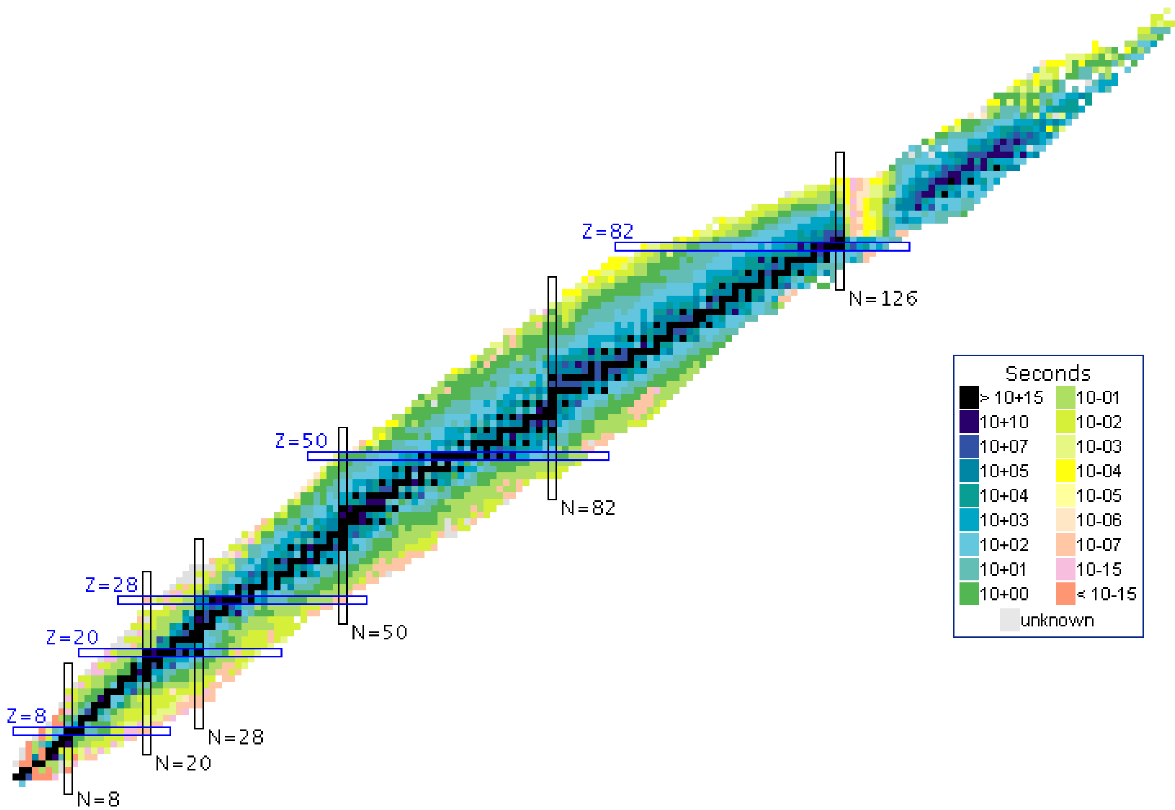


Figure 2.4: Table of the nuclides as a function of the number of neutrons (N, on the x-axis) and the number of protons (Z on the y-axis). Darker colours represent longer-lived nuclides, which can be found in the 'valley of stability'. The 'Magic Numbers' indicate particularly stable nuclei and are described in the shell model (see appendix 2.5). From http://www.nndc.bnl.gov/nudat2/.

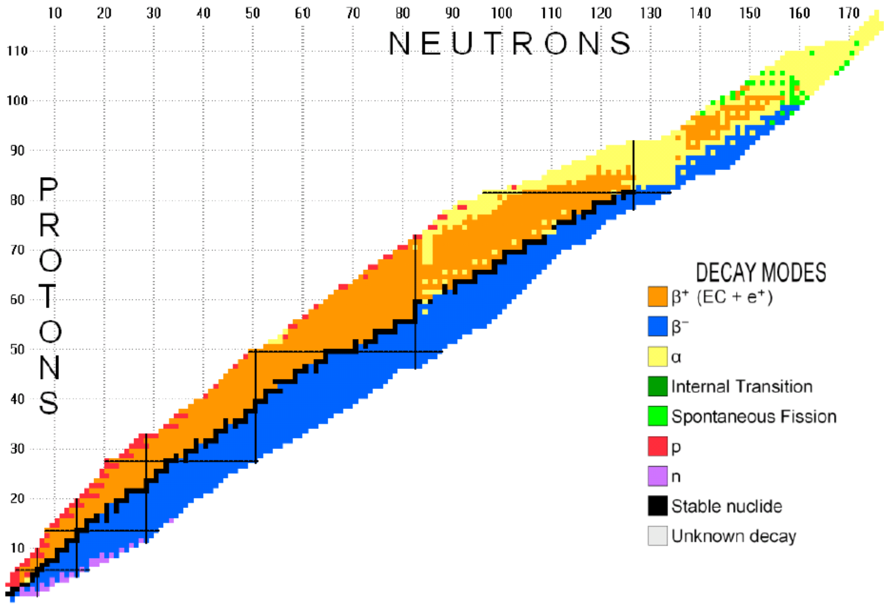


Figure 2.5: Decay modes of the nuclei. From [1].

The process of decay of heavy nuclei is often via sequential chains involving both alpha and beta decays. Since $\Delta A = -4$ for α decays and $\Delta A = 0$ for β and γ decays we see that for any nucleus starting with mass number A (and if it only decays via alpha, beta or gamma decay processes) all other nuclei in that chain must have some other set of mass numbers $A' = A - 4m$, where m is an integer indicating the number of alpha decays that have occured.

The helium nuclei emitted by alpha decay are highly ionising, and so travel only a small distance through matter – they can be stopped, for example, by a piece of card.

Rate calculation for α decays

We can model the α decay as a process in which 'proto α particles' are pre-formed inside the nucleus. Each is assumed to have a large number of collisions with the edge of the nucleus, but a small probability on each collision of **tunnelling** through the Coulomb barrier and escaping.

If the Q value of the decay is positive, then the decay is energetically favourable, but it may still be suppressed by a large tunnelling factor. Let us try to model the probability of tunnelling through the barrier. We will assume that the large exponential in the quantum tunnelling factor will dominate the calculation of the rate of decay, so we will neglect differences in the probability of formation of the proto-alpha particle, and its rates of hitting the barrier.

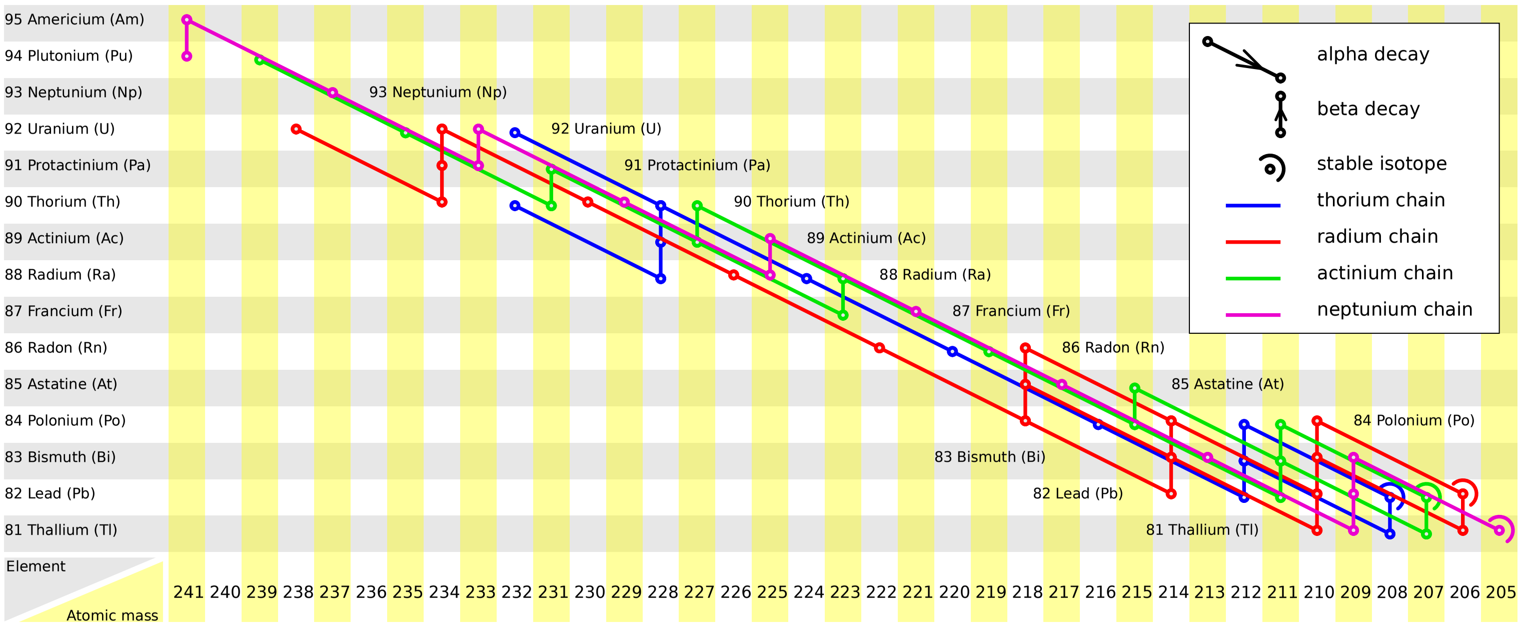


Figure 2.6: There are therefore *four* non-overlapping decay chains for the heavy elements, corresponding to $A = n$, $A = n+1$, $A = n+2$, and $A = n + 3$ respectively, where n is an integer.

The time independent Schrödinger equation defines the energy eigenstate $|\Psi\rangle$,

$$E|\Psi\rangle = \left(\frac{p^2}{2m} + V\right)|\Psi\rangle.$$

where E is the energy of the alpha particle, p is the momentum operator, m is its mass, and V is the potential in which it moves.

For simplicity, we will ignore the spherical geometry and treat the problem as one-dimensional in the radial direction r. In doing so we neglect a term $l(l+1)/(2mr^2)$ in the effective potential, where l is the angular momentum of the outgoing alpha particle. This term will in fact turn out to be an important correction to the potential if conservation of angular momentum requires an emission with $l > 0$.

Returning to the 1-dimensional case, we assume that the heavy daughter nucleus has negligible recoil energy, so that the outgoing alpha particle has kinetic energy Q. The Schrödinger equation then becomes

$$Q\langle r|\Psi\rangle = \left(-\frac{1}{2m}\frac{\partial^2}{\partial r^2} + V(r)\right)\langle r|\Psi\rangle, \tag{2.6}$$

where we use natural units such that $\hbar = \frac{h}{2\pi} = c = 1$ (see appendix 2.A). In the Dirac notation $\langle r|\Psi\rangle$ represents the wave function — that is the amplitude to find the alpha particle located between r and $r + dr$, and is often written $\Psi(r)$. Without losing any generality we can write the wave function as the exponential of some other function $\eta(r)$,

$$\langle r|\Psi\rangle = \exp[\eta(r)]. \tag{2.7}$$

After inserting (2.7) into (2.6) and dividing by $\exp(\eta)$ we find

$$Q = -\frac{1}{2m}\left[\eta'' + (\eta')^2\right] + V(r),$$

where the primes indicate derivatives by r. Placing the nucleus at the origin, assuming the nucleus to be much heaver than the alpha particle, and working in its rest frame, we can model the potential $V(r)$ felt by any α particle by the function

$$V(r) = \begin{cases} \text{const} & r < R_a \\ \frac{zZ\alpha_{\text{EM}}}{r} & r > R_a \end{cases}$$

where inside the nucleus V is large and negative, and outside the nucleus it is given by the Coulomb potential and hence characterised by the charges z and Z of the α-particle and the daughter nucleus respectively. The constant α_{EM} in the Coulomb potential is the dimensionless electromagnetic fine structure constant

$$\alpha_{\text{EM}} = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}.$$

Within the barrier the potential is smoothly varying, so η should be a smoothly varying function of (r). We then expect $\eta'' \ll (\eta')^2$, and we can safely neglect the η'' term compared to the $(\eta')^2$. [3]

[3] This is known as the WKBJ approximation, named after Wentzel, Kramers, Brillouin and Jeffreys, is a good approximation if many wave-lengths (or in the classically forbidden regions, as here, many factors of $1/e$) of the wave-function occur before the potential changes significantly.

The tunnelling probability can be found from the ratio of the mod-squared amplitudes:

$$P = \frac{|\langle R_b|\Psi\rangle|^2}{|\langle R_a|\Psi\rangle|^2} = e^{-2G}.$$

where the 'Gamow factor' G (> 0) is given by

$$-G = \eta(R_b) - \eta(R_a) = -\sqrt{2m}\int_{R_a}^{R_b} dr\,\sqrt{V(r) - Q}.$$

Choosing the minus sign before the radical ensures that we select the exponentially falling solution. The inner limit of the integration is the radius of the nucleus

The exponentially rising solution would be of interest only if the alpha particle were tunnelling into the nucleus. This alternative solution is the relevant one in fusion reactions, for example.

$$R_a \approx r_0 A^{\frac{1}{3}},$$

and the outer limit

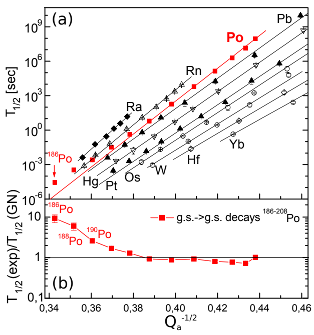
$$R_b = \frac{zZ\alpha_{\rm EM}}{Q},$$

the radius for which $Q > V$, i.e. where the α particle enters the classically allowed region.

The Gamow factor integral will be dominated by that part where the potential $V(r)$ is much larger than the released energy Q, so to a good approximation one can neglect Q in the integral. The rate dependence on Q is therefore dominated by the dependence of the upper integration limit R_b on Q, leading to the **Geiger-Nutall** equation

$$\log\Gamma = -c_1\frac{Z}{\sqrt{Q}} + c_2\ , \tag{2.8}$$

for the dependence of the $alpha$-decay rate on the charge Z of the daughter nucleus and the energy release, where c_1 and c_2 are constants.



The logarithms of experimental partial α-decay half-lives for the even-even Yb-Ra nuclei with neutron number $N < 126$ as a function of $Q_\alpha^{-1/2}$ (in MeV$^{-1/2}$). The straight lines show the description of the Geiger-Nuttall law with A and B values fitted for each isotopic chain. The sub-figure shows the deviation of the experimental α-decay half-lives from those predicted by the GN law for the light Po isotopes. From [5].

2.4.2 Beta decays, electron capture

There are three related nuclear decay processes which are all mediated by the **weak nuclear interaction**. Neutron-rich isotopes can decay through the emission of an electron e and an anti-neutrino $\bar{\nu}_e$ in the **beta decay** a process:

$${}^{A}_{Z}\mathrm{X} \longrightarrow {}_{Z+1}^{A}\mathrm{Y} + e^- + \bar{\nu}_e. \tag{2.9}$$

The effect is to increase the atomic number by one, but to leave the mass number unchanged. At the level of the individual nucleons the reaction is

$$n \longrightarrow p + e^- + \bar{\nu}_e. \tag{2.10}$$

The emitted electron can be observed and its energy measured. The associated anti-neutrino has a very small interaction probability, and so is expected to escape unobserved. Long before neutrinos were observed, Wolfgang Pauli realised that an additional, invisible, massless particle was required in order to conserve energy and momentum in the decay (2.9). His arguments ran as follows. The emitted electrons are observed to have a variety of different kinetic energies, up to Q. Meanwhile the mass difference between the parent and daughter nucleus is fixed to a single value. The energy given to the recoiling daughter nucleus is small and is fixed by momentum conservation, so it can't be responsible for the deficit observed when the electron has energy less than Q. Energy conservation is then only possible if the total energy Q can be shared between the electron and some other unobserved partice – the (anti-)neutrino.

Pauli also argued that without the neutrino the reaction (2.10) would violate angular momentum conservation. Adding the angular momenta of just the two observed final state spin-half particles – the electron and the proton – according to the rules of quantum mechanical angular momentum addition we would find

$$\frac{1}{2} \oplus \frac{1}{2} = 0 \text{ or } 1,$$

Neither of the possibilities of total angular momentum $s = 0$ or $s = 1$ match the spin of the initial neutron, which has $s = \frac{1}{2}$. However by adding a third spin-half particle to the final state – the $s = \frac{1}{2}$ anti-neutrino – we can reconstruct a state which has total angular momentum equal to that of the proton ($s = \frac{1}{2}$) since

$$\frac{1}{2} \oplus \frac{1}{2} \oplus \frac{1}{2} = \frac{1}{2} \text{ or } \frac{3}{2}.$$

When adding angular momenta j_1 and j_2 in quantum mechanics the permitted values of the total angular momentum range from $|j_1 - j_2|$ to $j_1 + j_2$ in integer steps.

Isotopes which have a surplus of protons can proceed via one of two processes. The first is the emission of a positively charge anti-electron. This is known as **positron emission** or β^+ decay

$${}^A_Z\mathrm{X} \longrightarrow {}_{Z-1}^{\;\;A}\mathrm{Z} + e^+ + \nu_e \tag{2.11}$$

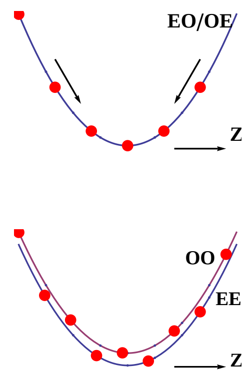
The positron is the anti-particle of the electron. It has the same mass as the electron, but positive charge.

The second method of decay of proton-rich nuclei is by the nucleus removing one of the atomic electrons, the **electron capture** process:

$${}^A_Z\mathrm{X} + e^- \longrightarrow {}_{Z-1}^{\;\;A}\mathrm{Z} + \nu_e \tag{2.12}$$

These two processes ((2.11) and (2.12)) result in the same change to the nucleus, and so compete with one another to reduce the Z number of proton-rich nuclei. When considering whether electron capture or β^+ decay will dominate we note that

- The Q value for positron emission is $2 \times m_e c^2$ smaller than that for the corresponding electron capture.
- Electron capture relies on there being a substantial overlap of an electron wave-function with the nucleus.



Mass as a function of Z for nuclides of the same A, for odd-A nuclei (above) and even-A nuclei (below). The even-A case has two curves separated by 2δ.

When viewed at the level of the nuclear consitutents, all three of the interactions above — β decay (2.9), β^+ decay (2.11) and electron capture (2.12) — involve the interaction of four particles: a proton, a neutron, an (anti-)electron and an (anti-) neutrino.

$$\begin{aligned} n &\longrightarrow p + e^- + \bar{\nu}_e & (2.13)\\ p &\longrightarrow n + e^+ + \nu_e, & (2.14)\\ p + e^- &\longrightarrow n + \nu_e. & (2.15) \end{aligned}$$

We note that all of the reactions (2.13)–(2.15) are assumed to be occurring inside the complex environment of the nucleus. Of these three reactions, only neutron decay (2.13) can occur in isolation, since it is the only one with $Q > 0$, (the neutron being about $1.3\,\mathrm{MeV}/c^2$ heavier than the proton). The other two reactions (2.14)–(2.15) occur only within a nucleus, when the energy released from the rearrangement of the nuclear constituents is sufficient to compensate for the endothermic nature of the reaction at the level of the individual nucleon.

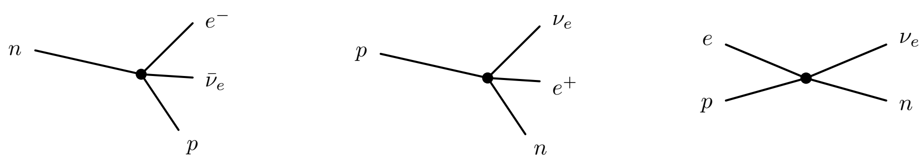
We note that all of three transitions — β^-, β^+, and e^- capture — leave the mass number A unchanged. The parent and daughter nuclei are isobars. The decay processes (2.13)–(2.15) allow transitions between isobars, and mean that for odd-A nuclei for any value of A there is usually only one stable isobar, that for which the mass of the system is minimum.

Even-A nuclei may have one stable isobar, but can also have two or very occasionally three. Multiple stable states are possible for A even because the binding curves for (even-Z, even-N) and (odd-Z odd-N) are separated by 2δ, where δ is the pairing energy in the SEMF. For an even-even nucleus, since all of the reactions (2.13)–(2.15) change both $|Z|$ and $|N|$ by one they result in an odd-odd nucleus, and so a transition to the higher of the two curves. None of the reactions permit a change in Z of two units, and the probability of two such reactions happening at once is extremely small, so an even-even nucleus with some Z can be stable provided that both the neighbouring nuclei with $Z + 1$ and $Z - 1$ have larger mass.

Fermi theory of beta decays

To understand the lifetimes of the nuclei, we wish to calculate the expected rates for $\beta^\pm$ decays. We follow the method and approximations of Enrico Fermi.

If we put the initial state particles on the left hand side of the diagram, and the final state particles on the right hand side, then we obtain the following three diagrams for the reactions (2.13)–(2.15).



We shall assume that each of these four-particle interactions will happen at a single point in space. The amplitude for each reaction is given by the same constant — a four-body coupling constant which tells us the amplitude for each interaction at that point in space. For each of the three diagrams that coupling is the **Fermi constant**,

$$\boxed{G_F \approx 1.17 \times 10^{-5}\,\mathrm{GeV}^{-2}}$$

which has units of inverse energy squared.

In using a single, constant factor, we implicitly make the simplifying assumption that the four-body interaction does not depend on the spins of the incoming or outgoing particles. To simplfy calculations we will also follow Fermi in assuming that the wave-functions of the electron and the anti-neutrino can be represented by plane waves. This ignores the effect of the Coulomb attraction between the electron and the nucleus, and is a good approximation provided that the electron energy is sufficiently high.

We recall that in quantum mechanics, the rate of some a transition from an initial state to a final state characterised by a continuum of energy levels is given by the **Fermi Golden Rule**[4]

$$\Gamma = \frac{2\pi}{\hbar}|A_{fi}|^2 \frac{dN}{dE_f} \tag{2.16}$$

Here the transition rate Γ is given in terms of the amplitude A_{fi} connecting the initial and the final states, and the density $\frac{dN}{dE_f}$ of states at the final energy. It is the A_{fi} and $\frac{dN}{dE_f}$ that we shall have to calculate.

To be concrete, let us consider the beta decay reaction (2.13). We denote the initial nuclear wave-function by $\langle \mathbf{x}|\Psi_i\rangle$, the final nuclear wave-function by $\langle \mathbf{x}|\Psi_f\rangle$. The electron and anti-neutrino wave-functions are approximated as plane waves

$$\langle \mathbf{x}|\phi_e\rangle \equiv \phi_e = \exp\left(i\,\mathbf{p}_e \cdot \mathbf{x}\right) \tag{2.17}$$

$$\langle \mathbf{x}|\phi_\nu\rangle \equiv \phi_\nu = \exp\left(i\,\mathbf{p}_\nu \cdot \mathbf{x}\right). \tag{2.18}$$

We can now write down the initial state $|\Psi_i\rangle$, which is just that of the parent nucleus

$$|\Psi_i\rangle = |\psi_i\rangle,$$

and final state $|\Psi_f\rangle$, which is the product of the daughter nucleus state $|\psi_f\rangle$, the electron state $|\phi_e\rangle$ and the anti-neutrino state $|\phi_{\bar{\nu}_e}\rangle$

$$|\Psi_f\rangle = |\psi_f\rangle \times |\phi_e\rangle \times |\phi_{\bar{\nu}_e}\rangle.$$

The matrix element $\mathcal{A}_{fi}$ controls the transition from the initial to the final state

$$\mathcal{A}_{fi} = \langle \Psi_f|\mathcal{A}|\Psi_i\rangle$$

It can be obtained by working in the position representation and recognising that the amplitude G_F associated with the point-like interaction (2.13) should be integrated over the volume of the nucleus,

$$\mathcal{A}_{fi} = \int d^3x\; G_F\; \phi_e^*\, \phi_\nu^*\, \psi_f^*\, \psi_i.$$

The ϕ and ψ terms are the position representations (wave functions) of the four particles, and in the final state are found in complex conjugate form. The integral sums over the amplitudes for the point-like reaction to occur anywhere in the nucleus, since the reaction could have occurred anywhere within.

To perform the integral we first Taylor expand the exponentials in the plane wave functions (2.17)–(2.18) for the electron and the neutrino. The expansion is useful

[4]For a refresher, see appendix 2.B.1

because the exponents $\mathbf{p}\cdot\mathbf{x}$ are small.[5] The product of ϕ_e^* and ϕ_ν^* can therefore be written

$$e^{-i(\mathbf{p}_e+\mathbf{p}_\nu)\cdot\mathbf{x}} \approx 1 - i(\mathbf{p}_e + \mathbf{p}_\nu)\cdot\mathbf{x} + \dots \tag{2.19}$$

Provided that the first term in this expression does not vanish when performing the integral, it can be expected to dominate, and the whole integral can be approximated by

$$\begin{aligned}\mathcal{A}_{fi} &= G_F \int d^3x\ \psi_f^*\,\psi_i \\ &\equiv G_F\ M_{\mathrm{nucl}}\end{aligned}$$

where in the lower line M_{nucl} denotes the overlap integral between the neutron in the parent nucleus and the proton in the daughter nucleus. The size of the quantity M_{nucl} depends on the participating nuclei, and is known as the **nuclear matrix element**.

In some particularly simple cases M_{nucl} can be calculated analytically. In particular, if the initial-state neutron, and the final-state proton happen to inhabit the same state within a nucleus, the overlap integral is maximal, i.e. for those nuclei

$$|M_{\mathrm{nucl}}| = 1.$$

An example of a maximum overlap integral is found for the simplest case of the isolated neutron decay. [6]

To complete the job of calculating Γ we need to find the density of states factor $\frac{dN}{dE_f}$. The density of states for the outgoing electron can be calculated from the density of states inside a box[7],

$$\mathrm{d}N = \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}.$$

Assuming spherical symmetry the angular integrals yield 4π so

$$\mathrm{d}N = \frac{4\pi\ p^2\mathrm{d}p}{(2\pi)^3}.$$

A similar result holds for the neutrino. The states allowed by the daughter nucleus are fixed by total momentum conservation, so provide no further contribution to the density of states. The recoil energy of the heavy daughter nucleus is negligible, so conservation of energy gives

$$E_e + E_\nu = Q,$$

where E_e is the kinetic energy of the electron. Hence the rate of decays that yield electrons with momenta between p_e and $p_e + \mathrm{d}p_e$ is

$$\mathrm{d}\Gamma(p_e) = G_F^2|M_{\mathrm{nucl}}|^2\frac{(Q-E_e)^2}{2\pi^3}p_e^2\mathrm{d}p_e. \tag{2.20}$$

[5]The size of $\mathbf{x}$ is of order the typical nuclear size, i.e. $\sim 10\,\mathrm{fm}$, which in natural units is $10\,\mathrm{fm}/(197\,\mathrm{MeV\,fm}) \sim 10^{-1}$ MeV^{-1}. The typical momenta of the out-going particles are of order MeV, so the dot products in the exponents are of order 10^{-1}.

[6]Such decays are called ‘super-allowed’.

[7]See appendix 2.B.2 for the source of this term.

In the relativistic limit where $E_e \gg m_e$ we can perform the integral and obtain the simple result

$$\boxed{\Gamma_\beta \propto Q^5.}$$

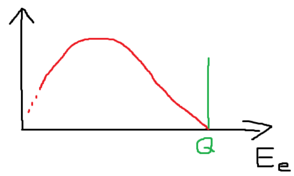
i.e. the rate depends on the fifth power of the available energy.

'Forbidden' decays (We have assumed that the first term in (2.19) will dominate, but it can vanish due to selection rules. For example, if the nuclear matrix element has odd parity the first term vanishes, since then we are integrating the product of an odd and an even function. In that case, the next term in the series is required, and the reaction rate is suppressed. Such decays are said to be 'first forbidden'. In general the larger the change in angular momentum required in the nuclear transition, the further along the series one will need to go to find a non-zero term, and the slower will be the decay.

From the differential rate (2.20), we see that the observed electrons will have energy E_e in the range

$$\boxed{0 < E_e < Q.}$$

The upper bound is when all the available energy is carried by the electron, while the lower one is when all that energy is carried by the unobserved neutrino. The kinetic energy of the heavy recoiling daughter nucleus is assumed to be negligible.



Spectrum of electron kinetic energy expected for beta decays.

2.4.3 Gamma decays

Gamma decays are electromagnetic transitions, and are found when excited nuclear states relax to their ground states.

Similarly to the beta decay case, one can work out the rate using the Fermi golden rule. If one represents the initial nuclear wave-function by Ψ_a and the final nuclear wave-function by Ψ_b, then the appropriate matrix element is found to be

$$\langle \Psi_f | M | \Psi_i \rangle = \int d^3x\, \Psi_b^* (\mathsf{A} \cdot \hat{\mathsf{J}})\, e^{-i\mathbf{k}\cdot\mathbf{x}} \Psi_a$$

where A represents the electromagnetic 4-potential and $\hat{\mathsf{J}} = q\hat{\mathsf{P}}/m$ is the electric 4-current operator. The electromagnetic selection rules and transitions are analogous to those of atomic physics.

MORE

2.5 Shell Model

The SEMF (recall §2.2.1) provides a reasonable description of the binding energies of the nuclei for $A > 30$. However it assumes that the density of states is a continuous function, whereas in a quantum mechanical system the energy states will really take discrete levels. We can expect the continuum approximation to

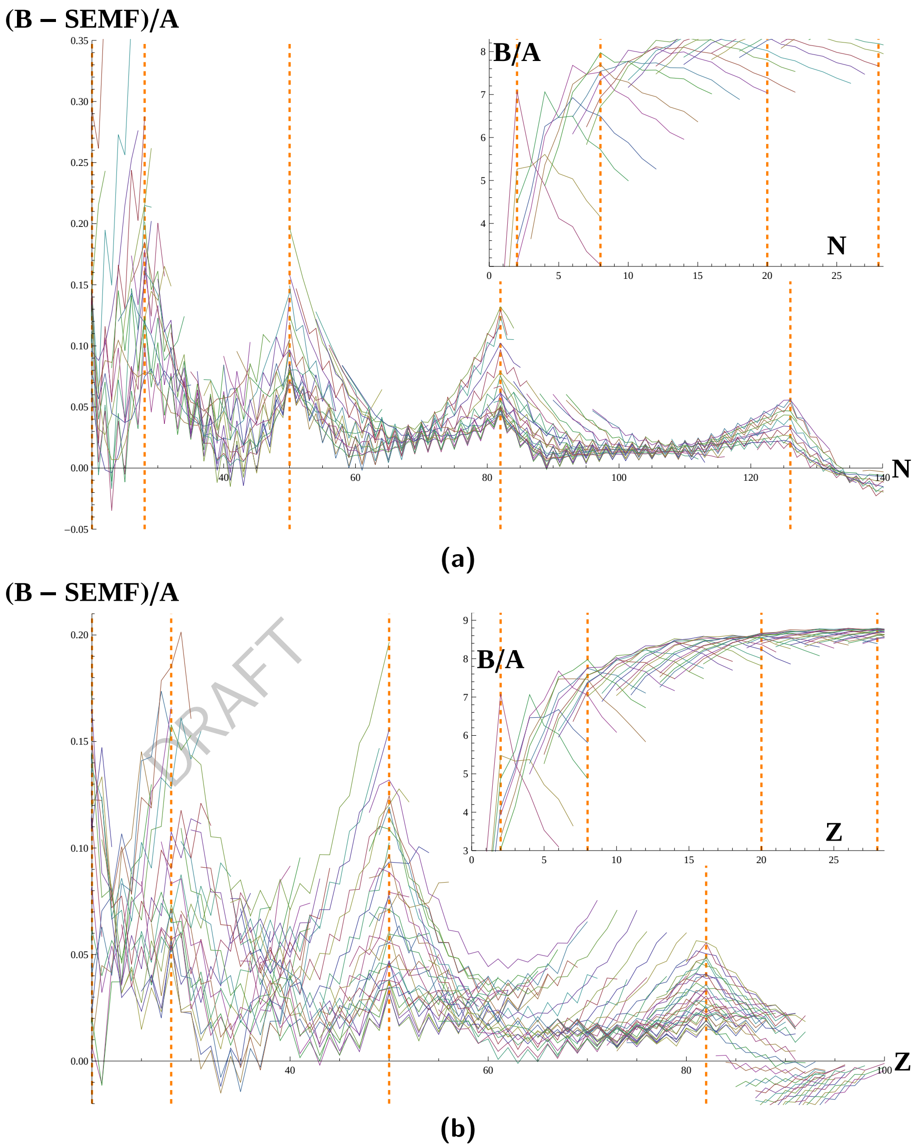
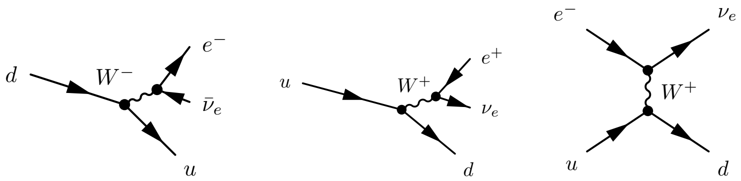


Figure 2.7: Difference between the measured binding energy (per nucleon) and the SEMF prediction. **(a)** The x-axis shows the number of neutrons in the nucleus; curves show isoto**p**es (same Z). **(b)** The x-axis shows the number of protons in the nucleus; curves show isoto**n**es (same N). In both cases the inset shows the binding energy per nucleon for the low-A nuclei. The magic numbers $\{2, 8, 20, 28, 50, 82, 126\}$ are marked with dashed lines.

We will later find that Beta decays are mediated at very small length scales ($\sim 10^{-18}$ m) by charged spin-1 force-carrying particles known as $W^{\pm}$ bosons.



The Feynman diagrams above show the $W^{\pm}$ bosons responsible for β^- decay, β^+ decay, and electron capture. For projectiles with wavelength $\lambda \gg \lambda_{c,W}$, where

$$\lambda_{c,W} = \frac{\hbar}{m_W c}$$

is the W boson Compton wavelength, or equivalently for projectiles with momentum $p \ll m_W$, the small-distance behaviour of the interaction is not apparent. We do not resolve the W boson and instead we get what appears to be a single four-body interaction.

break down, for example, when the number of nucleons is small, or when there are large gaps between energy levels.

Differences from the SEMF at small A (e.g. the tightly bound isotopes ${}^4_2\mathrm{He}$ and ${}^{16}_{8}\mathrm{O}$) are already obvious in Figure 2.1. The discrepancies are shown in more detail in Figure 2.7 which plots the measured binding energy (per nucleon) and the SEMF prediction. We can see that there are regions of particularly high stability — that is with anomalously large B/A — clearly visible. The particular stable nuclei are found around particular special values of N or Z:

$$\{2, 8, 20, 28, 50, 82, 126\}.$$

These are known as the **magic numbers**.

There are other interesting observations about these 'magic number' nuclei. Nuclei with a magic number of protons tend to have a large number of stable isotopes. Similarly those with a magic number of neutrons tend to have a large number of stable istones. Magic nuclei also tend to be more abundant than their neighbours in the periodic table of the nuclei.

In fact these special values are similar to the filled shells of electrons in atomic physics. We shall find that they correspond to configurations of nuclear shells that are exactly filled with either protons or neutrons.

In the next section, we consider more precisely how those discrete energy levels can change the binding energy.

2.5.1 Solution of the infinite spherical well

As a first approximation, and one which we can solve exactly, we can take a non-relativistic quantum mechanical model of a nucleus, treating the nucleons independently. We model the potential as being spherical, and centered on the origin, with the potential

$$V(r) = \begin{cases} V_0 & r \le R_0 \\ \infty & r > R_0 \end{cases}. \tag{2.21}$$

That is the potential takes a constant value everywhere inside the nucleus, and is infinite outside. Now we know this does not correspond exactly to the nuclear potential, which really has a finite potential barrier, and smooth edges rather than infinite cliff edges, but solving it will teach us something interesting, so let's give it a go.

We want to solve the time independent Schrödinger equation

$$H|\psi\rangle = E\psi$$

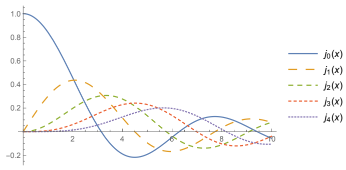
in the position basis, to find the energy levels E corresponding to solutions $\phi(\vec{x}) = \langle \vec{x}|\phi\rangle$. The Hamiltonian for the system of a single particle in a potential $V(\vec{x})$ is

$$H = \frac{-\hbar^2}{2m}\nabla^2 + V(\vec{x}).$$

Our nuclear potential $V(r)$, given by (2.21), means that problem has spherical symmetry, so let's try to solve in spherical polar coordinates (r, θ, ϕ). The solutions $\psi(\vec{x})$ can be separated into the product

$$\psi(\vec{x}) = R(r)\, Y(\theta, \varphi)$$

of a radial function $R(r)$ which depends only on the radial distance, and a second function $Y(\theta, \varphi)$ which depends only on the polar and azimuthal angles θ and φ respectively.


Spherical Bessel functions of the first kind $j_l(u)$

The angular part is solved in a standard way by the spherical harmonics $Y_{l,m}(\theta, \varphi)$. They are labelled by two integer quantum numbers,l and m, where $l \ge 0$ and $-l \le m \le l$.[8]

The radial equation is

$$-\frac{\hbar^2}{2mr^2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(rR(r)) + \frac{\hbar^2 l(l+1)}{2mr^2}R(r) + V(r)R(r) = ER(r).$$

It can be simpified using the substitution

$$k^2 = \frac{2(E - V_0)m}{\hbar^2}.$$

The solutions for the constant potential V_0 found inside our nucleus are the linear combinations

$$R(r) = a_l\, j_l(kr) + b_l\, y_l(kr),$$

where the a_l and b_l are constants, and the functions $j_l(u)$ and $y_l(u)$ known as spherical Bessel functions of the first kind and of the second kind respectively. The first few of these spherical Bessel functions are tabulated below.

[8] See e.g. Binney and Skinner "The physics of Quantum Mechanics".

l	$j_l(u)$	$y_l(u)$
0	$\frac{\sin u}{u}$	$-\frac{\cos u}{u}$
1	$\frac{\sin u}{u^2} - \frac{\cos u}{u}$	$-\frac{\cos u}{u^2} - \frac{\sin u}{u}$
2	$\frac{(3-u^2)\sin u}{u^3} - \frac{3\cos u}{u^2}$	$\frac{(u^2-3)\cos u}{u^3} - \frac{3\sin u}{u^2}$

Table 2.1: The first few Spherical Bessel functions.

The $y_l(u)$ are not finite at $u = 0$ so can't be part of our wave function. [9] That leaves us with the $j_l(u)$. For our potential, with its infinite step outside the nucleus, the wave function must vanish at $r = r_0$. Thus, our radial solutions should be of the form $R(r) = a_l\, j_l(kr)$ where k is chosen such that $j_l(kr_0) = 0$. The j_l are oscillatory, so many such solutions exist.

The solutions are therefore characterised by three numbers: the integers l and m associated with the spherical harmonics, and a further integer n which identifies which root of the Bessel function is used in the radial equation. It's worth taking note that with this infinite spherical well potential (2.21), large l solutions are possible even when n is small. This stands in contrast to the solutions of the Coulomb-like potential $V \propto r^{-1}$ common in atomic physics where $l \leq n$.

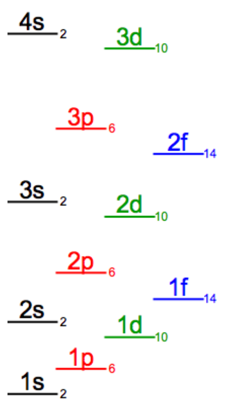
Let's first take the example for no angular momentum, $l = 0$. The roots $y_0(u) = 0$ occur where kr is an integer multiple of π, thought not at $kr = 0$ since $\sin u \to u$ as $u \to 0$. $kr = \pi, 2\pi, \ldots$ and so on. To satisfy that boundary condition we must have that $r_0 k = n\pi$ with n a positive integer. The $E_{n,l,m}$ with no angular momentum ($l = 0$) are therefore

$$E_{n,0,0} = V_0 + \frac{n^2\hbar^2\pi^2}{2mr_0^2}.$$

The solutions with larger l can also be determined from the appropriate zeros of the j_l, but they dont have closed-form solutions. In general it will be the case that

$$E_{n,l,m} = V_0 + \frac{\hbar^2 k_{n,l}^2}{2m},$$

where $k_{n,l}$ is the value of k which provides the nth zero of $j_l(kr_0)$. Part of the spectrum is shown in the margin, using the traditional spectroscopic notation where states with $l = 1, 2, 3$ and 4 are labeled s,p,d and f respectively.



Low energy part of the spectrum of the infinite spherical potential well. The levels are labeled with their spectroscopic labels above, and by their degeneracy to the side.

How many states can we expect at each energy level? The nucleons are spin-half fermions, so there is a spin degeneracy of $2s+1 = 2$ for each spatial state. For each value of n, l there are $2l + 1$ spatial states. So for a particular $E_{n,l}$ the multiplicity of states is $2(2l + 1)$. We would therefore expect the magic numbers of full shells to start with a two for a filled 1s level. Adding the six 1p states give the next magic number of eight. Adding the two 2s and the ten 1d states at the same time, which we might attempt since they have fairly similar energies. would give us a further twelve states bringing the total to 20, the next magic number. So we have a magic number sequence of (2, 8, 20,...). These first few magic numbers are well predicted by this simple model.

[9] More precisely we require the probability density $|\psi|^2 \propto r^2 y_l(kr)^2$ to be finite. For the y_l this can be true only for $y_0(kr)$, but that function suffers from a kink in $|\phi|^2$ at the origin so it can't fit our needs either.

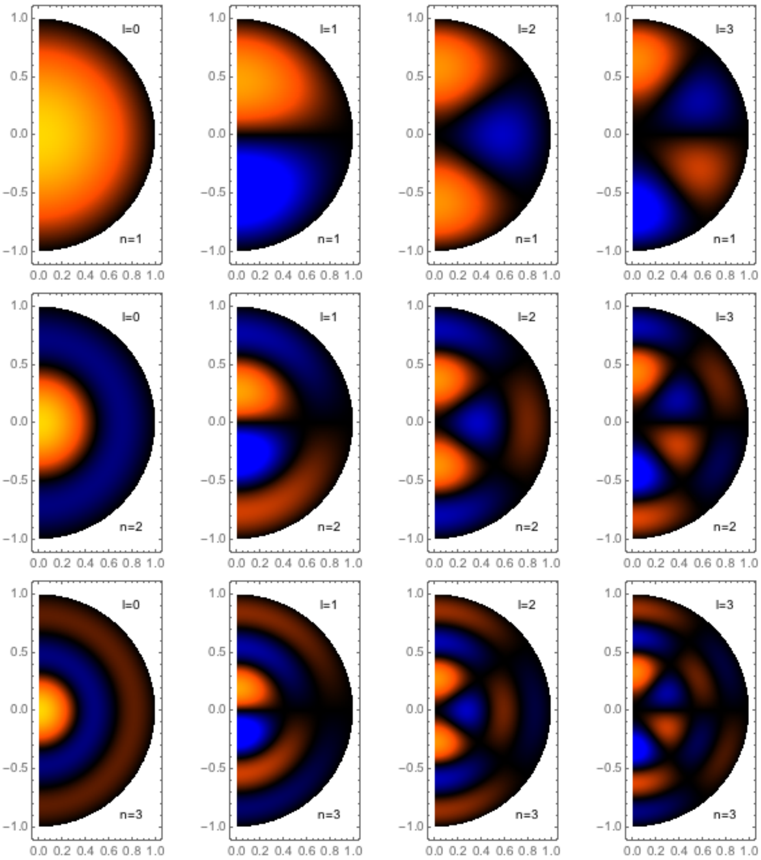


Figure 2.8: The plots show $\psi_{n,l,m}$ for an infinte spherical well. The slices are in the x, z plane at $\varphi = 0$. The quantum number n increases down the column of plots, while and l increasing across rows. In all cases $m = 0$.

2.5.2 Spin orbit interaction

The predictions from our simple model of the nucleus would not keep working. Adding the $1f$ level, which has $l = 3$ would give us a further 14 states, bringing the total to 34. However the next magic number is only 28, so we wanted to add only 8 more states, rather than the 14 predicted by the model.

The naive model has too many degenerate states because it missed the spin-orbit factor in the Hamiltonian

$$H_{\text{S-O}} = V_{\text{S}-\text{O}}(r)\,\mathbf{L}\cdot\mathbf{S}.$$

This term accounts for the interaction between the spin of the nucleons and the field in which they are moving. A similar term arises in atomic physics from the coupling between the electron magnetic moment and the magnetic field resulting from its movement around the nucleus.

We can use the standard result for the quantum mechanical angular momentum operator $\mathbf{J}$ that

$$\mathbf{J}^2 = \mathbf{L}^2 + \mathbf{S}^2 + 2\mathbf{L}\cdot\mathbf{S}$$

Figure 2.9: Shell model prediction of the magic numbers.

to determine the average value of $\mathbf{L}\cdot\mathbf{S}$. For an eigenstate of the $\mathbf{J}^2$, $\mathbf{L}^2$ and $\mathbf{S}^2$ operators,

$$\langle \mathbf{L}\cdot\mathbf{S}\rangle = \frac{1}{2}\big(j(j+1) - l(l+1) - s(s+1)\big).$$

Each nucleon has a fixed value of $s = \frac{1}{2}$, so the possible values of j are $l \pm \frac{1}{2}$. We then obtain

$$\langle \mathbf{L}\cdot\mathbf{S}\rangle = \begin{cases} \frac{1}{2}l & \text{for } \ j = l + \frac{1}{2} \\ -\frac{1}{2}(l+1) & \text{for } \ j = l - \frac{1}{2}. \end{cases}$$

The $j = l \pm \frac{1}{2}$ states therefore receive corrections to their energies which for positive V_{SO} increase the energy of $j = l + \frac{1}{2}$ and decrease the energy of the $j = l - \frac{1}{2}$ states. The required sign of V_{SO} does indeed turn out to be positive.[10]

Returning to the magic numbers it can be seen that the $1f$ states which have $l = 3$ will split into into eight states with $j = 7/2$ and six with $j = 5/2$. It is the additional eight lower-energy $1f_{7/2}$ states which are required to make the next magic number. The remaining magic numbers can be obtained in a similar manner, as can be seen in Figure 2.9.

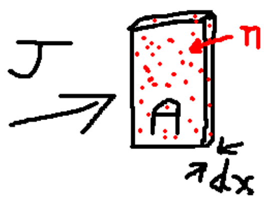
spin and parity of odd-A nuclei

mention more sophisticated techniques for nuclear structure

2.6 Nuclear Scattering

The structure of the nucleus can be probed by scattering projectiles from it. Those projectiles might be protons, electrons, muons, or indeed other nuclei.

2.6.1 Cross sections



Many experiments take the form of scattering a beam of projectiles into a material. Inside the material there are many scattering centres (e.g. nuclei) of a similar type.

Provided that the material is sufficiently thin that the flux is approximately constant within it, the rate of any reaction W_i will be proportional to the flux of incoming projectiles J (number per unit time) the number density of scattering centres n in the material (i.e. the number per unit volume), and the width δx of the material. We can write this as

$$W_i = \sigma_i \, n \, J \, \delta x. \tag{2.22}$$

[10] That this is the opposite sign from that obtained from the Coulomb potential in atomic physics can be attributed to the different functional form of the potential $V(r)$ in the two cases.

The constant of proportionality σ_i has dimensions of area. It is known as the **cross section** for process i and is defined by

$$\boxed{\sigma_i = \frac{W_i}{n\,J\,\delta x}} \tag{2.23}$$

We can get some feeling for why this is a useful quantity if we rewrite (2.22) as

$$W_i = \underbrace{(n\,A\,\delta x)}_{N_{\text{scat}}}\,J\,\underbrace{\frac{\sigma}{A}}_{P_{\text{scatt}}}$$

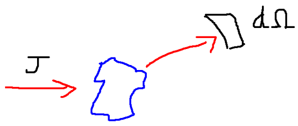
where A is the total area of the material. Here N_{scat} is the total number of scattering centres in the target material. The cross section can be interpreted as the **effective area** presented to the beam per scattering centre for which a particular reaction can be expected to occur. For example if all particles coming within distance d of a nucleus were scattered then the scattering cross section would be pid^2, the area of a disk of radius d, and a fraction $\pi d^2/A$ of the projectiles will be scattered.

When many different scattering processes are available the total rate of loss of beam is given by $W = \Sigma W_i$, and the corresponding total cross section is therefore

$$\sigma = \sum_i \sigma_i .$$

We could choose to quote cross sections in units of e.g. fm^2 or in natural units of GeV^{-2}, however the most common unit used in nuclear and particle physics is the so-called **barn** (b) where

$$\boxed{1\,\text{barn} = 10^{-28}\,\text{m}^2}$$



We can convert the barn to natural units of MeV^{-2} using the $\hbar c$ conversion constant as follows

$$\begin{aligned} 1\,\text{barn} &= 10^{-28}\,\text{m}^2 \\ &= 100\,\text{fm}^2/(197\,\text{MeV fm})^2 \\ &= 0.00257\,\text{MeV}^{-2}. \end{aligned}$$

The *differential cross section* $\frac{d\sigma}{d\Omega}$ is the cross section per unit solid angle of scattered particle. It is defined to be the rate of scattering per target per unit incoming flux density per unit solid angle $(d\Omega)$ of deflected particle.

2.6.2 Resonances and the Breit-Wigner formula

Sometimes the projectile can be absorbed by the nucleus to form a compound state, then later reemitted. The time-energy uncertainty relationship of quantum mechanics tells us that if a state has only a finite lifetime (of order Δt), then it has an uncertaintity on its energy ΔE given by

$$\Delta E \Delta t \sim \hbar .$$

As well as scattering experiments, the size of various nuclei can be determined by other methods, including:

- Shifts in the energy-levels of atomic electrons from the change of their Coulomb potential caused by the finite size of the nucleus.
- Muonic equivalents of the above. Muons are about 200 times heavier than electrons, so their "Bohr raduis" is about 200 times smaller. One observes the series of x-rays from the atomic (muonic) transitions

The decay width can be generalised to a particle which has many different decay modes. The rate of decay into mode i is given Γ_i. The total rate of decay is given by the sum over all possible decay modes

$$\Gamma = \sum_{i=1\ldots n} \Gamma_i.$$

The fraction of particles that decay into final state i, is known as the **branching ratio**

$$\mathcal{B} = \frac{\Gamma_i}{\Gamma}.$$

The quantity Γ_i is known as the **partial width** to final state i, whereas the sum of all partial widths is known as the **total width**.

Using the fact that the decay rate $\Gamma = 1/\tau$, and using natural units to set $\hbar = 1$, we find that

$$\Delta E \sim \Gamma. \tag{2.24}$$

In these units, the uncertainty in the rest-energy of a particle is equal to the rate of its decay. This means that if we take a set of identical unstable particles, and measure the mass of each, we will expect to get a range of values with width of order Γ.

Long-lived intermediate states have small Γ and hence well-defined energies. We tend to think of these reasonably long-lived intermediate states as 'particles'. The neutron is an example of an unstable state that lives long enough for the word 'particle' to be meaningfully applied to it.

Short-lived intermediate states have large widths and less well defined energies. When the intermediate state is so short-lived that its width Γ is similar to its mass, then the decay is so rapid that it is no longer useful to think of it as a particle — it's really some transition through which the state happens to be momentarily passing.

We can develop these ideas more quantitatively by considering the general process

$$A + B \longrightarrow O \longrightarrow C + D. \tag{2.25}$$

The initial particles A and B collide to form an unstable intermediate O, which then decays to the final state C and D (which may or may not have the same particle content as the initial state). An example of a familiar process is the absorption and then emission of a photon by an atom, with an intermediate excited atomic state,

$$A + \gamma \longrightarrow A^* \longrightarrow A + \gamma.$$

Alternatively the reaction could represent an inelastic nuclear interaction, for example the nuclear absorption of a neutron to form a heavier isotope followed by its de-excitation

$$^{25}\mathrm{Mg} + n \longrightarrow {}^{26}\mathrm{Mg}^* \longrightarrow {}^{26}\mathrm{Mg} + \gamma.$$

Other reactions can create and annihilate other types of particle.

The reaction will proceed most rapidly when the energies of the incoming particles are correctly tuned to the mass of the intermediate. The reaction rate will be fastest when the total energy (as measured in the centre-of-mass frame) of $A+B$ is equal to the rest-mass energy E_0 of the intermediate state. The energy need only match E_0 to within the uncertainty Γ in the energy of the intermediate.

Under the condition that the transition from $A+B$ to $C+D$ proceeds **exclusively** via the intermediate state '0', of mass m_0 and that the width of the intermediate is not too large ($\Gamma \ll m_0$), the probability for the scattering process, as a function of total energy E takes the familiar Lorentzian shape

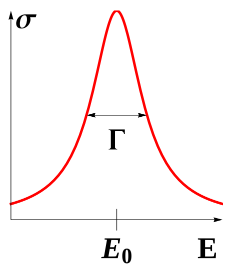
$$p(E) \propto \frac{1}{(E-E_0)^2 + \Gamma^2/4}. \tag{2.26}$$

This is the same peaked shape as seen in resonances in situations involving oscillators, and so the excited intermediate state is often called a **resonance**, and the process is known as resonant scattering.

Derive Breit-Wigner shape

Taking into account density of states and flux factors, and the possibilities of decay into multiple different final states, the overall cross-section for the process (2.25) is given by the **Breit-Wigner** formula (a full derivation of which can be found in Appendix **??**.

$$\boxed{\sigma_{i\to 0\to f} \;=\; g\frac{\pi}{k^2}\frac{\Gamma_i\Gamma_f}{(E-E_0)^2+\Gamma^2/4}.} \tag{2.27}$$



The Breit-Wigner line shape.

Since excited states are very common, this is an important result not just in nuclear and particle physics, but also in any process where excitations are found. The terms in this equation are as follows:

- Γ_i is the *partial width* of the resonance to decay to the initial state $A+B$
- Γ_f is the *partial width* of the resonance to decay to the final state $C+D$
- Γ is the *full width* of the resonance at half-maximum (and equal to the sum $\sum_j \Gamma_j$ over all possible decay modes)
- E is the centre-of-mass energy of the system
- E_0 is the characteristic rest mass energy of the resonance
- k is the wave-number of the incoming projectile in the centre-of-mass frame which is equal to its momentum in natural units.
- $g = \frac{(2J+1)}{(2s_1+2)(2s_2+1)}$ is the probability that the initial particles have the correct angular momentum to form the resonance. J is the spin of the resonance, and s_1 and s_2 are the spins of the two projectiles. This factor averages over the $(2s+1)$ spin states of each of the initial state particles (which are assumed to be unpolarised), and sums over the $(2J+1)$ spin states of the resonant intermediate state.

The cross-section is non-zero at any energy, but has a sharp peak at energies E close to the rest-mass-energy E_0 of the intermediate particle. Longer lived intermediate particles have smaller Γ and hence sharper peaks.

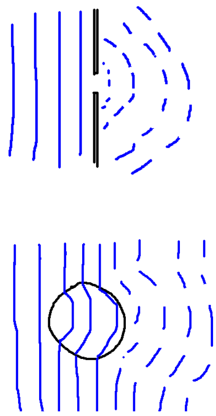
Resonant scattering experiments can tell us about the excited states of nuclei, and hence provide further information about nuclear structure and interactions. All sorts of particles which are too short-lived to travel macroscopic distances can nevertheless be created as intermediate states and studied from the properties of their Breit-Wigner peaks.

2.6.3 Nuclear scattering and form factors

When a target only slightly perturbs the wave-function of the projectile, the resulting scattering behaives rather like optical diffraction.

In optics, the properties of a microscopic aperture can be understood from the pattern obtained when light, of wavelength similar to the size of the aperture, is diffracted by that apperture. Far from the aperture, the optical pattern observed is the two dimensional Fourier transformation of the aperture function. This is true even if the optical aperture is too small to observe directly.

Now consider scattering a wave from a three-dimensional projectile. Again, the observed diffraction pattern comes from a Fourier transform of the object, but now the aperture function is replaced with the potential $V(\mathbf{x}')$.



Diffractive scattering from a slit and from an object.

In the Born approximation, which is valid for weak potentials, the amplitude $f(\Delta\mathbf{k})$ for scattering a projectile such that its change in momentum is $\Delta\mathbf{k}$, is proportional to the 3D Fourier transform of the scattering potential V,

$$f(\Delta\mathbf{k}) = A \int \mathrm{d}^3x' \; V(\mathbf{x}') \; e^{-i\Delta\mathbf{k}\cdot\mathbf{x}'} \tag{2.28}$$

where A is a normalising constant. The probability to scatter into some small angle $\mathrm{d}\Omega$ is then proportional to $|f(\Delta\mathbf{k})|^2$.

Let us consider the scattering of a projectile of charge z from a nucleus with charge Z and spherically symmetric local charge density $\rho(r)$, centred at the origin. The potential at some point $\mathbf{x}'$ is given by summing over the Coulomb potentials from distributed charges at all other locations $\mathbf{x}''$,

$$\begin{aligned} V(\mathbf{x}') &= \frac{ze^2}{4\pi\epsilon_0} \int \mathrm{d}^3x'' \, \frac{\rho(\mathbf{x}'')}{|\mathbf{x}' - \mathbf{x}''|} \\ &= z\alpha \int \mathrm{d}^3x'' \, \frac{\rho(\mathbf{x}'')}{|\mathbf{x}' - \mathbf{x}''|} \end{aligned}$$

where in the second step we again use the relation (valid in natural units) that the electromagnetic fine structure constant $\alpha = \frac{e^2}{4\pi\epsilon_0}$.

Substituting this form of the potential into the Born relation (2.28) we find

$$f(\Delta\mathbf{k}) = z\alpha A \int \mathrm{d}^3x' \int \mathrm{d}^3x'' e^{-i(\Delta\mathbf{k})\cdot\mathbf{x}'} \, \frac{\rho(\mathbf{x}'')}{|\mathbf{x}' - \mathbf{x}''|}.$$

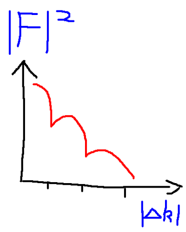
We may simplify this expression by defining a new variable $\mathbf{X} = \mathbf{x}' - \mathbf{x}''$. This change of variables allows us to factorize the two integrals, giving the result

$$f(\Delta\mathbf{k}) = \underbrace{\left[\frac{1}{Z}\int \mathrm{d}^3x''\rho(x'')e^{-i\mathbf{\Delta k}\cdot\mathbf{x}''}\right]}_{\text{Form Factor}} \times Zz\alpha A \underbrace{\left[\int \mathrm{d}^3X \frac{e^{-i\mathbf{\Delta k}\cdot\mathbf{X}}}{|\mathbf{X}|}\right]}_{\text{Rutherford}}. \tag{2.29}$$

The scattering amplitude from a distributed charge is therefore equal to the product of two terms. The second term can be recognised as the Rutherford scattering amplitude – the amplitude that would be obtained from scattering from a point charge density $\rho(\mathbf{x}) = Z\delta(\mathbf{x})$. The second term therefore tells us nothing about the internal structure of the nucleus. All of the interesting information about the nuclear structure is encapsulated in the first term,

$$\boxed{F_{\text{nucl}}(\Delta\mathbf{k}) = \frac{1}{Z}\int d^3x''\ \rho(\mathbf{x}'')\ e^{-i\mathbf{\Delta k}\cdot\mathbf{x}''}}$$

which is known as the **nuclear form factor**. The form factor is the three-dimensional Fourier transform of the normalised charge density $\rho(\mathbf{x})/Z$. All of the interesting information about the size and structure of the nucleus is found in $F_{\text{nucl}}(\Delta\mathbf{k})$. We will find interesting scattering — that is interesting 'diffraction patterns' — if the exponent is of order unity. For this to be true the de Broglie wave-length of the projectile must be of the same order as the nuclear size, as was noted in the introduction to this chapter.



Sketch of a nuclear form factor diffraction pattern.

The mod-squared of the optical amplitude gives the intensity. Similarly it is $|F|^2$ which is important when considering the flux of scattered projectile particles. The rate at which particles are scattered into unit solid angle is given by

$$\frac{dN}{d\Omega} = |F_{\text{nucl}}(|\mathbf{\Delta k}|)|^2 \left(\frac{dN}{d\Omega}\right)_{\text{Rutherford}}. \tag{2.30}$$

This equation is more often written in terms of the differential cross section for scattering

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |F_{\text{nucl}}(|\mathbf{\Delta k}|)|^2 \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Rutherford}}.}$$

By examining the form factor for particles scattered with various changes in momentum $|\Delta\mathbf{k}|$ we can infer information about $\rho(\mathbf{x})$ and hence about the size and shape of the nuclear potential $V(\mathbf{x})$.

2.7 Key points

- In **natural units** (appendix 2.A), $\hbar = c = 1$ and

$$\boxed{[\text{Mass}] = [\text{Energy}] = [\text{Momentum}] = [\text{Time}]^{-1} = [\text{Distance}]^{-1}}$$

A useful conversion constant is

$$\boxed{\hbar c \approx 197 \text{ MeV fm}}$$

- The nuclear mass is well described by the **semi-empirical mass formula**

$$\boxed{M(A,Z) = Zm_p + (A-Z)m_n - \alpha A + \beta A^{\frac{2}{3}} + \gamma \frac{(A-2Z)^2}{A} + \epsilon \frac{Z^2}{A^{\frac{1}{3}}} - \delta(A,Z).}$$

- The binding energy leads to a **valley of stability** in the (A, Z) plane where the stable nuclei lie

- In a reaction or decay, the Q-value is the energy released in a decay

$$\boxed{Q = \sum M_i - \sum M_f}$$

If $Q > 0$ the reaction is **exothermic** – it gives out energy, whereas if $Q < 0$ the reaction is **endothermic**, and energy must be supplied for it to proceed.

- Alpha decay rates are dominated by quantum **tunnelling** through the Coulomb barrier.

- Beta decay rates and electron capture are governed by the **Fermi coupling constant**

$$\boxed{G_F \approx 1.17 \times 10^{-5}\,\text{GeV}^{-2}}$$

- The **cross section** is defined by:

$$\boxed{\sigma_i = \frac{W_i}{n\,J\,\delta x}} \tag{2.31}$$

The **differential cross section** is the cross section per unit solid angle

$$\boxed{\frac{d\sigma_i}{d\Omega}}$$

- Cross sections for sub-atomic physics are often expressed in the unit of **barns**.

$$\boxed{1 \text{ barn} = 10^{-28} \text{ m}^2}$$

- The **Breit Wigner formula** for resonant scattering is

$$\boxed{\sigma_{i\to 0\to f} = \frac{\pi}{k^2}\frac{\Gamma_i\Gamma_f}{(E-E_0)^2 + \Gamma^2/4}.}$$

- In elastic nuclear scattering the **form factor**

$$\boxed{F(|\Delta k|) = \int d^3x \, \left(\frac{\rho(x)}{Z}\right) \, e^{-i\mathbf{\Delta k}\cdot\mathbf{x}},}$$

is the 3D Fourier transform of the normalised charge density, and is related to the Rutherford scattering differential cross section by

$$\boxed{\frac{d\sigma}{d\Omega} = |F_{\mathrm{nucl}}(|\mathbf{\Delta k}|)|^2 \left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Rutherford}}.}$$

Appendix

Appendix 2.A Natural units

In the S.I. system of units, times are measured in seconds and distances in meters. In those units the speed of light takes the value close to $3 \times 10^8\,\mathrm{ms}^{-1}$.

We could instead have chosen to use unit of time such that $c = 1$. For example we could have used units in which time is measured in seconds and distance in light-seconds. In those units the speed of light is one (one light-second per second). Using units in which $c = 1$ allows us to leave c out of our equations (provided we are careful to remember the units we are working in). Such units are useful in relativistic systems, since now the relativistic energy-momentum-mass relations are simplified to

$$\begin{aligned} E &= \gamma m \\ p &= \gamma m v \\ E^2 - p^2 &= m^2. \end{aligned}$$

So for a relativistic system setting $c = 1$ means that energy, mass and momentum all have the same dimensions.

Since we are interested in quantum systems, we can go further and look for units in which $\hbar$ is also equal to one. In such units the energy E of a photon will be equal to its angular frequency ω

$$E = \hbar\omega = \omega.$$

Setting $\hbar = 1$ therefore means that the units of energy are the same as the units of inverse time. Units with $\hbar = 1$ imply that time (and via $c = 1$ distance too) must have the same dimensions as inverse energy, E^{-1}.

So in our system **natural units** with $\hbar = c = 1$ we have have that all of the following dimensions are the same:

$$\boxed{[\mathrm{Mass}] = [\mathrm{Energy}] = [\mathrm{Momentum}] = [\mathrm{Time}]^{-1} = [\mathrm{Distance}]^{-1}}$$

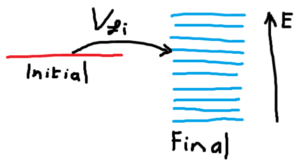
We are still free to choose a convenient unit for all of these quantities. In subatomic physics it is common to use units of energy (or inverse energy). The nuclear energy levels have typical energies of the order of 10^6 electron-volts, so we shall measure energies, momenta and masses in MeV, and lengths and times in MeV^{-1}. At the end of a calculation we might wish to recover, for example, a "real" length from one measured in our MeV^{-1} units. To do so we can make use of the conversion factor

$$\boxed{\hbar c \approx 197\ \mathrm{MeV\,fm}}$$

which tells us that one of our MeV^{-1} length units corresponds to 197 fm where $1\,\mathrm{fm} = 10^{-15}\,\mathrm{m}$.

Appendix 2.B Tools for cross-section calculations

2.B.1 Decays and the Fermi Golden Rule



In subatomic physics we are interested in the decays of unstable particles, such as radioactive nuclei, muons from the atmosphere, or Higgs bosons. Using time-dependent perturbation theory in quantum mechanics it is possible to show that that the transition rate of an unstable state into a continuum of other states is given by the **Fermi Golden Rule**:

$$\boxed{\Gamma = \frac{2\pi}{\hbar} |V_{fi}|^2 \frac{\mathrm{d}N}{\mathrm{d}E_f},} \tag{2.32}$$

where

- Γ is the rate of the decay
- $V_{fi} = \langle f|V|i\rangle$ is the matrix element of the Hamiltonian coupling the initial and the final states, obtained by acting on the initial state with the perturbing Hamiltonian, then finding the inner product of that pertubed state with the final state.
- $\frac{dN}{dE_f}$ is the density of final states per unit energy, which tells us how many states are accessible from the continuum.

A derivation of this result can be found in all good quantum mechanics textbooks.

2.B.2 Density of states

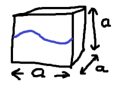
The density of states for a single particle within a cubic box with sides length a can be calculated as follows. The plane wave solution is of form

$$\langle \mathbf{x}|\Psi\rangle \propto \exp\left(i\mathbf{k}\cdot\mathbf{x}\right).$$

If we require periodic boundary conditions, with period a equal to the side of the box, then the values of the wavenumber k_x are constrained to $k_x = 2\pi n/a$ for integer n. Similar conditions hold for k_y and k_z. The number of momentum states within some range of momentum $d^3\mathbf{p} = d^3\mathbf{k}$ (for $\hbar = 1$) is therefore given by

$$dN = \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\mathcal{V}$$

where $\mathcal{V} = a^3$ is the volume of the box.



2.B.3 Fermi G.R. example

Consider the isotropic decay of a neutral spin-0 particle A into two massless daughters, B and C

$$A \longrightarrow B + C.$$

The Fermi G.R. gives the decay rate (in natural units) of A as

$$\Gamma = 2\pi \left|V_{fi}\right|^2 \frac{dN}{dE_f}.$$

The density of final states can be found from the allowed momenta $\mathbf{p}_B$ of particle B.

$$dN = \frac{d^3\mathbf{p}_B}{(2\pi)^3}\mathcal{V}$$

When $\mathbf{p}_B$ is fixed there is no further freedom for $\mathbf{p}_C$ since the sum of the momenta of the two final state particles is fixed by total momentum conservation. This constraint means that for the two body final state there is no additional term in the density of states for $\mathbf{p}_C$.[11] Since all decay angles are equally probable, the integrals over the angles contribute 4π, leading to

$$\Gamma = 2\pi \left|V_{fi}\right|^2 \frac{4\pi\, p_B^2}{(2\pi)^3} \frac{dp_B}{dE_f}\mathcal{V}.$$

The relativistic decay products each have momentum $|\mathbf{p}_B| = E_f/2$ so $\frac{dp_B}{dE_f} = \frac{1}{2}$. Normalising to one unstable particle in our unit volume gives $\mathcal{V} = 1$, and results in a decay rate

$$\begin{aligned}\Gamma &= \frac{1}{2\pi}\left|V_{fi}\right|^2 p_B^2 \\ &= \frac{1}{8\pi}\left|V_{fi}\right|^2 m_A^2.\end{aligned}$$

2.B.4 Lifetimes and decays

The number of particles remaining at time t is governed by the decay law[12]

$$\frac{dN}{dt} = -\Gamma N,$$

[11] For a three-body final state there would be terms in dN of the form $\frac{d^3p}{(2\pi)^3}$ for two of the three particles, the third again being fixed by momentum conservation.

[12] The decay law was discovered experimentally by Frederick Soddy (1877-1956). Soddy, who had been a scholar at Merton, was also first person to understand that radioactivity led to the transmutation of the elements — in effect making him the first true alchemist.

where the constant Γ is the decay rate per nucleus. The equation is easily integrated to give

$$N(t) = N_0 \exp\left(-\Gamma t\right).$$

We can calculate the particles' average proper lifetime τ, using the probability that they decay between time t and $t + \delta t$

$$p(t)\,\delta t = -\frac{1}{N_0}\frac{dN}{dt}\,\delta t = \Gamma \exp\left(-\Gamma t\right)\,\delta t.$$

The mean lifetime is then

$$\begin{aligned}\tau &= \langle t \rangle \\ &= \frac{\int_0^\infty t\,p(t)\,dt}{\int_0^\infty p(t)\,dt} \\ &= \frac{1}{\Gamma}\end{aligned}$$

The decay law can be justified from precise experimental verification. In essence it represents a statement that the decay rate is independant of the history of the nucleus, its method of preparation and its environment. These are often excellent approximations, provided that the nucleus lives long enough that has mass $m \gg \Gamma$ where Γ is its decay width, and provided it it not bombarded with particularly disruptive projectiles, such as high-energy strongly interacting particles.

2.B.5 The flux factor

When calculating a cross section σ from a rate Γ, we need to take into account that for scattering from a single fixed target

$$\sigma = \frac{W}{\mathcal{J}}$$

where $\mathcal{J}$ is the flux density of incoming particles. The flux density is itself given by

$$\mathcal{J} = n_p v$$

where n_p is the number density of projectiles and v is their speed. If we normalise to one incoming particle per unit volume, then $n_p = 1$ and the cross section is simply related to the rate by

$$\sigma = \frac{W}{v}$$

2.B.6 Luminosity

In a **collider** — a machine which collides opposing beams of particles — the rate of any particular reaction will be proportional to the cross section for that reaction and on various other parameters which depend on the machine set-up. Those parameters will include the number of particles in each collding bunch, their spatial distributions, and their frequency of bunch crossings.

We can define a parameter called the **luminosity** $\mathcal{L}$ which encapsulates all the relevant machine parameters. It is related to the rate W and the cross section σ by

$$\mathcal{L} = \frac{W}{\sigma}.$$

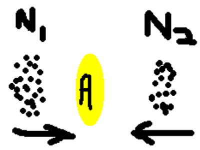
For any collider the luminosity tells us the instantaneous rate of reaction for any cross section. The product of the time-integrated luminosity and the cross section tell us the expected count of the events of that type

$$N_{\mathrm{events,\,i}} = \sigma_i \int \mathcal{L}\, dt\ .$$

For a machine colliding trains of counter-rotating bunches containing N_1 and N_2 particles respectively at a bunch-crossing rate f, we can show that the luminosity is

$$\boxed{\mathcal{L} = \frac{N_1 N_2 f}{A},}$$

where A is the cross-sectional area of each bunch (perpendicular to the beam direction).



We have assumed above that the distributions of particles within each bunch is uniform. If that is not the case (e.g. in most real experiments the beams have approximately Gaussian profiles) then we will have to calculate the effective overlap area A of the bunches by performing an appropriate integral.

Further Reading

- *"An Introduction to Nuclear Physics"*, W. N. Cottingham and D. A. Greenwood, 2001 for the basics
- *"Nuclear Physics"*, M.G. Bowler, Pergamon press, 1973
- *"Introductory Nuclear Physics"*, P.E. Hodgeson, E. Gadioli and E. Gadioli Erba, OUP, 2003
- The BNL table of the nuclides provides good reference data http://www.nndc.bnl.gov/nudat2/.

Bolwer and Hodgeson et. al. are good books which go well beyond the material in this introduction.

Chapter 3

Applications of Nuclear Physics

Chapter 4

Modern Nuclear Physics

Chapter 5

Hadrons

Chapter 6

Non-relativistic scattering theory

Chapter 7

Feynman diagrams

Chapter 8

The Standard Model

Chapter 9

Accelerators and detectors

Appendix A

Examples

A.1 Radioactivity and nuclear stability

Solutions

Three vectors are written in bold e.g. $\mathbf{x}$.
Four vectors are written sans-serif, e.g. P.
The metric is $\mathrm{diag}(1,-1,-1,-1)$ so that with $\mathsf{X}\cdot\mathsf{X} = c^2t^2 - \mathbf{x}\cdot\mathbf{x}$, time-like intervals have positive signs, and propagating particles satisfy $\mathsf{P}\cdot\mathsf{P} = +m^2$.

1.1. a) What assumptions underlie the radioactivity law

$$\frac{dN}{dt} = -\Gamma N \;?$$

b) If some number N_0 of nuclei are present at time $t = 0$ how many are present at some later time t?

c) Calculate the mean life τ of the species.

d) Relate the half-life $t_{\frac{1}{2}}$ to τ.

e) How do things change when the particle moves relativistically?

a) Assumptions: rate of loss depends only on number of number of nuclei, not on environment. Approximately valid because the typical energy scales in the environment are usually much too small to induce transitions.

b) Integrating the differential equation

$$\int_{N_0}^{N} \frac{dN'}{N'} = -\int_0^t \Gamma\, dt'$$

we find the familiar $N = N_0 e^{-\Gamma t}$.

c) The mean life is

$$\tau = \langle t \rangle = \frac{\int t\, P(t)\, dt}{\int P(t)\, dt} = \frac{\int_0^\infty t\, e^{-\Gamma t}\, dt}{\int_0^\infty e^{-\Gamma t}\, dt} = \frac{1}{\Gamma}$$

since the probability of decaying between times t and $t + dt$ is $P(t) = \Gamma e^{-\Gamma t}$.

d) Taking logs:

$$-t/\tau = \ln n/n_0$$

and setting $n/n_0 = \frac{1}{2}$, giving

$$t_{\frac{1}{2}} = \tau \ln 2$$

e) Particles which are travelling relativistically are time dilated by factor γ

$$n(x) = n_0 \exp\left[-\frac{x}{\beta\gamma c\tau}\right].$$

1.2. A sample consists originally of nucleus A only, but subsequently decays according to

$$A \xrightarrow{\Gamma_A} B \xrightarrow{\Gamma_B} C.$$

Write down differential expressions for $\frac{dA}{dt}$, $\frac{dB}{dt}$ and $\frac{dC}{dt}$. Solve for, and then sketch, the fractions of, $A(t)$, $B(t)$ and $C(t)$. At what time is the decay rate of B maximum?

The differential equations are

$$\begin{aligned}\frac{d}{dt}A &= -\Gamma_A A \\ \frac{d}{dt}B &= \Gamma_A A - \Gamma_B B \\ \frac{d}{dt}C &= \Gamma_B B.\end{aligned}$$

The first is not coupled:

$$A = A_0 e^{-\Gamma_A t}.$$

Trying solution $B = \alpha e^{\Gamma_A t} + \beta e^{\Gamma_B t}$ with boundary condition $\alpha + \beta = 0$ and we find $\alpha = \frac{A_0 \Gamma_A}{\Gamma_B - \Gamma_A}$, and $\beta = -\alpha$, so

$$B = \frac{A_0 \Gamma_A}{\Gamma_B - \Gamma_A}\left(e^{-\Gamma_A t} - e^{-\Gamma_B t}\right).$$

We can get C by direct integration of B, with B.C. $B(0) = 0$

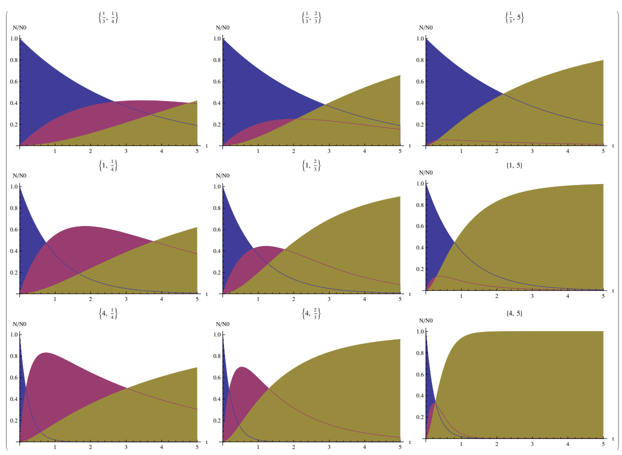
$$C = \int \Gamma_B B \, dt = \frac{A_0}{\Gamma_B - \Gamma_A}\left[\Gamma_A \left(e^{-\Gamma_B t} - 1\right) - \Gamma_B \left(e^{-\Gamma_A t} - 1\right)\right].$$

Decays of B are maximum when B is maximum. We can calculate $\dot{B}$ using the expression for B above, which is zero when

$$0 = \Gamma_A e^{-\Gamma_A t^*} - \Gamma_B e^{-\Gamma_B t^*}.$$

So $t^* = (\Gamma_A - \Gamma_B)^{-1} \ln(\Gamma_A/\Gamma_B)$.

Check for different $\{\Gamma_A, \Gamma_B\}$:



1.3. Consider Compton scattering $\gamma + e^- \to \gamma + e^-$ of a photon of energy E_γ ($\sim$ MeV) from a stationary electron.

a) Show that the energy of the scattered photon is

$$E'_\gamma = \frac{m_e E_\gamma}{m_e + E_\gamma(1-\cos\theta)},$$

where θ is the angle through which the photon is scattered.

b) If an incoming photon is scattered through an angle $\sim 180^\circ$ at the surface of a material, how much of the original photon's energy would you expect *not* to be deposited in the material?

a) Letting the photon momentum be P and the electron momentum be Q, with primes after the scatter. Then energy-momentum conservation is given by

$$\mathsf{P} + \mathsf{Q} = \mathsf{P}' + \mathsf{Q}'$$

To eliminate the components of Q′,

$$\mathsf{Q}'^2 = \left(\mathsf{P} + \mathsf{Q} - \mathsf{P}'\right)^2 \ .$$

The Q^2 and Q'^2 terms cancel on each side, and the P^2 and P'^2 terms are zero. The cross terms must give

$$0 = \mathsf{P}\cdot\mathsf{Q} - \mathsf{P}'\cdot\mathsf{Q} - \mathsf{P}\cdot\mathsf{P}'.$$

Without loss of generality the four-vectors can be expressed as

$$\mathsf{P} = \begin{pmatrix} E \\ E \\ 0 \\ 0 \end{pmatrix} \qquad \mathsf{Q} = \begin{pmatrix} m_e \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathsf{P}' = \begin{pmatrix} E' \\ E'\cos\theta \\ E'\sin\theta \\ 0 \end{pmatrix}.$$

So we have that

$$0 = Em_e - E'm_e - eE'(1-\cos\theta)$$

which can be rearranged into the form given.

b) You don't see the the energy from the scattered photon, so you fail to see $mE/(m+2E)$ of the original energy.

1.4. Briefly explain the origin of each of the terms in the semi-empirical mass formula (SEMF)

$$M(N,Z) = Zm_p + Nm_n - \alpha A + \beta A^{\frac{2}{3}} + \gamma\frac{(N-Z)^2}{A} + \epsilon\frac{Z^2}{A^{\frac{1}{3}}} + \delta(N,Z)$$

and obtain a value for ϵ.

Show that we can include the gravitational interaction between the nucleons by adding a term to the SEMF of the form

$$-\zeta A^{5/3}$$

and find the value of ζ.

Use this modified SEMF to obtain a lower bound on the mass of a gravitationally-bound 'nucleus' consisting only of neutrons (a neutron star).

Obtain value for ϵ by building uniform sphere of charge from spherical shells, out to the nuclear radius of

$$R = r_0 A^{\frac{1}{3}}$$

with $r_0 \approx 1.25$ fm. The shell of width dr at radius r makes a contribution to the energy of

$$dE = \frac{1}{4\pi\epsilon_0 r} Q(r) dQ$$

where $Q(r) = Q_0(r/R)^3$ and

$$dQ = Q_0 \frac{4\pi r^2 dr}{\frac{4}{3}\pi R^3} = 3Q\frac{r^2 dr}{R^3}$$

so that the total electrostatic energy is given by

$$E = \frac{3Q^2}{4\pi\epsilon_0} \int_0^R \frac{r^4}{R^6} dr = \frac{3}{5} \frac{Q^2}{4\pi\epsilon_0 R}$$

Comparing with the SEMF we see that

$$\varepsilon = \frac{3}{5} \frac{e^2}{4\pi\epsilon_0} \frac{1}{r_0} = \frac{3}{5} \frac{1}{137} \frac{197\,\text{MeV fm}}{1.25\,\text{fm}} \approx 0.69\,\text{MeV}.$$

which compares well with the value of 0.71 MeV obtained by fitting the data.

To include the contribution from gravity we need to add an attractive gravitational term calculated in much the same way as the Coulomb term, except with $\frac{Q^2}{4\pi\epsilon_0}$ replaced by $-G_N M^2$, so assuming $m_n = m_p = m_N$, taking $M \approx m_N A$ and $R = r_0 A^{1/3}$ we have

$$\zeta = \frac{3}{5} \frac{G_N m_N^2}{r_0} \approx 5.6 \times 10^{-37}\,\text{MeV}.$$

For a neutron star with $Z = 0$ and $A = N$ we then have binding energy

$$B = \alpha N - \beta N^{2/3} - \gamma N + \zeta N^{5/3}.$$

For large N we can ignore the surface term which has coefficient of the same order as α and γ but is $\propto N^{2/3}$. We have zero binding energy when $B = 0$ i.e. for

$$(\gamma - \alpha)N = \zeta N^{5/3}.$$

Since $\gamma - \alpha \approx (23.2 - 15.8) = 7.4$ MeV we obtain a lower bound on the size of a neutron star

$$N > \left(\frac{\gamma - \alpha}{\zeta}\right)^{3/2} = \left(\frac{7.4}{5.6 \times 10^{-37}}\right)^{3/2} = 4.8 \times 10^{55},$$

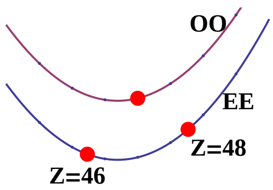
meaning that the mass must be larger than 8.0×10^{28} kg which is $0.06\,\mathrm{M_{sun}}$. Observed neutron stars typically contain 1.3 to 2 solar masses.

1.5. An analysis of a chart showing all stable ($t_{\frac{1}{2}} > 10^9$ years) nuclei shows that there are 177 even-even, 121 even-odd and 8 odd-odd stable nuclei and, for each A, only one, two or three stable isobars. Explain these observations qualitatively using the SEMF. Energetically ${}^{106}_{48}\mathrm{Cd}$ could decay to ${}^{106}_{46}\mathrm{Pd}$ with an energy release of greater than 2 MeV. Why does ${}^{106}_{48}\mathrm{Cd}$ occur naturally?

e-e energetically most favourable, o-o least favourable

Only 2-3 stable nuclei due to asymmetry and Coulomb terms. For const A energy is parabolic in Z, so isobars beta $\pm$ decay towards bottom of parabola.

For $^{106}_{48}\mathrm{Cd}$ to decay requires $\Delta Z = 2$, which must proceed via either (a) two beta decays the first of which is e-e to o-o and is energetically forbidden or (b) a double beta decay which has a very low rate.



1.6. The radius r of a nucleus with mass number A is given by $r = r_0 A^{\frac{1}{3}}$ with $r_0 = 1.2\,\mathrm{fm}$. What does this tell us about the nuclear force?

a) Use the Fermi gas model (assuming $N \approx Z$) to show that the energy ϵ_F of the Fermi level is given by

$$\epsilon_F = \frac{\hbar^2}{2mr_0^2}\left(\frac{9\pi}{8}\right)^{\frac{2}{3}}.$$

b) Estimate the total kinetic energy of the nucleons in an $^{16}\mathrm{O}$ nucleus.

c) For a nucleus with neutron number N and proton number Z the asymmetry term in the semi-empirical mass formula is

$$\frac{\gamma(N-Z)^2}{A}.$$

Assuming that $(N - Z) \ll A$ use the Fermi gas model to justify this form and to estimate the value of γ. Comment on the value obtained.

The proportionality $r \propto A^{1/3}$ tells us that the every nucleon gives the same contribution to the volume ('close-packed'). If the nuclear force extended beyond nearest neighbours then the nucleons in large-A nuclei could stack up in the higher energy levels in a smaller volume. So 'close packing' is evidence in favour of nearest-neighbour forces.

a) In terms of the Fermi momentum p_X the total number of states (including factor of 2 for spin) is

$$X = 2 \times \frac{4\pi}{3}\frac{p_X^3}{(2\pi\hbar)^3}V.$$

Eliminating $V = \frac{4\pi}{3}Ar_0^3$ the Fermi momentum for either species is

$$p_X = \frac{\hbar}{r_0}\left(\frac{9\pi X}{4A}\right)^{\frac{1}{3}}.$$

Assuming $X/A \approx \frac{1}{2}$ and using $\epsilon_F = p_N^2/2m$ we get the result given.

b) The Fermi energy is 33.5 MeV. The density of states $\frac{dN}{dE}$ is $\propto E^{\frac{1}{2}}$, so

$$\langle E \rangle = \frac{\int^{\epsilon_F} E\frac{dN}{dE}\,dE}{\int^{\epsilon_F}\frac{dN}{dE}\,dE} = \frac{\int^{\epsilon_F} E^{\frac{3}{2}}\,dE}{\int^{\epsilon_F} E^{\frac{1}{2}}\,dE} = \frac{3}{5}\epsilon_F.$$

c) The total energy is

$$\begin{aligned} E &= N\langle E_N\rangle + Z\langle E_Z\rangle \\ &= \frac{3}{10m}(Np_N^2 + Zp_Z^2) \\ &= \frac{3}{5}\epsilon_F \frac{N^{\frac{5}{3}} + Z^{\frac{5}{3}}}{(A/2)^{\frac{2}{3}}}. \end{aligned}$$

Assume $Z \approx N$. Let $(Z-N)/A = \delta$ so that $N = A/2(1-\delta)$ and $Z = A/2(1+\delta)$. Expanding for $\delta \ll 1$, $(1\pm x)^P \approx 1 \pm Px + P(P-1)x^2/2!$. The quadratic term (with $P = \frac{5}{3}$) must provide the asymmetry term

$$E_{\rm asym} = \frac{3}{5}\epsilon_F \frac{A}{2}\frac{10}{9}\delta^2 = \frac{1}{3}\epsilon_F A\delta^2 = \frac{1}{3}\epsilon_F \frac{(N-Z)^2}{A}$$

so the asymmetry term prefactor $\gamma = \frac{1}{3}\epsilon_F \approx 11.2$ MeV. This is smaller than the typical value of about 23 MeV because we have not taken into account the change in the depth of the potential.

1.7. Alpha decay rates are determined by the probability of tunnelling through the Coulomb barrier. Draw a diagram of the potential energy $V(r)$ as a function of the distance r between the daughter nucleus and the α particle, and of the wave function $\langle r|\psi\rangle$.

The decay rate can be expressed as $\Gamma = fP$, where f is the frequency of attempts by the alpha particle to escape and $P = \exp(-2G)$ is the probability for the alpha particle to escape on any given attempt.

By using a one-dimensional Hamiltonian

$$H = \frac{p_r^2}{2m} + V,$$

with a 1D momentum operator $p_r = -i\hbar\frac{\partial}{\partial r}$, and by representing the wave function by

$$\langle r|\Psi\rangle \equiv \Psi(r) = \exp[\eta(r)],$$

for some function $\eta(r)$ with $r = x$ show that

$$G = \frac{\sqrt{2m}}{\hbar}\int_a^b \sqrt{V(r) - Q}\,\mathrm{d}r.$$

Since the maximum potential barrier height is usually much larger Q, we can approximate the integrand by neglecting Q inside the integrand. Show that this approximation leads to a transition rate λ of the form $\ln\lambda = C - D/\sqrt{Q}$.

[Hint: can you convince yourself that $\eta'' \ll (\eta')^2$?]

Substituting $\Psi = \exp[\eta(r)]$ into Schrödinger and dividing by $\exp(\eta)$ we find that the $\eta(r)$ must satisfy

$$Q = -\frac{\hbar^2}{2m}\left[\eta'' + (\eta')^2\right] + V(r).$$

We can assume $\eta'' \ll (\eta')^2$ if we are in the semi-classical regime where the potential is "smooth", meaning that there are many e-folds of the wave function before the potential changes significantly. This should be true except close to the nuclear boundary.

If you need to convince yourself of this, consider that for the very low tunnelling probabilities we are dealing with there must be very many e-foldings of the wave-function within the classically forbidden region. We can then see that the potential is approximately constant over each e-folding, so that a e^{-kr} form (with k approximately constant over an *individual* e-folding) is a decent approximation.

The potential is

$$V(r) = \begin{cases} -V_0 & r < r_a \\ \frac{zZ\alpha}{r} & r > r_a. \end{cases}$$

We can estimate the tunnelling probability (neglecting the ∇ in the flux) by simply finding the amplitude for the particle to make its way from the nucleus into the classically allowed region. This amplitude is

$$\frac{|\Psi(r_b)|}{|\Psi(r_a)|} = e^{\eta_b - \eta_a} = e^{-G}$$

where the exponent is given by

$$-G = \eta_b - \eta_a = \sqrt{\frac{2m}{\hbar^2}} \int_{r_a}^{r_b} \sqrt{V(r) - Q}\, \mathrm{d}r$$

The lower bound is the nuclear radius

$$r_a \approx r_0 A^{\frac{1}{3}}$$

and the upper bound is the edge of the classically allowed region

$$r_b = \frac{Zz\alpha}{Q}.$$

Neglecting Q will be reasonable for any reasonably long-lived nucleus, since if $Q \sim V(r_a)$ then tunnelling will occur rapidly, and the nucleus will decay very quickly. With this approximation

$$-G = \sqrt{\frac{2m}{\hbar^2}} \int_{r_a}^{Zz\alpha/Q} \sqrt{\frac{Zz\alpha}{r}}\, \mathrm{d}r$$

which yields

$$-G = 2\sqrt{\frac{2mZz\alpha}{\hbar^2}} \left(\sqrt{\frac{Zz\alpha}{Q}} - r_a \right).$$

The rate $\lambda \propto \exp -2G$ so $\log \lambda \propto C + D/\sqrt{Q}$ as required.

1.8. Complete the integral from the previous question without making the approximation that Q is small. You might wish to consider the substitution $r = r_b \cos^2\theta$ helps. Show that

$$G = \frac{\pi}{2} Zz\alpha \sqrt{\frac{2mc^2}{Q}} \mathcal{F}(r_a/r_b)$$

where the dimensionless function

$$\mathcal{F}(r) = \frac{2}{\pi}\left(\cos^{-1}\sqrt{r} - \sqrt{r(1-r)}\right)$$

lies in the range between 0 and 1, and for small Q approaches 1.

Compare this solution to that from the previous question.

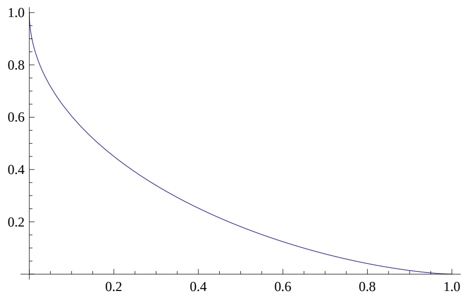
We can rewrite

$$G = \sqrt{\frac{2mQ}{\hbar^2}} \int_{r_a}^{r_b} \sqrt{\frac{r_b}{r} - 1}\,\mathrm{d}r$$

The integral can be solved by substituting $r = r_b \cos^2\theta$ giving

$$\begin{aligned} G &= \sqrt{\frac{2mQ}{\hbar^2}} r_b \int_0^{\theta_a} 2\sin^2\theta\, d\theta \\ &= \sqrt{\frac{2mQ}{\hbar^2}} r_b [\theta_a - \sin\theta_a \cos\theta_a] \end{aligned}$$

which can be rewritten in the form given, with $\mathcal{F}$ taking the form below.



1.9. a) What are the basic assumptions of the Fermi theory of beta decay?

b) The Fermi theory predicts that in a beta decay the rate of electrons emitted with momentum between p and $p + dp$ is given by

$$\frac{d\Gamma}{dp_e} = \frac{2\pi}{\hbar} G^2 |M_{fi}^{\text{nuc}}|^2 \frac{1}{4\pi^4\hbar^6 c^3} (E - Q)^2 p^2,$$

where E is the energy of the electron, and Q is the energy released in the reaction. Justify the form of this result.

c) Show that for $Q \gg m_e c^2$ the total rate is proportional to Q^5

d) What spin states are allowed for the combined system of the electron + neutrino?

e) ‡ Why are transitions between initial and final nuclei with angular momenta differing by more than $\hbar$ suppressed?

[Hint: what form does the Schrödinger equation take for angular momentum quantum number $l \neq 0$?]

a) Point-like interaction, matrix element constant, heavy nucleus, plane-wave solution for electron and neutrino.

b) $\frac{2\pi}{\hbar}$ and $|M_{fi}|^2$ come from Fermi's Golden Rule. The other terms come from the density of states

$$dN_\nu = \frac{4\pi q^2\, dq}{(2\pi\hbar)^3} \qquad dN_e = \frac{4\pi p^2\, dp}{(2\pi\hbar)^3}$$

A heavy recoil nucleus allows any values for p and q without gaining energy itself. Energy conservation gives $E_e = Q - E_\nu$ so the neutrino momentum is $q = (E - Q)/c$.

For a fixed value of dp_e we have that $dE_f = dE_\nu$ so the density of states

$$\frac{dN}{dE_f} = \frac{dN}{dE_\nu} = \frac{(E-Q)^2 p^2\, dp}{4\pi^4 \hbar^6 c^3}$$

giving the desired result.

c) For $E \gg m_e$, we can approximate p by E and dp by dE giving

$$\Gamma = A \int_0^Q (E-Q)^2 E^2\, dE = \frac{A}{30} Q^5.$$

d) The electron and neutrino are each spin-half so can be in a combined $S = 0$ (Fermi) or $S = 1$ (Gamow-Teller) state.

e) The matrix element is

$$M_{fi} = \langle \Psi_f \Psi_\nu \Psi_e | \mathcal{O} | \Psi_i \rangle$$

The leptonic wave functions $\langle \mathbf{r}|\Psi_e\rangle$, $\langle \mathbf{r}|\Psi_\nu\rangle$ are plane waves (only approximately true for the e^-) which we can expand as

$$\langle \mathbf{r}|\Psi\rangle \propto \exp(i\mathbf{k}\cdot\mathbf{r}) = 1 + i\mathbf{k}\cdot\mathbf{r} + \ldots$$

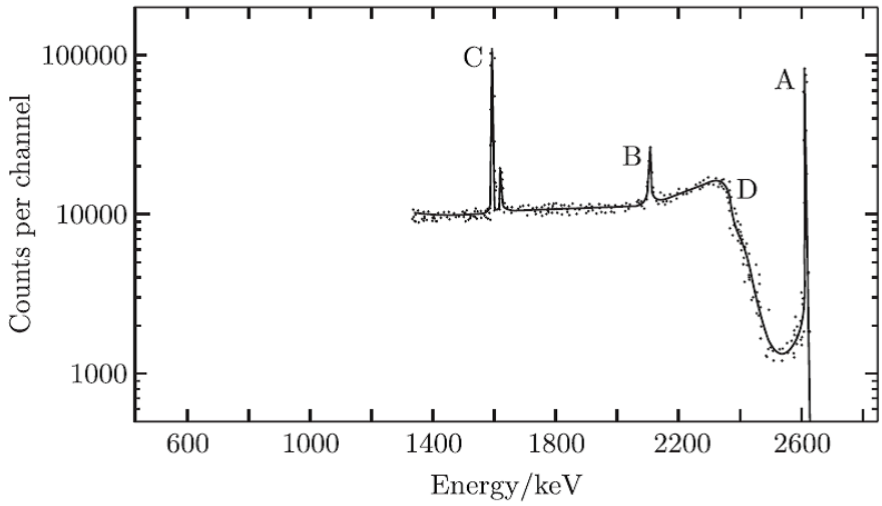
since for typical energies $\mathbf{k}\cdot\mathbf{r} \ll 1$.

'Allowed' transitions have non-zero matrix elements for the first term in this expansion, which is only non-zero for no lepton angular momentum $L = 0$. The decay is 'first forbidden' if the nuclear matrix element is such that the the first term in the expansion is zero and the second $\propto \mathbf{k}\cdot\mathbf{r}$ is non-zero (i.e. if we require lepton $L = 1$). And so on for second, third, ... forbidden transitions. The decay is 'super-allowed' when nuclear matrix elements also have maximal overlap integrals.

1.10. A high energy photon can create an electron-positron pair within the material. When a positron comes to rest it will annihilate against an electron from the material

$$e^+ + e^- \to \gamma + \gamma.$$

What will be the energy of these secondary photons?



The figure shows the energy spectrum from the gamma decay of $^{24}\mathrm{Na}$ as measured in a small Ge(Li) detector. Suggest the origins of the peaks A, B, C and the edge D. For such a detector describe the stages by which gamma ray energy is converted into a measurable voltage pulse.

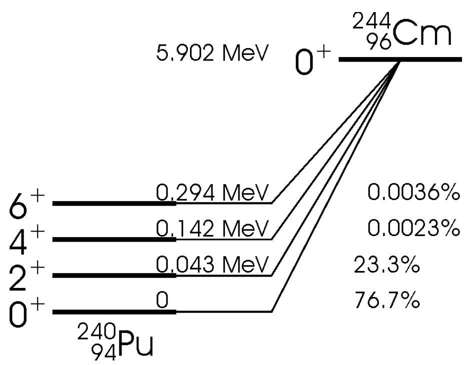
Each photon has 511 keV of energy. In the plot, peak A comes from complete absorption of a photon in the material. Comment that peak width comes mostly from resolution of detector.

All other features come from loss of some fraction of energy from the absorption material. D is the Compton edge from reverse scattering of the photon, near the boundary of the material. Edge occurs at position corresponding to $\theta = \pi$.

Some of the other features result from pair creation followed by annihilation to two 511 keV photons. If one photon escapes then we get peak B. If both escape or the e^+ escapes we get the *small* peak next to C.

The large peak at C itself seems to be a second gamma emission line of $^{24}\mathrm{Na}$ [see spectrum]

1.11. The figure shows the α-decay scheme of $^{244}_{96}\mathrm{Cm}$ and $^{240}_{94}\mathrm{Pu}$.



Show — either by (a) using a suitable approximation of $\mathcal{F}$ calculated in a previous question or (b) by redoing the corresponding integral neglecting Q — that we would expect the rates to satisfy and equation of the form

$$\log \Gamma = A - \frac{BZ}{\sqrt{Q}}.$$

The Q value for the ground state to ground state transition is 5.902 MeV and for this transition $A = 132.8$ and $B = 3.97(\text{ MeV})^{1/2}$ when Γ is in s^{-1}. The branching ratio for this transition is given in the figure. Calculate the mean life of ^{244}Cm.

Estimate the transition rate from the ground state of ^{244}Cm to the 6^+ level of ^{240}Pu using the same A and B and compare to the branching ratio given in the figure.

Suggest a reason for any discrepancy.

For these decays $r_b = Zz\alpha/Q \approx 47$ fm which is much larger nuclear radius $r_a \approx 1.25A^{1/3} = 8$ fm, so we can assume $\mathcal{F} \approx 1$, leaving $G \propto Z/\sqrt{Q}$. (Alternatively redo the integral from the previous question omitting Q.

For the values given, the decay constant for the 0^+ state is

$$\Gamma_{0^+} = \exp\left(132.8 - \frac{3.97 \times 94}{5.902}\right) \text{ s}^{-1}$$

So

$$\tau_{0^+} = 1/\Gamma_0^+ = 1.09 \times 10^9 \text{ s} = 34.6 \text{ years}.$$

However this only accounts for 76.7% of the total rate so the lifetime is

$$\tau = 0.767 \times 34.6 \text{ years} = 26.5 \text{ years}$$

The same values applied to the 6^+ state give

$$\tau_{6^+} \approx 5.8 \times 10^{10} \text{ s}$$

which is $\approx 53 \times \tau_{0^+}$. We'd then expect a branching ratio of $76.7\%/53 = 1.4\%$; however we observe only 0.0036%.

When there is angular momentum we must modify the potential in the 1d Schrödringer equation

$$\frac{Zz\alpha}{r} \to \frac{Zz\alpha}{r} + \frac{\hbar^2}{2mr^2} l(l+1)$$

meaning that the effective potential barrier is larger when $l \neq 0$. This supresses larger δJ transitions beyond what would be expected from the reduced Q value alone.

1.12. Discuss the evidence for shell structure in the atomic nucleus. What is meant by the 'magic numbers', and what values do they take?

Consider a model for the nucleus with a three-dimensional harmonic potential,

$$V(\mathbf{x}) = \tfrac{1}{2}\kappa^2 x^2.$$

By solving the single-particle Schrödinger equation in Cartesian coordinates, or otherwise, show that the degeneracy of the nth energy level is $(n+1)(n+2)$. Calculate the degeneracies of the first five energy levels and compare this model with the magic numbers. How could the model be improved?

1.13. Deduce from the shell model the spins and parities of the ground states of the following nuclei, stating any assumptions you make: ${}^{7}_{3}\mathrm{Li}$, ${}^{17}_{8}\mathrm{O}$, ${}^{20}_{10}\mathrm{Ne}$, ${}^{27}_{13}\mathrm{Al}$, ${}^{14}_{7}\mathrm{N}$, ${}^{39}_{19}\mathrm{K}$, ${}^{41}_{21}\mathrm{Sc}$.

Evidence: abundances; deviation of B/A from SEMF; spins (as below)

Magnetic moments measured from NMR match expectation from unpaired nucleons. Calculate by using Hamiltonian

$$H_\mu = -\boldsymbol{\mu} \cdot \mathbf{B}$$

with $\boldsymbol{\mu} = \boldsymbol{\mu}_L + \boldsymbol{\mu}_S$ with $j = l \pm \frac{1}{2}$. *[See e.g. Cottingham & Greenwood §5.6.]*

From something like the Saxon-Woods potential (rounded off square well), with spin-orbit interation $V_{ls}(r) \propto \mathbf{L} \cdot \mathbf{S}$, the energy levels are ordered such that

$$E(l + \tfrac{1}{2}) < E(l - \tfrac{1}{2})$$

Order of shells is:

$$\underbrace{1s_{\frac{1}{2}}}_{2}\ \underbrace{1p_{\frac{3}{2}}\ 1p_{\frac{1}{2}}}_{+6=8}\ \underbrace{1d_{\frac{5}{2}}\ 2s_{\frac{1}{2}}\ 1d_{\frac{3}{2}}}_{+12=20}\ \underbrace{1f_{\frac{7}{2}}}_{+8=28}\ \underbrace{2p_{\frac{3}{2}}\ 1f_{\frac{5}{2}}\ 2p_{\frac{1}{2}}\ 1g_{\frac{9}{2}}}_{+22=50}$$

The next magic numbers are 82 and 126 (see e.g. Cottingham & Greenwood).

Spin – parities:

${}^{7}_{3}\mathrm{Li}$	p in $1p_{\frac{3}{2}}$	$\frac{3}{2}^-$
${}^{17}_{8}\mathrm{O}$	n in $1d_{\frac{5}{2}}$	$\frac{5}{2}^+$
${}^{20}_{10}\mathrm{Ne}$	even-even	0^+
${}^{27}_{13}\mathrm{Al}$	p in $1d_{\frac{5}{2}}$	$\frac{5}{2}^+$
${}^{14}_{7}\mathrm{N}$	n and p in $1p_{\frac{1}{2}}$	0^+ or $\underline{\mathbf{1}^+}$
${}^{39}_{19}\mathrm{K}$	p in $1d_{\frac{3}{2}}$	$\frac{3}{2}^+$
${}^{41}_{21}\mathrm{Sc}$	p in $1f_{\frac{7}{2}}$	$\frac{7}{2}^-$

A.2 Quarks and scattering

Solutions

2.1. What is meant by the 'cross section' and the 'differential cross section'?

Consider classical Rutherford scattering of a particle with mass m and initial speed v_0 from a potential

$$V(r) = \frac{\alpha}{r}$$

a) Show from geometry that the change in momentum is given by

$$|\Delta \mathbf{p}| = 2p\sin(\Theta/2).$$

b) Considering the symmetry of the problem, show that

$$bv_0 = r^2\frac{d\theta}{dt}$$

where b is the impact parameter, $\mathbf{r}$ is the location of the particle from the origin and θ is the angle $\angle(\mathbf{r}, \mathbf{r}^*)$ where $\mathbf{r}^*$ is the point of closest approach.

c) Starting from Newton's second law show that

$$|\Delta \mathbf{p}| = \frac{2\alpha}{v_0 b}\cos\left(\frac{\Theta}{2}\right).$$

d) Show that the scattering angle Θ is given by

$$\tan(\Theta/2) = \frac{\alpha}{2bT} \tag{A.1}$$

where b is the impact parameter (the closest distance of the projectile to the nucleus if it were to be undeflected) and $T = \frac{p^2}{2m}$ is the initial kinetic energy.

e) Calculate the Rutherford scattering cross section σ for scattering of projectiles by angles greater than $\Theta_{\min}$.

f) Show that the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{16}\left(\frac{\alpha}{T}\right)^2\frac{1}{\sin^4(\Theta/2)}$$

where T is the kinetic energy of the particle.

Why is it not possible to calculate the total cross section for this reaction?

The cross section for an interaction i is given by:

$$\sigma_i = \frac{W_i}{J}$$

where W_i is the rate the interaction i and J is the incoming flux density of projectiles.

The differential cross section for the same reaction

$$\frac{d\sigma_i}{d\Omega} = \frac{1}{J}\frac{dW_i}{d\Omega}$$

where the second term is the rate at which scattered particles impinge on some solid angle $d\Omega$.

a) Conservation of energy gives $p = p'$, after which $|\Delta\mathbf{p}| = 2|\mathbf{p}|\sin(\Theta/2)$ by geometry.

b) Conservation of angular momentum gives $mbv_0 = j = mr^2\frac{d\theta}{dt}$.

c) Newton II gives

$$|\Delta\mathbf{p}| = \int dt\cos\theta\frac{\alpha}{r^2}.$$

We can use the angular momentum relationship to rephrase the integral as

$$\begin{aligned}|\Delta\mathbf{p}| &= \frac{\alpha}{bv_0}\int_{-\frac{\pi}{2}+\frac{\Theta}{2}}^{\frac{\pi}{2}-\frac{\Theta}{2}} d\theta\,\cos\theta \\ &= \frac{2\alpha}{v_0 b}\cos\left(\frac{\Theta}{2}\right)\end{aligned}$$

d) Equating the two values obtained for $|\Delta\mathbf{p}|$ gives

$$\tan\left(\frac{\Theta}{2}\right) = \frac{\alpha}{2bT}.$$

e) Θ decreases monotonically with b so $\sigma(\Theta > \Theta_0) = \sigma(b < b_0) = \pi b_0^2$ with

$$b_0 = \frac{\alpha}{2T}\cot\left(\frac{\Theta}{2}\right).$$

f) $d\Omega = \sin\Theta\,d\Theta\,d\phi$ and from geometry $d\sigma = b\,d\phi\,db$, so

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= -\frac{b}{\sin\Theta}\frac{db}{d\Theta} \\ &= \frac{1}{16}\left(\frac{\alpha}{T}\right)^2\frac{1}{\sin^4(\Theta/2)}\end{aligned}$$

where in the last step we substitute $-\frac{db}{d\Theta} = \frac{\alpha}{4T}\operatorname{cosec}^2(\Theta/2)$ and $\frac{1}{\sin\Theta} = 1/(2\sin(\Theta/2)\cos(\Theta/2)$.

The total cross section diverges as $b \to \infty$, or equivalently as $\Theta \to 0$.

2.2. The $J^P = \frac{3}{2}^+$ decuplet contains the following baryons:

$$\begin{gathered}\Delta^-\ \Delta^0\ \Delta^+\ \Delta^{++} \\ \Sigma^{*-}\ \Sigma^{*0}\ \Sigma^{*+} \\ \Xi^{*-}\ \Xi^{*0} \\ \Omega^-\end{gathered}$$

What is the quark content of each of these baryons?

The constituent quarks have no relative orbital angular momentum. How does the Ω^- baryon state behave under exchange of any pair of quarks? Explain how this is achieved in terms of the space, spin, flavour and colour parts of the state vector.

Account for the absence of sss, ddd and uuu states in the $J^P = \frac{1}{2}^+$ octet.

What is meant by *quark confinement*?

Quarks only found in hadrons (qqq, baryons or $q\bar{q}$ mesons) which are colour singlet states. Existence of sub-structure is inferred from (e.g.)

- structures in hadron multiplets
- high-energy probes scattering off point-like objects (deep inelastic scattering)
- energy levels in quarkonium
- ratio of cross sections of e^+e^- to hadrons compared to $\mu^+\mu-$.

The isospin mutiplets have quark content

$$\begin{aligned}\Delta &= (ddd, ddu, duu, uuu)\\ \Sigma^* &= (dds, uds, uus)\\ \Xi^* &= (dss, uss)\\ \Omega &= (sss)\end{aligned}$$

The Ω state vector can be written as a direct product

$$|\Omega\rangle = |\Psi_{\text{space}}\rangle|\Psi_{\text{spin}}\rangle|\Psi_{\text{flavour}}\rangle|\Psi_{\text{colour}}\rangle.$$

The flavour, spin and space parts of the $|\Omega\rangle$ are symmetric under exchange of any two quark labels. $|\Psi\rangle_{\text{space}}$ is symmetric because all are in spherically symmetric $L = 0$ states. The possible values of $|\Psi\rangle_{\text{spin}}$ with different m_S can found by successive application of $S_- = S_{1-} + S_{2-} + S_{3-}$ to $|\uparrow_1\uparrow_2\uparrow_3\rangle$, so all possible values of $|\Psi\rangle_{\text{spin}}$ are symmetric under exchange of any labels. The colour part is the totally antisymmetric combination

$$|\psi_c\rangle = \left\| \begin{pmatrix} |r_1\rangle & |g_1\rangle & |b_1\rangle \\ |r_2\rangle & |g_2\rangle & |b_2\rangle \\ |r_3\rangle & |g_3\rangle & |b_3\rangle \end{pmatrix} \right\|$$

so the combination satisfies the required antisymmetry under exchange of labels of any pair of identical fermions.

For $J^P = \frac{1}{2}^+$ we still require antisymmetry under exchange of any two quarks. Since again the product $|\Psi_{\text{space}}\rangle|\Psi_{\text{flavour}}\rangle|\Psi_{\text{colour}}\rangle$ is $\mathrm{S} \times \mathrm{S} \times \mathrm{A/S}$ under any exchange, we would require the spin part to be symmetric under any exchange, which isn't possible for total spin $J = \frac{1}{2}$.

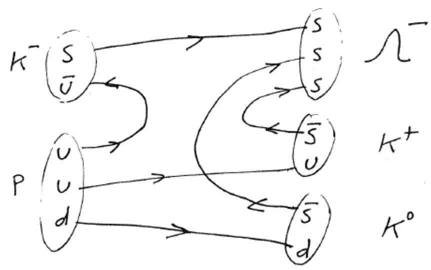
2.3. Write down the valence quark content for each of the different particles in the reactions below and check that the conservation laws of electric charge, flavour, strangeness and baryon number are satisfied throught.

$$\begin{array}{lrcl} (1) & \pi^- + p & \to & K^0 + \Lambda \\ (2) & K^- + p & \to & K^0 + \Xi^0 \\ (3) & \Xi^- + p & \to & \Lambda + \Lambda \\ (4) & K^- + p & \to & K^+ + K^0 + \Omega^- \end{array}$$

Draw a quark flow diagram for the last reaction.

By the magic of the quark model, if we check conservation of flavour, then conservation of electric charge, strangeness and baryon number follow automatically.

(1)	$\pi^- + p$ $d\bar{u} + uud$	$\rightarrow$	$K^0 + \Lambda$ $d\bar{s} + uds$
(2)	$K^- + p$ $s\bar{u} + udd$	$\rightarrow$	$K^0 + \Xi^0$ $d\bar{s} + uss$
(3)	$\Xi^- + p$ $dss + uud$	$\rightarrow$	$\Lambda + \Lambda$ $uds + uds$
(4)	$K^- + p$ $s\bar{u} + uud$	$\rightarrow$	$K^+ + K^0 + \Omega^-$ $u\bar{s} + d\bar{s} + sss$



2.4. Consider the decay of the ρ^0 meson ($J^P = 1^-$) in the following decay modes:

a) $\rho^0 \rightarrow \pi^0 + \gamma$

b) $\rho^0 \rightarrow \pi^+ + \pi^-$

c) $\rho^0 \rightarrow \pi^0 + \pi^0$

For case (b) and (c), draw a diagram to show the quark flow.

Consider the symmetry of the wave-function required for $\pi^0 + \pi^0$ and explain why this decay mode is forbidden.

From consideration of the relative strength of the different fundamental forces, determine which of the other two decay modes will dominate.

Pions are spin-0 so to conserve angular momentum $\pi^0\pi^0$ would have to be in a $L = 1$ state, which has negative parity. However $\pi^0\pi^0$ is a state of identical bosons so must be symmetric under exchange of labels. But in the CM frame the operation of exchange of labels is equivalent to the parity operation. It is not possible to write down an $L = 1$ state which is symmetric under interchange of the identical pions.

The strong decay $\rho^0 \rightarrow \pi^+\pi^-$ dominates. The electromagnetic decay contains of factor of α relative to the strong decay. In fact the branching ratio is 6×10^{-4}, meaning that the form factor must also be important.

2.5. A wave function is modelled as the sum of the incoming plane wave and an outgoing (scattered) spherical wave,

$$\langle \mathbf{x}|\Psi^{(+)}\rangle = A\left[e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{e^{ikr}}{r}f(\mathbf{k}',\mathbf{k})\right].$$

Calculate the flux associated with the plane wave and the spherical wave separately. Hence show that the cross section into solid angle $d\Omega$ is

$$\frac{d\sigma}{d\Omega} = |f(\mathbf{k}',\mathbf{k})|^2 .$$

justifying any assumptions you make.

[Hint: Remember that the flux is given by $\frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*)$*.]*

> The cross section is defined by
> $$W = \sigma n J.$$
> For an outgoing spherical wave, the rate of particles making their way into solid angle $d\Omega$ is
> $$W_{d\Omega} = J_o r^2 d\Omega$$
> where J_o is the outgoing flux at radius r.
>
> We ignore the interference terms since while the incoming wave is *approximately* a plane wave close to the scattering center, but can be assumed to be negligible sufficiently far from the beam.
>
> For the incoming wave $e^{i\mathbf{k}\cdot\mathbf{x}}$ the flux is
> $$J_i = |A|^2\frac{\hbar k}{m}.$$
>
> For the outgoing flux, $\nabla_r = \frac{\partial}{\partial r}$. The derivatives which hit the $1/r$ terms cancel, so
> $$J_o = |A|^2\frac{\hbar k}{mr^2}|f|^2.$$
> The contribution to the cross section from a small part of solid angle $d\Omega$ is the ratio of the outgoing rate of particles scattered into $d\Omega$ to the incident flux. The area over which the outgoing flux should be integrated to get the rate is $r^2 d\Omega$, so
> $$d\sigma = \frac{J_o(r^2 d\Omega)}{J_i} = |f|^2 d\Omega .$$

2.6. In the Born approximation, the scattering amplitude is given by

$$f^{(1)}(\mathbf{k}',\mathbf{k}) = -\frac{1}{4\pi}(2m)(2\pi)^3\langle\mathbf{k}'|V|\mathbf{k}\rangle$$

Explain the terms in this equation, and state the conditions for which it is valid.

b) The Yukawa potential is given by

$$V(r) = \frac{g^2}{4\pi}\frac{e^{-\mu r}}{r}.$$

Show that for this potential

$$\langle \mathbf{k}'|V|\mathbf{k}\rangle = \frac{g^2}{4\pi}\frac{1}{(2\pi)^3}\int d^3x\, e^{-i\mathbf{\Delta k}\cdot\mathbf{x}}\frac{e^{-\mu r}}{r}$$

where $\Delta\mathbf{k} = \mathbf{k}' - \mathbf{k}$.

c) Hence show that

$$|\langle \mathbf{k}'|V|\mathbf{k}\rangle| = \frac{g^2}{(2\pi)^3}\frac{1}{q^2+\mu^2}$$

where $q = |\Delta\mathbf{k}|$.

d) Show that within the Born approximation

$$\frac{d\sigma}{d\Omega} = \frac{g^4}{(4\pi)^2}\frac{4m^2}{\left[2k^2(1-\cos\Theta)+\mu^2\right]^2} \tag{A.2}$$

where Θ is the scattering angle.

e) Find the total cross section for the case when $\mu \neq 0$.

f) When $\mu \to 0$, $V(r) \propto 1/r$. Compare the differential cross section (A.2) to the classical Rutherford scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{16}\left(\frac{\alpha}{T}\right)^2\frac{1}{\sin^4(\Theta/2)}.$$

What must be the relationship between g and α for Born approximation to reproduce the classical Rutherford formula for electron–proton scattering?

a) In the Born approximation the interaction is modelled as a single interaction of the incoming wave with the potential. This is valid provided that the potential is sufficiently weak.
$\mathbf{k}$ is the incoming wave-vector. $\mathbf{k}'$ is a wave-vector with the same magnitude as $\mathbf{k}$ pointing towards the detector. m is the mass of the projectile (or the reduced mass for a two-body system). V is the scattering potential. b) We need the amplitude for the states

$$\langle \mathbf{k}'|V|\mathbf{k}\rangle = \int d^3x \int d^3x'\, \langle \mathbf{k}'|\mathbf{x}'\rangle\langle \mathbf{x}'|V|\mathbf{x}\rangle\langle \mathbf{x}|\mathbf{k}\rangle$$

The conventionally-normalised wave-functions for the states with definite momentum are

$$\begin{aligned}\langle \mathbf{x}|\mathbf{k}\rangle &= (2\pi)^{-\frac{3}{2}}\, e^{+i\mathbf{k}\cdot\mathbf{x}} \\ \langle \mathbf{k}'|\mathbf{x}'\rangle &= (2\pi)^{-\frac{3}{2}}\, e^{-i\mathbf{k}'\cdot\mathbf{x}'}.\end{aligned}$$

and the potential is local so that

$$\langle \mathbf{x}'|V|\mathbf{x}\rangle = \delta^3(\mathbf{x}-\mathbf{x}')V(\mathbf{x}').$$

Using the delta function to do the $\mathbf{x}'$ integral

$$\langle \mathbf{k}'|V|\mathbf{k}\rangle = \frac{g^2}{4\pi}\frac{1}{(2\pi)^3}\int d^3x\, e^{-i\mathbf{\Delta k}\cdot\mathbf{x}}\frac{e^{-\mu r}}{r}$$

where $\Delta\mathbf{k} = \mathbf{k}' - \mathbf{k}$.

c) In spherical polars, with θ relative to the $\mathbf{q}$ direction the integral is given by

$$\begin{aligned}
I &= \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \int_0^\infty dr\, \frac{r^2}{r} e^{-iqr\cos\theta} e^{-\mu r} \\
&= 2\pi \int_0^\infty dr\, \frac{1}{-iq} \left[r e^{-iqr\cos\theta} e^{-\mu r} \right]_{\cos\theta=-1}^{+1} \\
&= \frac{2\pi}{-iq} \int_0^\infty dr\, \left(e^{-iqr-\mu r} - e^{+iqr-\mu r} \right) \\
&= \frac{2\pi}{-iq} \left[\frac{e^{-iqr-\mu r}}{-iq-\mu} - \frac{e^{iqr-\mu r}}{iq-\mu} \right]_{r=0}^{\infty} \\
&= \frac{2\pi}{-iq} \left(\frac{-1}{-iq-\mu} - \frac{-1}{iq-\mu} \right) \\
&= \frac{2\pi}{+iq} \frac{(iq-\mu)-(-iq-\mu)}{(-iq-\mu)(iq-\mu)} \\
&= \frac{2\pi}{iq} \left(\frac{2iq}{q^2+\mu^2} \right) \\
&= \frac{4\pi}{\mu^2+q^2}
\end{aligned}$$

Hence

$$\langle \mathbf{k}'|V|\mathbf{k}\rangle = \frac{g^2}{(2\pi)^3} \frac{1}{\mu^2+\Delta k^2}.$$

d) The scattering amplitude is given by

$$\begin{aligned}
f^{(1)}(\mathbf{k}',\mathbf{k}) &= -\frac{1}{4\pi}(2m)(2\pi)^3 \langle \mathbf{k}'|V|\mathbf{k}\rangle \\
&= -\frac{g^2}{4\pi} \frac{2m}{\mu^2+\Delta k^2}.
\end{aligned}$$

The differential cross section is then

$$\frac{d\sigma}{d\Omega} = |f^{(1)}(\mathbf{k}',\mathbf{k})|^2 = (2m)^2 \left(\frac{g^2/(4\pi)}{\mu^2+\Delta k^2} \right)^2.$$

From geometry

$$\Delta k^2 = (2k\sin(\Theta/2))^2$$

e) The cross section is $\sigma = \int d\Omega\, \frac{d\sigma}{d\Omega}$.

$$\sigma = (2m)^2 \left(\frac{g^2}{4\pi} \right)^2 2\pi \int_0^\pi d\Theta\, \frac{\sin\Theta}{\left(\mu^2 + 4k^2\sin^2(\Theta/2)\right)^2}.$$

The integral can be written

$$\begin{aligned}
I &= \int_0^\pi d\Theta\, \frac{2\sin(\Theta/2)\cos(\Theta/2)}{\left[\mu^2+4k^2\sin^2(\Theta/2)\right]^2} \\
&= \left[\frac{1/(2k)^2}{\mu^2+4k^2\sin^2(\Theta/2)} \right]_0^\pi \\
&= \frac{2}{4k^2\mu^2+\mu^4}.
\end{aligned}$$

Putting this together we get

$$\sigma = \frac{m^2 g^4}{\pi(4k^2\mu^2 + \mu^4)}.$$

f) For massless intermediate particles (such as photons) the total cross section $\sigma \to \infty$. The differential cross section

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= (2m)^2 \left(\frac{g^2/(4\pi)}{4k^2 \sin^2(\Theta/2)} \right)^2 \\ &= \frac{1}{16T^2} \left(\frac{g^2}{4\pi} \right)^2 \frac{1}{\sin^4(\Theta/2)} \end{aligned}$$

Comparing this to the classical cross section for electrostatic scattering we find that

$$\alpha_e = \frac{e^2}{4\pi\epsilon_0} = \frac{g_{\mathrm{EM}}^2}{4\pi}$$

meaning that g_{EM} is a dimensionless measure of the electric charge.

2.7. Consider the scattering of an electron from a nucleus with extended spherical charge density $N(|\mathbf{x}''|)$ which is normalised such that $\int d^3x'' N(|\mathbf{x}''|) = 1$. The potential at any point is then

$$V(x') = zZ\alpha \int d^3x'' \frac{N(|\mathbf{x}''|)}{|\mathbf{x}' - \mathbf{x}''|},$$

where Z and z are the charges of the nucleus and the projectile respectively.

a) By expanding in the position basis, and defining a new variable $\mathbf{X} = \mathbf{x}' - \mathbf{x}''$, show that the Born approximation to the scattering amplitude can be expressed in the form

$$f(\mathbf{k}', \mathbf{k}) = f(\mathbf{k}', \mathbf{k})_{\mathrm{point}} \times F_{\mathrm{nucl}}(\mathbf{\Delta k})$$

where $f(\mathbf{k}', \mathbf{k})_{\mathrm{point}}$ is the scattering amplitude from a point charge,the *nuclear form factor*

$$F_{\mathrm{nucl}}(\mathbf{\Delta k}) = \int d^3x' \, e^{-i\mathbf{\Delta k}\cdot\mathbf{x}'} N(|\mathbf{x}'|)$$

is the 3D Fourier transform of the charge density distribution, and $\mathbf{\Delta k}$ is the change in momentum of the projectile.

Hence show that

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega} \right)_{\mathrm{Rutherford}} |F_{\mathrm{nucl}}(\mathbf{\Delta k})|^2.$$

b) Consider the example form factors in Figure A.1 (i) and (ii). How would these form factors scale along the Δk-axis if the radius r of the corresponding sphere of charge was doubled? By relating the momentum transfer to the scattering angle, use the data to estimate the size of the silver nucleus. Compare to the expectation for an incompressible nucleus, $r = r_0 A^{\frac{1}{3}}$ with $r_0 = 1.25\,\mathrm{fm}$.

c) How might one accelerate protons to kinetic energy of 17 MeV, and subsequently detect the scattered protons experimentally?

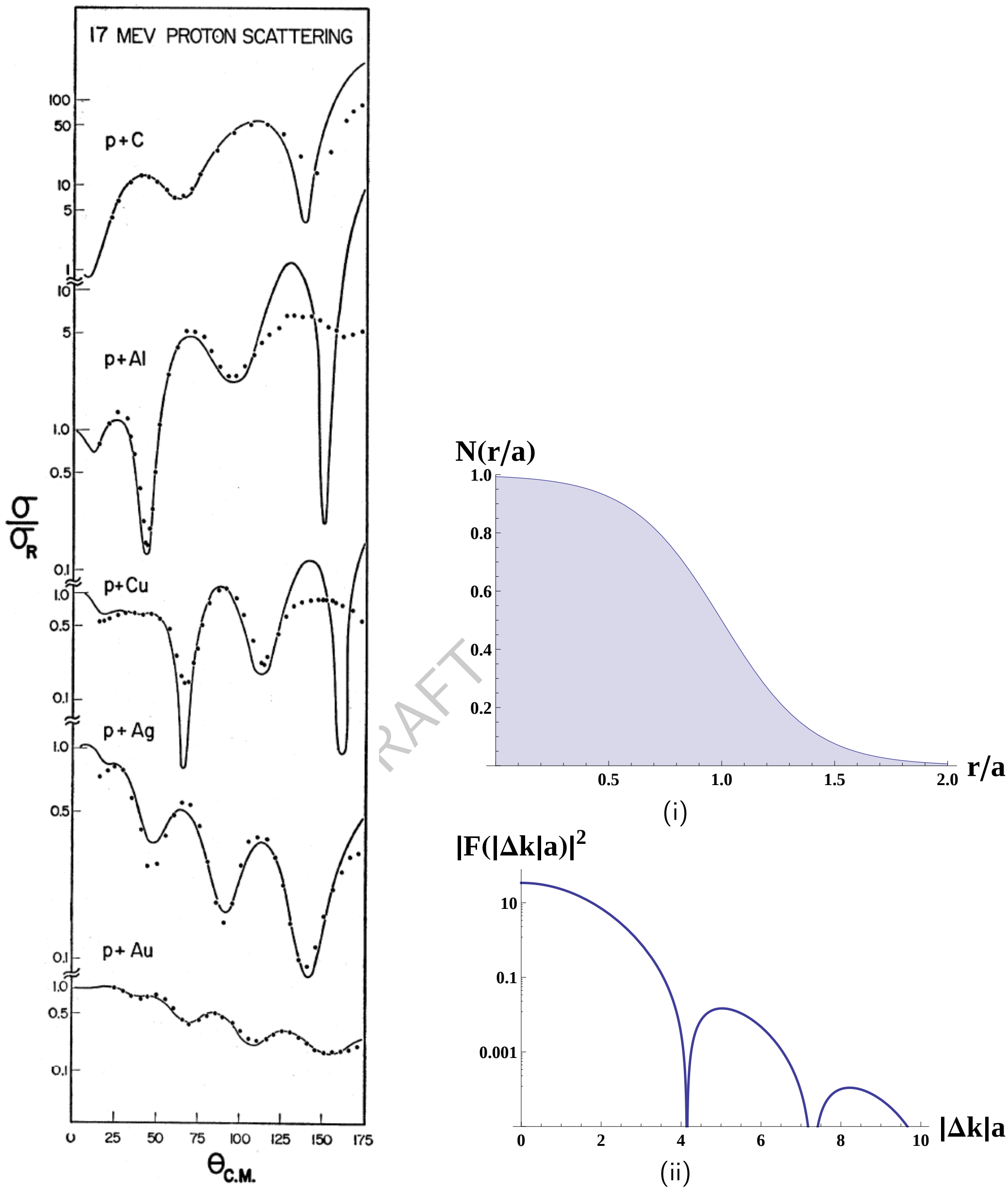


Figure A.1: **(LHS)** Scattering cross section for K.E. = 17 MeV protons normalized to the Rutherford scattering cross section (from [2]).
(RHS) (i) Normalised charge density distribution $\rho(r/a) \propto [1+\exp((r/a-1)/\delta)]^{-1}$ with $R = 1$, $\delta = 0.2$ (This is the so-called Saxon-Woods form [6]) (ii) The corresponding nuclear form factor $|F(|\mathbf{\Delta k}|a)|^2$. The constant a is the approximate radius of the nucleus.

a) Expanding in the position basis,

$$\langle \mathbf{k}'|V|\mathbf{k}\rangle = \frac{1}{(2\pi)^3}\int d^3x' \int d^3x'' e^{-i\mathbf{k}'\cdot\mathbf{x}'} \frac{Z\alpha}{|\mathbf{x}'-\mathbf{x}''|} N(x'') e^{+i\mathbf{k}\cdot\mathbf{x}'}.$$

Defining a new variable $\mathbf{X} = \mathbf{x}' - \mathbf{x}''$ and defining $\mathbf{\Delta k} = \mathbf{k}' - \mathbf{k}$ we have that

$$\begin{aligned}\langle \mathbf{k}'|V|\mathbf{k}\rangle &= \left[\int d^3x'' N(x'') e^{-i\mathbf{\Delta k}\cdot\mathbf{x}''}\right] \times \frac{Z\alpha}{(2\pi)^3}\left[\int d^3X \frac{e^{-i\mathbf{\Delta k}\cdot\mathbf{X}}}{|\mathbf{X}|}\right] \\ &= F_{\mathrm{nucl}}(\mathbf{\Delta k}) \times \langle \mathbf{k}'|V|\mathbf{k}\rangle_{\mathrm{Ruth}}.\end{aligned}$$

The differential cross section ratio is then

$$\left(\frac{d\sigma}{d\Omega}\right) \Big/ \left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Ruth}} = \frac{|\langle \mathbf{k}'|V|\mathbf{k}\rangle|^2}{|\langle \mathbf{k}'|V|\mathbf{k}\rangle_{\mathrm{Ruth}}|^2} = |F_{\mathrm{nucl}}(|\mathbf{\Delta k}|)|^2 .$$

b) The form factor calculation depends on $\mathbf{k}\cdot\mathbf{r}$, so if one doubles the radius of the sphere, the form factor distribution is squeezed into a factor of two smaller in k.

The momentum of the projectile is $p = \sqrt{2m_pE_p} = \sqrt{2\times 938 \times 17} \approx 179\,\mathrm{MeV}/c$. From the plot the short scale structure shows minima for Ag about every 40° between which the momentum transfer $|\Delta k| = 2p\sin\Theta/2 \approx 122\,\mathrm{MeV}/c$. From the comparison plots, $|F(|\Delta k|)|^2$ has minima around $|\Delta k| r_0 \approx 4$ so

$$r_0 \approx \frac{4\hbar c}{|\Delta k|c} = \frac{4\times 197\,\mathrm{MeV\,fm}}{122\,\mathrm{MeV}} = 6.5\,\mathrm{fm}.$$

This rough estimate compares with our expectation $1.25\,\mathrm{fm} \times (108)^{1/3} \approx 5.95\,\mathrm{fm}$.

c) A 17 MeV proton beam can be generated from a cyclotron, or by a linac. The scattered protons could be detected with (for example) a crystal scintillators connected to photomultiplier tubes.

2.8. The cross section for the production of γ-rays by neutrons incident on a certain nucleus (N, Z) is dominated by a resonance and given by the Breit-Wigner formula,

$$\sigma(n,\gamma) = \frac{\pi}{k^2}\frac{\Gamma_n\Gamma_\gamma}{(E-E_0)^2 + \Gamma^2/4}. \tag{A.3}$$

a) Define the symbols in this formula and explain the physical principles that underlie it, and the conditions under which it applies.

b) What are the mass and lifetime of the resonant state?

c) All spin effects have been ignored in (A.3). How would the formula differ if spins are included?

d) On the same plot draw how the inelastic (n,γ) and elastic (n,n) cross sections would behave close to the resonance, when $\Gamma_\gamma = 4\Gamma_n$, labeling important quantities including the peak cross section values.

[In (d) you may assume that resonant scattering dominates both cross-sections, and that decays other than to n and γ are negligible.]

a) σ is the cross section, W/j. k is the wave-number. E is the total energy of the system. E_0 is the resonance energy. $1/\tau = \Gamma = \sum \Gamma_i =$ FWHM provided $\Gamma \ll E_0$. Applies for scattering dominated by transitions through a long-lived intermediate state $A \to B^* \to C$ so that the width is small compared to the mass and also small compared to the energy spacing between resonances.

b) $m_R = E_0^2/c^2$, $\tau = \hbar/\Gamma$

c) When spins are included one must include for $a + b \to R \to b + c$ a factor

$$g = \frac{2J_R + 1}{(2J_a + 1)(2J_b + 1)}$$

which gives the probability that $a + b$ collide in the correct spin state to form the resonance.

d) Relative heights should be in correct proportion. Labels should include E_0, FWHM$=\Gamma$. The maximal cross sections are

$$\sigma_{\max}(n, n) = \frac{\pi}{k^2} 4g(\Gamma_n/\Gamma)^2$$

with $\Gamma_n/\Gamma = 1/5$. The (n, γ) cross section is 4 times larger.

2.9. The cross section for the reaction $\pi^- p \to \pi^0 n$ shows a prominent peak when measured as a function of the π^- energy. The peak corresponds to the Δ resonance which has a mass of 1232 MeV, with $\Gamma = 120$ MeV. The partial widths for the incoming and outgoing states are $\Gamma_i = 40$ MeV, and $\Gamma_f = 80$ MeV respectively for this reaction.

At what pion beam energy will the cross section be maximal for a stationary proton?

Describe and explain the similarities and differences you would expect between the cross section for $\pi^- p \to \pi^0 n$ and the one for $\pi^- p \to \pi^- p$, for centre-of-mass energies not far from 1.2 GeV. Giving values for the variables in the Breit-Wigner formula where possible. Use quark-flow diagrams to explain what is happening.

By considering the quark content of the intermediate states, discuss whether you would expect similar peaks in the cross sections for the reactions (a) $K^- p \to \text{products}$ and (b) $K^+ p \to \text{products}$.

Using the partial widths for the Δ^0 resonance from above, the reaction $\pi^- p \to \pi^- p$ has the same FWHM (since it has the same Γ), same E_0 and *half* the cross section as $\pi^- p \to \pi^0 n$.

At resonance the CM energy must be such that

$$m_\Delta^2 = (\mathbf{P}_\pi + \mathbf{P}_p)^2 = m_\pi^2 + m_p^2 + 2E_\pi m_p$$

So

$$E_\pi = \frac{m_\Delta^2 - m_\pi^2 - m_p^2}{2m_p} = 329.7\,\text{MeV}.$$

Thus the kinetic energy of the pion must be 190.1 MeV.

$K^+ + p$ has quark content $(u\bar{s}) + (uud)$ which fits no baryon. $K^- + p$ has quark content $(s\bar{u}) + (uud) = (uds)$ for which one would expect to find Λ^0 and Σ^0 resonances.

2.10. Consider the Hamiltonian $H = H_0 + V$, where H_0 is the free-particle Hamiltonian, and V is some localised potential. Let the ket $|\phi\rangle$ represent an eigenstate of H_0, and the ket $|\psi\rangle$ represent an eigenstate of H which shares the same energy eigenvalue as the $|\phi\rangle$ in the limit $\epsilon \to 0$.

Show that the Lippmann-Schwinger equation

$$|\psi^{(\pm)}\rangle = |\phi\rangle + \frac{1}{E - H_0 \pm i\epsilon} V |\psi^{(\pm)}\rangle.$$

is consistent with the states defined above by multiplying it by the operator $(E - H_0 \pm i\epsilon)$ and taking the limit $\epsilon \to 0$.

> After multiplying and $\epsilon \to 0$ we find $(E - H_0 - V)|\psi\rangle = (E - H_0)|\phi\rangle$, both of which are null kets by definition.

2.11. Show that the Green's function

$$G_\pm(\mathbf{x}, \mathbf{x}') \equiv \frac{\hbar^2}{2m} \left\langle \mathbf{x} \middle| \frac{1}{E - H_0 \pm i\epsilon} \middle| \mathbf{x}' \right\rangle$$

can be written

$$G_\pm(\mathbf{x}, \mathbf{x}') = \frac{i}{4\pi^2 \Delta} \int_0^\infty dq\, q \frac{e^{iq\Delta} - e^{-iq\Delta}}{q^2 - k^2 \mp i\delta} \tag{A.4}$$

where ϵ and δ are infinitesimals, $E = \frac{\hbar^2 k^2}{2m}$ and $\Delta = |\mathbf{x} - \mathbf{x}'|$.

[Hint: start by inserting identity operators $\int d^3p' |\mathbf{p}'\rangle\langle\mathbf{p}'|$ and $\int d^3p'' |\mathbf{p}''\rangle\langle\mathbf{p}''|$ on each side of the operator, and changing $\hat{H}_0 \to \hat{\mathbf{p}}^2/2m$.]

> After inserting the completeness relations we have to calculate the amplitude
>
> $$\left\langle \mathbf{p}' \middle| \frac{1}{E - (\hat{\mathbf{p}}^2/2m) \pm i\epsilon} \middle| \mathbf{p}'' \right\rangle = \frac{\delta^3(\mathbf{p}' - \mathbf{p}'')}{E - (\mathbf{p}''^2/2m) \pm i\epsilon}$$
>
> The δ^3 allows us to do one of the momentum integrals, leaving
>
> $$G_\pm = \frac{1}{(2\pi)^3} \int d^3q \frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}}{k^2 - q^2 \pm i\epsilon}$$
>
> where $E = \frac{\hbar^2 k^2}{2m}$ and $\mathbf{p}' = \hbar\mathbf{q}$. If we write this out in polar coordinates
>
> $$\frac{1}{(2\pi)^3} \int_0^\infty dq \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) q^2 \frac{e^{iq\Delta\cos\theta}}{k^2 - q^2 \pm i\epsilon},$$
>
> then doing angular integrals we get the desired result.

2.12. If you have done the course on functions of a complex variable, finish the integral in (A.4) using appropriate contour integrals to obtain

$$G_\pm(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi} \frac{e^{\pm ik\Delta}}{\Delta}.$$

The integral from 0 to ∞ is half the integral along the whole real axis

$$G_{\pm}(\mathbf{x},\mathbf{x}') = \frac{i}{8\pi^2\Delta}\int_{-\infty}^{\infty} dq\, q \frac{e^{iq\Delta} - e^{-iq\Delta}}{q^2 - k^2 \mp i\epsilon}.$$

Since ϵ is a vanishingly small number we can rewrite the denominator

$$(q^2 - k^2 \mp i\epsilon) = (q^2 - [k\sqrt{1 \pm i\epsilon/k^2}]^2) \approx (q^2 - [k \pm i\epsilon']^2).$$

The poles in the complex q-plane are therefore at

$$\begin{aligned} q = &\ \pm(k + i\epsilon') \qquad (G_+) \\ q = &\ \pm(k - i\epsilon') \qquad (G_-). \end{aligned}$$

Do G_+ first. Closing the $e^{iq\Delta}$ integral in the upper half-plane we pick up a pole for G_+ at $q = k + i\epsilon'$ with residue $\frac{ke^{ik\Delta}}{2k}$. The $e^{-iq\Delta}$ integral when closed in the lower half-plane gives an equal contribution. Cauchy then tells us that

$$\begin{aligned} G_+(\mathbf{x},\mathbf{x}') &= 2\,\frac{i}{8\pi^2\Delta}(2\pi i)\left(\frac{k}{2k}e^{ik\Delta}\right) \\ &= -\frac{1}{4\pi}\frac{e^{ik\Delta}}{\Delta}. \end{aligned}$$

We can get G_- by making the replacement $k \mapsto -k$.

2.13. The pions can be represented in an isospin triplet $(I = 1)$ while the nucleons form an isospin doublet $(I = \frac{1}{2})$,

$$\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p \\ n \end{pmatrix}$$

while the Δ series of resonances have $I = \frac{3}{2}$.

By assuming that the isospin operators I, I_3, $I_{\pm}$ obey the same alegbra as the quantum mechanical angular momentum operators J, J_z, $J_{\pm}$, explain why the ratio of $\Gamma_i/\Gamma_f \approx \frac{1}{2}$ was found in a previous question.

[Hint: you will need the Clebsch-Gordon coefficients for $\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle$ for $J = \frac{3}{2}$, $j_1 = 1$, $j_2 = \frac{1}{2}$, $M = -\frac{1}{2}$.]

Isospin values for the dramatis personae are

Particle	π^-	p	π^0	n	Δ^0
(I, I_3)	$(1,-1)$	$(\frac{1}{2},\frac{1}{2})$	$(1,0)$	$(\frac{1}{2},-\frac{1}{2})$	$(\frac{3}{2},-\frac{1}{2})$

By analogy with the angular momentum case we shall need the C.G. coefficients for

$$\langle\ j_1 = 1,\, j_2 = \tfrac{1}{2},\, m_1,\, m_2 \mid J = \tfrac{3}{2},\, M = -\tfrac{1}{2},\, j_1,\, j_2\ \rangle$$

Where for $(\pi^-, p) \Rightarrow (m_1 = -1, m_2 = \frac{1}{2})$ the C.G. coefficient is $\sqrt{\frac{1}{3}}$ while for $(\pi^0, n) \Rightarrow (m_1 = 0,\, m_2 = -\frac{1}{2})$ it is $\sqrt{\frac{2}{3}}$. The matrix element for the decay contains these coefficients for each decay type, so the rate contains the mod-squared of the above, hence

$$\frac{\Gamma(\Delta^0 \to p\,\pi^-)}{\Gamma(\Delta^0 \to n\,\pi^0)} = \frac{\left|\sqrt{\frac{1}{3}}\right|^2}{\left|\sqrt{\frac{2}{3}}\right|^2} = \frac{1}{2}.$$

2.14. A proton is travelling through a material and scattering the electrons in the material.

a) Express the scattering angle in terms of the impact parameter b, the reduced mass μ, the relative speed v, and the scattering angle in the ZMF. Hence show that the momentum transfer is

$$q = \frac{2\mu v}{\sqrt{1+z^2}}.$$

where $z = b\mu v^2/\alpha$.

b) Write down the energy given to an electron for a collision for a given impact parameter b. Integrate this up with area element $2\pi b\, db$ to show that the average energy lost by the projectile per distance travelled is

$$-\left\langle \frac{dE}{dx} \right\rangle = \frac{4\pi n_e \alpha^2}{m_e v^2} \int_{z_{\min}}^{z_{\max}} \frac{z\,dz}{1+z^2},$$

where n_e is the number density of electrons.

[Hint: recycle results from the Rutherford scattering question.]

a) From the question on Rutherford scattering, for a two-body system in the ZMF frame, we now have v as the relative velocity and $\mu \approx m_e$ replacing m. Thus we change p to $m_e v$ and T to $\frac{1}{2}m_e v^2$. The angular relationships are then

$$\tan(\Theta^*/2) = \frac{\alpha}{b\mu v^2} = \frac{1}{z}$$

and

$$\sin(\Theta^*/2) = \frac{q}{2\mu v}.$$

where $\mu \approx m_e$ is the reduced mass. Eliminate Θ^* using $\cot^2 + 1 = \operatorname{cosec}^2$,

$$z^2 + 1 = \left(\frac{2\mu v}{q}\right)^2,$$

leaving

$$q = \frac{2\mu v}{\sqrt{1+z^2}}.$$

b) The energy loss by the proton per collision in the lab frame is the energy gained by the electron which is

$$E_e = \frac{q^2}{2m_e}.$$

The average energy loss in a material with electron number density n_e is then

$$\begin{aligned} -\left\langle \frac{dE}{dx} \right\rangle &= n_e \langle \sigma\, \Delta E \rangle \\ &= n_e \int_{b_{\min}}^{b_{\max}} db\, (2\pi b) E_e(b) \\ &= 4\pi n_e m_e v^2 \int_{b_{\min}}^{b_{\max}} \frac{b\, db}{1+z^2} \\ &= \frac{4\pi n_e \alpha^2}{m_e v^2} \int_{z_{\min}}^{z_{\max}} \frac{z\, dz}{1+z^2}. \end{aligned}$$

It's easy to do the integral. The limits can be found from the points where $T_{\min}$ is of the order of the ionisation energy ($z_{\max}$); and at $T_{\max}$ when relativistic effects become important ($z_{\min}$).

2.15. a) A projectile travels through a medium of thickness x with n targets per unit volume. Show that the fraction absorbed or deflected by the medium is

$$P_{\text{absorb}}(x) = 1 - e^{-n\sigma x},$$

where σ is the absorption cross section.

b) Estimate, stating any assumptions you make, the thickness of lead that would be required to have a 50% chance of stopping a 2.3 MeV neutrino coming from a solar nuclear fusion reaction.

The cross section for the scattering of a neutrino from a stationary target is approximately

$$\sigma_{\text{tot}} = 2\pi G_F^2 \frac{4\pi\, p_{\text{CM}}^2}{(2\pi)^3} \frac{dp_{CM}}{dE_{\text{CM}}}$$

where E_{CM} is the centre-of-mass energy of the system, and p_{CM} is the momentum of the neutrino in the centre-of-mass frame.

c) Justify the form of this expression.

d) Explain how Figure A.2 supports a model in which the proton contains point-like constituents.

[The density of lead is about $11.3\,\text{g cm}^{-3}$. Some data for the cross section of neutrinos ν scattering from nucleons – meaning protons or neutrons – are shown in Figure A.2.]

a) The flux $j(x)$ will vary with distance since particles are absorbed. For a thin slice of material with area A at distance x and thickness δx the number of targets is $A\delta x$ and the fraction of the flux absorbed is

$$-\frac{\delta j}{j} = \frac{W_{\text{absorb}}}{W_{\text{arrive}}} = \frac{n(A\,\delta x)\,\sigma j}{jA} = n\sigma\,\delta x.$$

Integrating, $j(x) = j(0)e^{-n\sigma x}$. $\sigma_T/E_\nu \approx 0.7 \times 10^{-38}\,\text{cm}^2/\,\text{GeV}$, so at 2.3 MeV the cross section is about $1.6 \times 10^{-41}\,\text{cm}^2$.

b) We require $j/j_0 = \frac{1}{2}$, so $\sigma n x = \ln 2$. Let's assume the scattering of each nucleon is incoherent. The number density of nucleons in lead is

$$n = \rho/m_N = \frac{11.3\,\text{g cm}^{-3}}{1.67 \times 10^{-24}\text{g}} = 6.7 \times 10^{24}\,\text{cm}^{-3}.$$

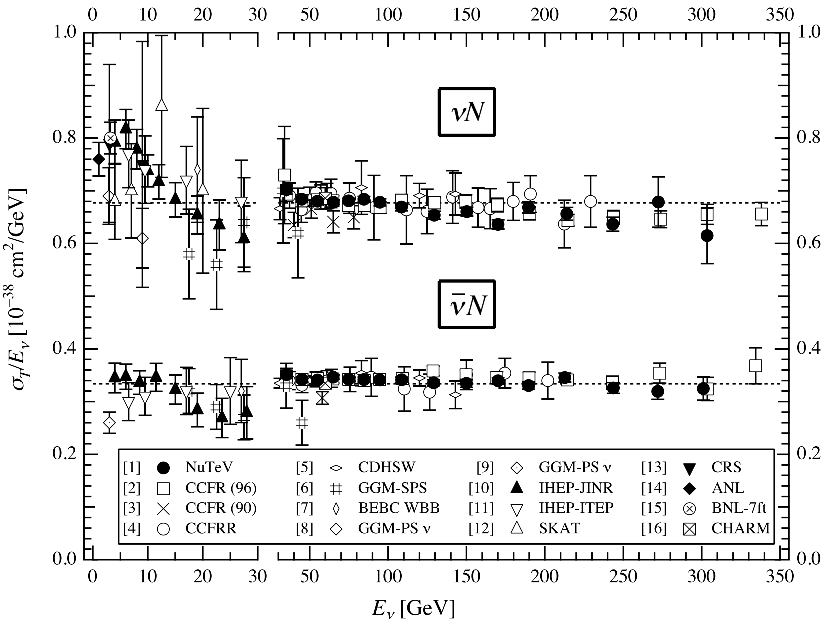


Figure A.2: $\sigma_{\text{tot}}/E_\nu$ for neutrino-nucleon νN (and anti-neutrino–nucleon $\bar{\nu}N$) interactions as a function of neutrino energy. From [4].

The distance x is then

$$\frac{\ln 2}{n\sigma} = \frac{0.69}{6.7\,10^{24} \times 1.6 \times 10^{-41}} = 6.3 \times 10^{15}\,\text{cm},$$

which is about 2.5 light-days.

We have extrapolated over many orders of magnitude. In fact the cross section falls rather faster than $\propto E_\nu$ at $\sim$ MeV energy levels due to nuclear shell effects inhibiting scattering.

c) Leaving aside the relativistic complications: 2π from FGR. Fermi matrix element G_F is squared in rate. Flux factor is unity for particles travelling close to c. Density of states for two-body final state is $\frac{d^3p}{(2\pi)^3} \approx \frac{4\pi p^2}{(2\pi)^3}\frac{dp}{dE}$, measured in CM frame. Subsituting numbers into the formula gives a prediction of 3.9×10^{-39} cm^2/ GeV, which is of the right order, even based on this simplified calculation.

d) Leaving constants aside, we consider $\mathsf{P}_\nu = (E_\nu, E_\nu, 0, 0)$ and $\mathsf{P}_q = x(m_p, 0, 0, 0)$, where x is the fraction of the momentum of the proton carried by the quark. For $E_\nu \gg m_p$ the centre-of-mass energy squared is

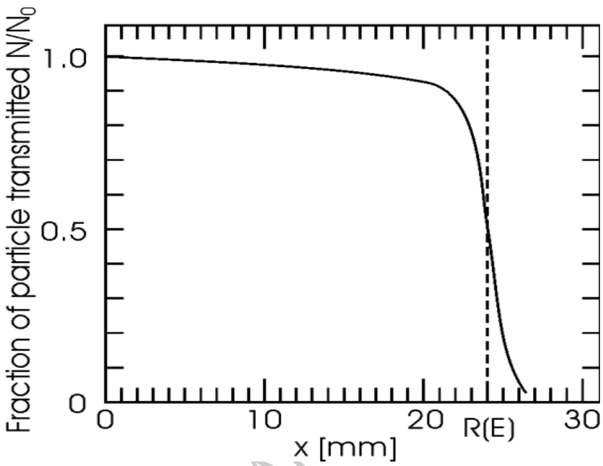
$$s = m_p^2 + 2x\,m_p E_\nu \approx 2x m_p E_\nu$$

and momentum of the neutrino in the centre-of-mass is $p_{\text{CM}} \approx E_{\text{CM}}/2$. We then have that $\sigma \propto E$, as found in the figure.

The absence of any other structure (the trivial form factor $F(q^2) \approx 1$) across the approximate energy range $3\,\text{GeV}^2 < s < 300\,\text{GeV}^2$ shows that the targets are point-like on length-scales of order $\frac{\hbar c}{\sqrt{s}}$. The absence of structure is observed in the length range 10^{-17} m$\to 10^{-16}$ m. If the quarks are not point-like then their typical sizes must therefore be smaller than 10^{-17} m.

In fact scattering of quarks from proton-proton interactions at the LHC further constrains any quark substructure to scales smaller than $\hbar c/\mathrm{TeV} \sim 10^{-19}$ m.

2.16. The figure shows the fraction N/N_0 transmitted when protons of kinetic energy $E = 140$ MeV impinge as a collimated beam on sheets of copper of various thicknesses x. By considering the 2-body kinematics of proton–electron collisions and proton–nucleus collisions, account for the attenuation for values of x between 0 and 20 mm, and for the sudden change in behaviour around the value of x marked $R(E)$.



Results similar to the figure were obtained for protons of $E = 100$ MeV, except that in this case a value of $R(E) = 14$ mm was obtained. Offer a brief explanation for the change in $R(E)$. What is the relative size of the nuclear scattering cross section σ_{Nucl} in copper compared to the geometric cross section?

[The density of copper is $8.9\,\mathrm{g\,cm^{-3}}$*, and it has relative atomic mass* 63.5*. You may assume that the nuclear radius is given by* $r = r_0\, A^{1/3}$ *with* $r_0 = 1.25$ fm *]*

Scattering from electrons will lead to gradual energy loss along path

$$-\left\langle \frac{dE}{dx} \right\rangle \approx 40\,\mathrm{MeV}/10\,\mathrm{mm}.$$

When all the energy is used up the flux stops. Some smearing at the edge since energy loss is a stochastic process.

Scattering from nuclei will lead to large change in momentum of the proton but with low probability (low x part of graph). From the graph the gradient at low x is $\delta N/N_0 \approx 0.05$ in 10 mm so

$$\begin{aligned} \sigma &= \frac{-\delta n}{n}\frac{1}{nl} \\ &= \frac{-\delta n}{n}\frac{m_{\mathrm{Cu}}}{\rho\sigma} \\ &= 0.05 \times \frac{63.5 \times 1.66 \times 10^{-27}\,\mathrm{kg}}{(8.9\,\mathrm{g\,cm^{-3}})(10\,\mathrm{mm})} \\ &= 0.6\,\mathrm{barn}. \end{aligned}$$

The geometric cross section is

$$\sigma_{\mathrm{geom}} = \pi r_0^2 A^{2/3} = \pi(1.25\,\mathsf{fm}^2)(64)^{\frac{2}{3}} = 0.79\,\mathsf{barn}.$$

A.3 Relativistic scattering

Solutions

3.1. The Klein-Gordon equation,

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\varphi(\mathbf{r},t) = 0. \tag{A.5}$$

is the relativistic wave equation for spin-0 particles.

a) Show that

$$\varphi = \frac{e^{-\mu r}}{r}$$

is a valid static-field solution.

b) Show that another possible solution to the Klein-Gordon equation is:

$$\phi(\mathsf{X}) = A\exp\left[i\mathsf{P}\cdot X\right]$$

Where P and X are the momentum and position four-vectors respectively. What restrictions does (A.5) place on the components of P?

What are the physical interpretation of these solutions?

> The radial part of the Laplacian in spherical polars is $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$ so
>
> $$\begin{aligned}\nabla^2\varphi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(-\mu\frac{e^{-\mu r}}{r} - \frac{e^{-\mu r}}{r^2}\right)\right] \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[(-\mu r - 1)e^{-\mu r}\right] \\ &= \frac{1}{r^2}\left[-\mu e^{-\mu r} - \mu(-\mu r - 1)e^{-\mu r}\right] \\ &= \mu^2 e^{-\mu r} = \mu^2\varphi.\end{aligned}$$
>
> The equation shows a possible scalar field configuration for particles of mass μ with a static source at the origin. For example the field of virtual pions around a nucleus. These are *virtual* particles which only travel a short distance. Though we have not calculated it here, these virtual particles do *not* satisfy the usual energy–momentum invariant: $E^2 - |\mathbf{p}|^2 \neq \mu^2$ for these guys.
>
> Expanding out the dot product
>
> $$\phi(\mathsf{X}) = Ae^{i(Et-\mathbf{p}\cdot\mathbf{x})}$$
>
> so $\frac{d^2}{dt^2}\phi = -E^2\phi$, and similarly $\nabla^2\phi = -|\mathbf{p}|^2\phi$. Therefore the K-G equation is satisfied provided $E^2 - |\mathbf{p}|^2 = m^2$. This is the relativistic propagation of a field containing *real* (i.e. non-virtual) particles of mass m, satisfying the usual energy–momentum invariant.

3.2. Draw all the tree-level electromagnetic Feynman diagram(s) for the following processes:

a) $e^- + e^+ \longrightarrow e^- + e^+$

Two diagrams: one for each of t-channel and s-channel photon

b) $e^- + e^- \longrightarrow e^- + e^-$

2 diagrams both with t-channel photon exchange, one with outgoing legs swapped.

c) $e^- + e^- \longrightarrow e^- + e^- + \mu^+ + \mu^-$

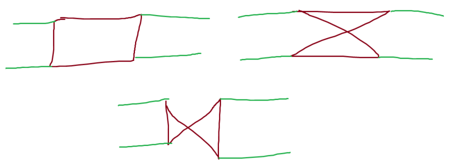
Electrons exchange a photon. Muon pair created from a virtual photon emitted from any of the charged legs (so 4 of them). Diagrams also exist where electrons are swapped.

d) $\gamma \longrightarrow e^+e^-$ in the presence of matter

2 diagrams. Virtual photon from Z attached to either e^+ leg or e^- leg.

e) $\gamma + \gamma \longrightarrow \gamma + \gamma$

Requires an electron loop. Three different topologies:



Two different directions for the electron to flow around each of those loops. Other particles may also enter in the loop. . .

3.3. Why does the ratio

$$\frac{\sigma(e^+ + e^- \to \mu^+ + \mu^-)}{\sigma(e^+ + e^- \to \tau^+ + \tau^-)}$$

tend to unity at high energies? Would you expect the same to be true for

$$\frac{\sigma(e^+ + e^- \to \mu^+ + \mu^-)}{\sigma(e^+ + e^- \to e^+ + e^-)}\ ?$$

Well above threshold the density-of-states factors become the same. The μ and τ production can progress through the s-channel photon channel so have very similar cross sections. The electron scattering cross section also has a t-channel diagram so has a different cross section.

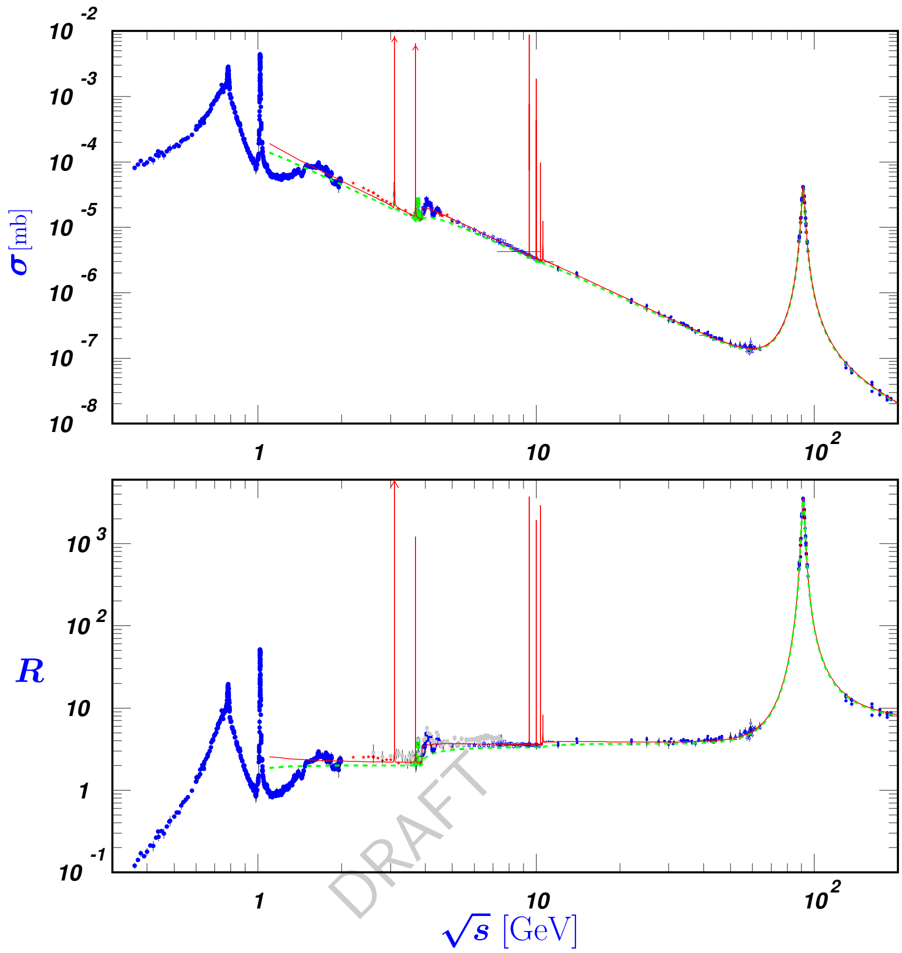


Figure A.1: The cross section $\sigma(e^+e^- \to \text{hadrons})$ and the ratio of cross sections $R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$ as a function of the center of mass energy $\sqrt{s}$.

3.4. Draw leading order electromagnetic Feynman diagrams for the processes

$$e^+ + e^- \to \mu^+ + \mu^- \quad \text{and} \quad e^+ + e^- \to q + \bar{q}.$$

How do the vertex and propagator factors compare?

Figure A.1 shows the ratio of the cross sections for the process of electron–positron annihilation to hadrons, and the corresponding cross section to the muon–antimuon final state as a function of $\sqrt{s}$, the centre-of-mass energy.

Considering the number of quarks that can be created at particular centre-of-mass energy, what values of R would you expect for centre-of-mass energy in the range $2\,\text{GeV} < \sqrt{s} < 20\,\text{GeV}$? How do your predictions match the data? How do these measurements support the existence of quark colour?

What is causing the sharp peaks in σ and R at centre-of-mass energy of $\approx 3\,\text{GeV}$, $10\,\text{GeV}$, and $100\,\text{GeV}$?

From the Feynman diagrams, the ratio is expected to be

$$N_c \sum q_i^2$$

where there are three colours and the sum is over the charges of the kinematically accessible quark states.

Below 3 GeV only u, d, s are accessible and so

$$R = 3\left(\frac{2}{3}^2 + \frac{1}{3}^2 + \frac{1}{3}^2\right) = 2$$

When $c\bar{c}$ becomes accessible above about 3 GeV we find

$$R = 3\left(\frac{2}{3}^2 + \frac{1}{3}^2 + \frac{1}{3}^2 + \frac{2}{3}^2\right) = \frac{10}{3}$$

and when a further $3 \times \frac{1}{3}^2$ is added for $\sqrt{s} > 10\,\text{GeV}$ we get $R = \frac{11}{3}$, all of which predictions (green lines on the plot) match the data rather well.

The peaks are for $c\bar{c}$, $b\bar{b}$ bound state resonances at $3\,\text{GeV}$ and $10\,\text{GeV}$, and for Z^0 at 92 GeV. The lower mass peaks come from the ω and ϕ mesons.

3.5. At the HERA collider $27\,\text{GeV}$ positrons collided with $920\,\text{GeV}$ protons. Why can these collisions can be considered to be due to positrons scattering off the quarks in the protons?

For these collisions draw **one** example of a Feynman diagram for **each** of the cases of weak charged-current, weak neutral-current and electromagnetic interaction.

Calculate the center-of-mass energy of the quark–positron system assuming that the 4-momentum of the quark P_q can be represented as a fixed fraction f of the proton 4-momentum P_p, in the approximation where both particles are massless.

What is the highest-mass particle that can be produced in such a collision in the approximation that a quark carries about $\frac{1}{3}$ of the proton momentum?

We can represent the quark 4-momentum by

$$\mathsf{P}_q = f\mathsf{P}_p = fE_p(1,\,0,\,0,\,1)$$

and the positron 4-momentum by

$$\mathsf{P}_e = E_e(1,\,0,\,0,\,-1).$$

The CM energy is then:

$$E_{\text{CM}}^2 = (\mathsf{P}_q + \mathsf{P}_e)^2 = \mathsf{P}_q^2 + \mathsf{P}_e^2 + 2\mathsf{P}_q \cdot \mathsf{P}_e \approx 0 + 0 + 4fE_pE_e,$$

so

$$E_{\text{CM}} = 2\sqrt{fE_pE_e}.$$

Assuming that the quark carries $\frac{1}{3}$ of the momentum of the proton the CM energy of the positron–quark system is so $E_{\text{CM}} = 2\sqrt{\frac{1}{3}E_pE_e} = 182\,\text{GeV}$ which would be sufficient to produce top quarks (via charge current interactions with virtual b quarks in the proton sea).

How does the propagator for the weak charged current and electromagnetic interactions vary with 4-momentum transfer P^2? Hence explain the fact that at low values of the momentum transfer it is found that the ratio of weak interactions to electromagnetic interactions is very small whereas at very high values it is found that the ratio is of the order of unity.

The positrons behave as if they scatter off individual quarks provided their de Broglie wavelength is much smaller than the size of the proton. Here we have 27 GeV positrons with resolution of order $\hbar c/p \approx 197\,\text{MeV fm}/27GeV$ which is a factor of about 140 smaller than the radius of the proton. One ought to do the calculation in the ZMF, but it is clear that the probe is going to be much smaller than the proton.
The propagators are

$$W : \frac{1}{\mathsf{P}_W^2 - m_W^2} \qquad \gamma : \frac{1}{\mathsf{P}_\gamma^2}$$

As with many texts we write the negative of the square of the momentum transfer $-\mathsf{P}^2 = Q^2$.

At low energy where the momentum transfer $-\mathsf{P}^2 = Q^2 \ll m_W^2$ the W propagator is suppressed relative to the electromagnetic one by a factor $\sim Q^2/m_W^2$. At large energies when $m_W \ll Q^2$ the term m_W^2 can be neglected and the relative cross sections are determined by relative coupling constants which are of order unity.

3.6. The J/Ψ has mass 3097 MeV, width 87 keV and equal branching ratios of 6% to $e^+ + e^-$ and $\mu^+ + \mu^-$ final states. What would you expect for these branching ratios if the J/Ψ decayed only electromagnetically? What does this tell you about the "strength" of the strong interaction in this decay? For comparison, the Ψ'' has mass 3770 MeV, width 24 MeV, but branching ratio to $e^+ + e^-$ of 10^{-5}.

Draw diagrams for the decays $D^0 \to K^- + \pi^+$ and $D^0 \to K^- + e^+ + \nu_e$. Disregarding the differences in the 2-body and 3-body density of states factors, what do you expect for the relative rates of these decays?

If J/Ψ decayed electromagnetically only then the expected final states, and their relative amplitudes and density-of-states factors are

	$u\bar{u}$	$d\bar{d}$	$s\bar{s}$	e^+e^-	$\mu^+\mu^-$
Q^2	$\left(\frac{2}{3}\right)^2$	$\left(\frac{1}{3}\right)^2$	$\left(\frac{1}{3}\right)^2$	1	1
DoS	3	3	3	1	1
Ratio	$\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	1

The factors of three in the quark density of states come from colour factors.

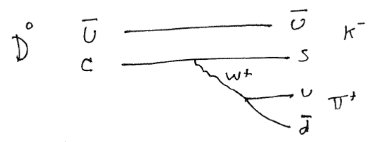
The sum of the above is 4, so the expected BR to ee (or $\mu\mu$) would be $\frac{1}{4} = 25\%$ for pure electromagnetic decays.

If the strong force was unsuppressed, we would expect a branching ratio to leptons of order

$$\frac{|g_{\mathrm{EM}}|^4}{|g_{\mathrm{S}}|^4}$$

from the vertex factors, which is of order 10^{-4}. This is close to the value observed for the $\Psi''(3770)$.

However the observed branching ratio is 6%, which is smaller than the value of 25% that we would expect in the absence of strong interactions, but larger than the value of $\sim 10^{-4}$ we would expect if strong interactions are turned on at their full strength. So we conclude that the strong interaction is present, but suppressed (known as 'OZI' suppression).



In the decay of the D^0 a colour factor should lead to enhancement of a factor of N_C (three) in the density of states for the pion case. However since the quark and anti-quark are in a colour-singlet object in the final state, it's not a good approximation to say that the quarks are free particles which can have any colour. The amplitude to find each colour-anticolour pair inside the pion will be $1/\sqrt{3}$, since the colour part of the meson state vector is $1/sqrt3\left(|r\bar{r}\rangle + |g\bar{g}\rangle + |b\bar{b}\rangle\right)$. Thus the net effect of colour is to add a factor of N_C from the d.o.s and a factor of $1/\sqrt{3}$ to the amplitude, hence to multiply the rate by

$$\mathrm{N}_C \times \left|1/\sqrt{N_C}\right|^2 = 1.$$

There is also a factor of $V_{ud}^2 \approx (0.97^2)$ from the CKM element in the decay to the pion, but this is fairly close to unity.

Looking at the data, the PDG gives

$$\mathcal{B}(D^0 \to K^- + \pi^+) = 3.88\%$$
$$\mathcal{B}(D^0 \to K^- + e^+ + \nu_e) = 3.55\%$$

which is close to 1:1 as predicted.

3.7. By conserving momentum at each vertex in the centre-of-mass frame (or otherwise) determine whether the propagator momentum is space-like ($\mathrm{P}^2 < m^2$) or time-like ($\mathrm{P}^2 > m^2$) for (i) $e^- + \mu^- \to e^- + \mu^-$ and for (ii) $e^+e^- \to \mu^+ + \mu^-$.

(i) For the elastic t-channel scattering in the CM frame no energy is exchanged, but momentum is exchanged so $|\mathbf{p}_\gamma| > 0$, but $E_\gamma = 0$ so the propagator is space-like.
(ii) For the s-channel process in the CM frame $\mathbf{p}_\gamma = 0$, but $E_\gamma > 0$ so the propagator is time-like.

3.8. Draw Feynman diagrams showing a significant decay mode of each of the following particles:
a) π^0 meson
b) π^+ meson
c) μ^-
d) τ^- to a final state containing hadrons
e) K^0

f) top quark

> Suggest:
> a) $\pi^0 \to \gamma + \gamma$
> b) $\pi^+ \to \mu^+ + \nu_\mu$
> c) $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$
> d) $\tau^- \to \pi^- + \nu_\tau$
> e) $K^0 \to \pi^+ + \pi^-$
> f) $t \to b + W^+$

A.4 The Standard Model

Solutions

4.1. a) Draw Feynman diagrams for the the production of $W^\pm$ bosons being produced in a $p\bar{p}$ collider. If the W^+ boson is close to its Breit-Wigner peak, what possible decays may it have? (Which final states are kinematically accessible?)

b) What fraction of W^+ decays would you expect to produce positrons?

c) Suggest why the W was discovered in the leptonic rather than hadronic decay channels.

d) How could the outgoing (anti-)electron momentum be determined? How might the components of the neutrino momentum perpendicular to the beam be determined?

a) Production is dominantly via valence quarks

$$u\bar{d} \to W^+ \quad \text{and} \quad d\bar{u} \to W^-$$

b) Kinematically accessible decays for W^+ are:

Final state	vertex factor	colour factor
$e^+\bar{\nu}_e$	1	1
$\mu^+\bar{\nu}_\mu$	1	1
$\tau^+\bar{\nu}_\tau$	1	1
$c\bar{s}$	$\cos\theta_c$	3
$c\bar{d}$	$\sin\theta_c$	3
$u\bar{d}$	$\cos\theta_c$	3
$u\bar{s}$	$\sin\theta_c$	3

where θ_c is the Cabibbo mixing angle. The fraction of decays to electrons (ignoring leptonic tau decays) is then

$$\frac{1}{1+1+1+3\cos^2\theta_c+3\sin^2\theta_c+3\cos^2\theta_c+3\sin^2\theta_c} = \frac{1}{9}$$

Some may realise that there is in principal also a decay mode to $c\bar{b}$ but this is very suppressed since V_{cb} is about $0.04^2 \sim 10^{-3}$.

c) The hadronic decays compete with scattering via the strong interaction which will dominate.

d) The electron momentum can be determined from e.g. its bending in a magnetic field

$$p_\perp = 0.3BR.$$

for $p_\perp$ in GeV, B in Tesla and R in m.

The transverse components of the neutrino momenta can be determined from momentum conservation in the directions perpendicular to the beam. The same cannot be done for the longitudinal component since the longitudinal momenta of the partons within the protons in the initial state are not known.

4.2. Write down Feynman diagrams for the decays of the muon and the tau lepton. Are hadronic decays possible? By considering the propagator factor in each case explain why one might expect on dimensional grounds that lifetimes should be in the ratio

$$\frac{\Gamma(\tau^- \to e^- + \nu + \bar{\nu})}{\Gamma(\mu^- \to e^- + \nu + \bar{\nu})} = \left(\frac{m_\tau}{m_\mu}\right)^5 .$$

Using the following data

$$m_\tau = 1777.0\,\mathsf{MeV} \qquad \tau_\tau = 2.91 \times 10^{-13}\,\mathsf{s}$$
$$m_\mu = 105.66\,\mathsf{MeV} \qquad \tau_\mu = 2.197 \times 10^{-6}\,\mathsf{s}$$
$$BR(\tau^- \to e^- + \nu + \bar{\nu}) = 17.8\%$$

test this prediction.

> As well as the two leptonic modes, the τ decays to a variety of hadronic modes e.g.
>
> $$\tau^- \to \pi^- \nu_\tau$$
> $$\tau^- \to \pi^- \pi^0 \nu_\tau$$
> $$\tau^- \to \pi^- \pi^+ \pi^- \nu_\tau$$
>
> The propagator factor in each case is
>
> $$\frac{1}{\mathsf{P}^2 - m_W^2}.$$
>
> Since $E_W \ll m_W$ the P^2 term can be ignored so the partial rate must be proportional to m_W^{-4}. By dimensional analysis, and since the mass of the decaying particle is the only relevant scale in the problem, we again find Sargent's rule
>
> $$\Gamma \propto \frac{m_f^5}{m_W^4}$$
>
> Our theoretical value for the τ lepton lifetime is
>
> $$\tau_\tau^{\text{theory}} = 0.178 \left(\frac{m_\mu}{m^\tau}\right)^5 \tau_\mu = 2.91 \times 10^{-13}\,\mathsf{s}$$
>
> precisely matching the experimental value.

4.3. Which of the Standard Model fermions couple to the Z^0 boson? To which final states may a Z^0 boson decay?

Explain why for the Z^0 the sum of the partial widths to the observed states (e^+e^-, $\mu^+\mu^-$, $\tau^+\tau^-$, hadrons) does not equal the FWHM of the Breit-Wigner.

By referring to the properties of the Breit-Wigner forumla, suggest how the LEP e^+e^- collider operating at centre-of-mass energies in the range $80\,\mathsf{GeV}$ to $100\,\mathsf{GeV}$ could have inferred that there are three neutrino species with $m_\nu < m_Z/2$, even though the detectors were unable to detect those neutrinos.

The Z^0 couples to all of the SM fermions (though the couplings are not equal).

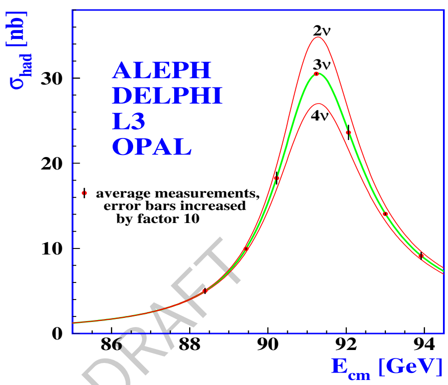
The first four terms represent the visible final states, so

$$\Gamma = \Gamma_{\text{vis}} + \Gamma_{\nu\bar{\nu}}$$

If Γ_{ν_i} can be independently inferred (e.g. from the rate of neutral current neutrino scattering) then the number of light neutrinos can be determined from

$$n_\nu = \frac{\Gamma - \Gamma_{\text{vis}}}{\Gamma_{\nu_i}}$$

$$\Gamma = \Gamma_{e^+e^-} + \Gamma_{\mu^+\mu^-} + \Gamma_{\tau^+\tau^-} + \Gamma_{q\bar{q}} + \sum_i \Gamma_{\nu_i\bar{\nu}_i}$$



LEP's Z^0 lineshape as measured and as predicted for different number of light neutrino species. From [3].

4.4. Consider a model with two neutrino mass eigenstates ν_2 and ν_3 with masses m_2 and m_3 and energies E_2 and E_3, mixed so that

$$|\nu_\mu\rangle = |\nu_2\rangle \cos\theta + |\nu_3\rangle \sin\theta$$
$$|\nu_\tau\rangle = -|\nu_2\rangle \sin\theta + |\nu_3\rangle \cos\theta.$$

Consider a beam of neutrinos created from from $\pi^- \to \mu^-\nu$ decays. Show that the observed flux of muon neutrinos observed at a distance L from such a source is

$$J(L) = J(L=0) \times \left[1 - \sin^2(2\theta)\sin^2\left\{\left(\frac{E_3 - E_2}{2\hbar}\right)\frac{L}{c}\right\}\right].$$

If m_2 and m_3 are much less than the neutrino momentum, $|\mathbf{p}|$, show that

$$|\nu_\mu(L)|^2 \approx |\nu_\mu(0)|^2 \times \left[1 - \sin^2(2\theta)\sin^2\left\{A\left(m_2^2 - m_3^2\right)\frac{L}{|\mathbf{p}|}\right\}\right].$$

What is the first length L^* at which the ν_μ detection rate is at a minimum?

If a range of neutrino energies are present what will be the ratio of the rate (per neutrino) of $\nu + n \to \mu^- + p$ for $L \ll L^*$ and $L \gg L^*$.

What would be the corresponding ratio for neutral-current scattering?

Solar neutrinos emitted in p–p fusion have been detected via the processes

$$\begin{aligned} \nu_e + d &\rightarrow p + p + e^- \quad \text{and,} \\ \nu_x + d &\rightarrow p + n + \nu_x. \end{aligned}$$

Suggest why the charged-current reaction showed only a third of the neutrino flux of the neutral-current reaction.

See notes for oscillation formula.

For a wide range of energies present and when $L \gg L^*$, the term involving $\sin^2(\ldots L)$ will average to $\frac{1}{2}$, in the two-body case, and we will get a survival probability of $1 - \frac{1}{2}\sin^2 2\theta$.

For three-state maximal mixing we will similarly expect to find $\frac{1}{3}$ of the neutrinos in each flavour eigenstate. The neutral current measures all neutrino species, whereas the charged current process (as described) is sensitive *only* to ν_e.

4.5. Draw all leading Feynman diagrams for the following processes:

a) $\nu_\mu + n \rightarrow p + \mu^-$

t-channel W exchange

b) $\nu_\mu + e^- \rightarrow \nu_\mu + e^-$

t-channel Z^0 exchange

c) $\bar{\nu}_e + e^- \rightarrow \bar{\nu}_e + e^-$

Z^0 in t-channel; also W in s-channel.

d) $\bar{\nu}_e + p \rightarrow e^+ + n$

t-channel W exchange

4.6. Write brief notes on:
a) The evidence that there are three and only three families of quarks and leptons.
b) The Cabibbo angle and quark mixing.
c) The evidence for confinement of quarks in hadrons.

4.7. How does helicity of a state change on application of the parity operator (which reverses the coordinate axes: $\mathbf{x} \overset{\mathbb{P}}{\mapsto} -\mathbf{x}$)?

If $|\Phi\rangle$ is an eigenstate of the parity operator, what can be said about the parity of the state $(1 + a\,\mathbf{S}\cdot\mathbf{p})|\Phi\rangle$?

$^{60}\mathrm{Co}$ nuclei ($J^P = 5^+$) are polarised by immersing them at low temperature in a magnetic field. When these nuclei β decay to $^{60}\mathrm{Ni}$ (4^+) more electrons are emitted opposite to the aligning B field than along it[1]. Explain carefully why this demonstrates parity violation in the weak interaction.

Justify the direction of the parity-violating effect.

The helicity observable is

$$h = \frac{\mathbf{S}\cdot\mathbf{p}}{|\mathbf{p}|}$$

Under the parity operator, the momentum and spin observables

$$\mathbf{p} \overset{\mathbb{P}}{\mapsto} -\mathbf{p} \qquad \mathbf{S} \overset{\mathbb{P}}{\mapsto} +\mathbf{S}$$

so we must have that

$$h \overset{\mathbb{P}}{\mapsto} -h.$$

The eigenvalues of the helicity operator change must sign under $\mathbb{P}$ whereas those of the unit operator are unchanged. The operators do not commute $[\mathbb{P}, 1 + a\mathbf{S}\cdot\mathbf{p}] \neq 0$. so the state

$$(1 + a\,\mathbf{S}\cdot\mathbf{p})|\Phi\rangle$$

cannot be an eigenstate of parity for general a. It must be a mixed parity state, which we can write as

$$|\psi\rangle = |\psi_+\rangle + |\psi_-\rangle.$$

The effect of parity violation will show up in the interference between the $|\psi_+\rangle$ and the $|\psi_-\rangle$.

$$^{60}\mathrm{Co} \to {}^{60}\mathrm{Ni} + e^- + \bar{\nu}_e,$$

For the $^{60}\mathrm{Co}$ decays $\mathbb{P}$ leaves $\mathbf{B}$ unchanged, but reverses $\mathbf{p}_e$. The fact that the electron fluxes

$$J_e(\mathbf{p}_e\cdot\mathbf{B}) \neq J_e(-\mathbf{p}_e\cdot\mathbf{B})$$

show that the final state has *mixed* parity, whereas the initial $^{60}\mathrm{Co}$ nuclei were in a *definite* (+ve) parity states. So clearly the parity operator does not communte with the evolution operator (and hence neither with the Hamiltonian operator).

If we put the magnetic field up the page

$$\mathbf{B}\Uparrow \qquad \boldsymbol{\mu}\Uparrow \qquad \mathbf{J}_{\mathrm{Co}}\Uparrow$$

when conserving momentum and spin we will favour the final state with the electron moving down the page

$$\begin{array}{c} \bar{\nu}\uparrow\Uparrow \\ e^-\downarrow\Uparrow \end{array} \quad \text{more than} \quad \begin{array}{c} e^-\uparrow\Downarrow \\ \nu\downarrow\Downarrow \end{array}.$$

[1] Reported in [7].

4.8. Neutrino and anti-neutrino states have only ever been observed with the following eigenvalues of the helicity operator respectively:

$$\nu : -\tfrac{1}{2}\hbar \qquad \bar{\nu} : +\tfrac{1}{2}\hbar$$

What values of the projection operators

$$\mathcal{P}_{\pm} = \frac{1}{2}\left(1 \pm \boldsymbol{\sigma} \cdot \frac{\mathbf{p}}{|\mathbf{p}|}\right)$$

must be present in weak processes for (anti-)neutrinos reactions?

What is the implication for parity in the weak interaction?

Let $|\nu_{\pm}\rangle$ be the neutrino eigenstates of helicity with eigenvalues $\pm\frac{1}{2}$.

$$\begin{matrix} \nu \\ \Leftarrow \\ \longrightarrow \end{matrix} \qquad \begin{matrix} \bar{\nu} \\ \Rightarrow \\ \longrightarrow \end{matrix}$$

For the neutrino interactions we need $\mathcal{P}_-$ operators

$$\mathcal{P}_-|\nu_+\rangle = 0 \qquad \mathcal{P}_-|\nu_-\rangle = 1$$

whereas for anti-neutrinos we need $\mathcal{P}_+$

$$\mathcal{P}_+|\bar{\nu}_+\rangle = 1 \qquad \mathcal{P}_+|\bar{\nu}_+\rangle = 0$$

[The operators for the weak interaction are of course really $\frac{1}{2}(1 \pm \gamma^5)$*, but first we'd need to solve Dirac. . . so this is the best we can do for now.]*

The complete non-existence (or possibly non-interaction) of positive helicity neutrinos and negative helicity anti-neutrinos means that the weak interaction violates parity *maximally*.

4.9. Write down Feynman diagrams and for the processes $\pi^+ \to e^+\nu_e$ and $\pi^+ \to \mu^+\nu_\mu$. By considering the helicities of the final state particles, suggest why the π^+ $(J^P = 0^-)$ decays dominantly to $\mu^+\nu_\mu$.

The neutrino is -ve helicity, so considering conservation of linear and angular momentum in the ZMF, the charged lepton must be -ve helicity too. The amplitude to find a right-chiral spin-$\frac{1}{2}$ object in a -ve helicity state is $\sqrt{1-\beta}$ (again, need Dirac). So the amplitude is smaller for the (more relativistic) electron, and for the (less relativistic) muon.

4.10. The Large Hadron Collider has been designed to accelerate counter-rotating beams of protons to energies of 7 TeV, and to collide those beams at a small number of interaction points.

a) Use dimensional analysis to estimate the smallest length scale which this machine could be used to resolve. How does this compare to the size of e.g. atoms, nuclei and protons?

b) The LHC beam pipe is evacuated to reduce loss of beam from collisions with gas moleceules. If less than 5% of the beam protons are to be lost from collisions with gas nuclei over a ten hour run, estimate the maximum permissible number density of H gas atoms in the beam pipe.

c) The machine collides counter-rotating bunches of protons, each bunch having circular cross section with radius 17 μm (in the direction perpendicular to travel). How many protons are required in a bunch to have an average of ten interactions per bunch crossing?

d) If such bunches collide every $25\,\mathrm{ns}$, what is the *luminosity* of the machine? (Express your answer in units of cm^{-2} s^{-1}.)

e) If the cross section for producing a Higgs Boson is $50\,\mathrm{pb}$, how many will be made each second?

f) What is the kinetic energy of each bunch of protons in the LHC?

[Some data for proton-proton cross sections can be found in Figure A.1.]

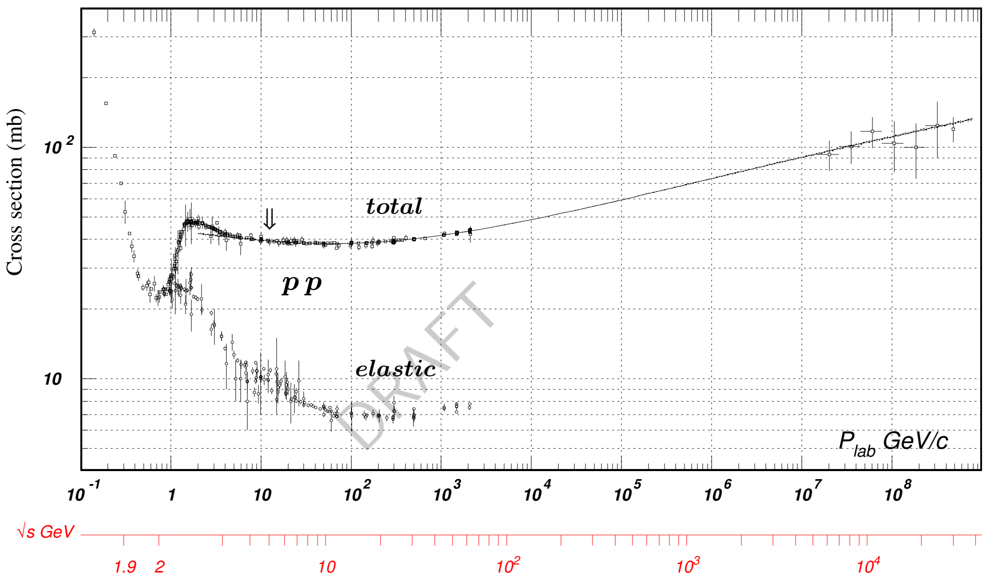


Figure A.1: Proton–proton scattering cross section data as a function of laboratory momentum (upper scale) and of the centre-of-mass energy (lower scale). From [4].

a) $\hbar c/E = 197\,\text{MeV fm}/14\,\text{TeV}$ so we can resolve $\approx 0.14 \times 10^{-19}\,\text{m}$ — i.e. about $80\,000$ times smaller than the proton.

b) We want the fraction lost to be small so $0.05 \approx n\sigma_{\text{tot}}ct_{\text{beam}}$. From the plot, $\sigma_{\text{tot}}(p_{\text{lab}} = 7\,\text{TeV}) \approx 50\,\text{mb} = 50 \times 10^{-31}\,\text{m}^2$, so the number density of stray H atoms should be

$$n_H < \frac{0.05}{\sigma_{\text{tot}}ct_{\text{beam}}} = \frac{0.05}{(50\ 10^{-31}\,\text{m}^2)(3\ 10^8\,\text{m}\,\text{s}^{-2})(10\ 60\ 60\,\text{s})} \approx 10^{15}\,\text{m}^{-3}.$$

c) At CM energy of 14 TeV the total pp cross section is about $100\,\text{mb}$. The number of interactions per bunch crossing is $\langle N_{\text{tot}}\rangle = N_p^2\sigma_{\text{tot}}/A$ so we require

$$N_p^2 = \langle N_{\text{tot}}\rangle \frac{A}{\sigma_{\text{tot}}} = 10\,\frac{\pi(17 \times 10^{-6}\,\text{m})^2}{100 \times 10^{-31}\,\text{m}^2} \approx 9 \times 10^{20}.$$

So the protons/bunch should be $N_p \approx 3 \times 10^{10}$.

d) The rate, $W = \sigma L$. For $\sigma_{\rm tot} = 100\,{\rm mb}$, we have specified that

$$W_{\rm tot} = 10/25\,{\rm ns} = 4 \times 10^8 \text{ s}^{-1}$$

therefore

$$L = W_{\rm tot}/\sigma_{\rm tot} = 4 \times 10^{33}\,\text{cm}^{-2}\,\text{s}^{-1}.$$

e) $W_{\rm Higgs} = \sigma_{\rm Higgs} L$ which with $\sigma_{\rm Higgs} = 50\,{\rm pb}$ gives

$$W_{\rm Higgs} = (50 \times 10^{-12} \times 10^{-24})(4 \times 10^{33}) = 0.2\,\text{s}^{-1}$$

f) $E_{\rm bunch} = N E_p = (10^{10})(7 \times 10^{12} \times 1.6\ 10^{-19}) \approx$ 11 kJ (macroscopic).

A.5 Energy production, stars

Solutions

5.1. a) Why does $^{235}\mathrm{U}$ fission with thermal neutrons whereas $^{238}\mathrm{U}$ requires neutrons with energies of order MeV?

b) The fission of $^{235}\mathrm{U}$ by thermal neutrons is asymmetric, the most probable mass numbers of fission fragments being 93 and 140. Use the semi-empirical mass formula to estimate the energy released in fission of $^{235}_{92}\mathrm{U}$ and hence the mass of $^{235}_{92}\mathrm{U}$ consumed each second in a 1 GW reactor.

c) In almost all uranium ores, the proportion of $^{235}\mathrm{U}$ to $^{238}\mathrm{U}$ is 0.0072. However, in certain samples from Oklo in the Gabon the proportion is 0.0044. Assuming that a natural fission reactor operated in the Gabon 2×10^9 years ago, estimate the total energy released from 1 kg of the then naturally occurring uranium. How might the hypothesis that $^{235}\mathrm{U}$ was depleted by fission be tested?

[$t_{1/2}(^{238}\mathrm{U}) = 4.5\times10^9$ *years,* $t_{1/2}(^{235}\mathrm{U}) = 7.0\times10^8$ *years.]*

a) The capture of a neutron changes $^{235}\mathrm{U}$ from O-E to E-E, so releases the energy from the pairing term. This energy is sufficient to overcome the energy barrier to fission.

b) We are interested in the process

$$n + {}^{235}_{92}\mathrm{U} \longrightarrow {}^{140}\mathrm{X} + {}^{93}\mathrm{Y} + 3n$$

The three neutrons are needed for conservation of baryon number. We need to work out the atomic numbers of the two isotopes X and Y. The quick and dirty way to do this is to assume about the same Z/N ratio as in the parent Uranium isotope, which would give $Z(A=140)\approx 55$ and $Z(A=93)\approx 37$. The better way is to look at the binding energy E_B as a function of Z for constant A and to find out the point where $\frac{dE_B}{dZ}=0$. (In fact to more precisely find the most stable isobar one should also account for the mass difference between neutrons and protons and the mass of any beta electron.) For odd-A nuclei (which have no pairing term) the value of Z at the minimum

$$Z_{\min} = \frac{4s + (m_n - m_p)A}{2(4s + dA^{\frac{2}{3}})}.$$

For $A=140$ this yields $Z=56$ and for $A=93$ we get $Z=40$, so we were reasonably close before. (Consulting a table of nuclides shows the stable isobars are $^{140}_{58}\mathrm{Ce}$ and $^{93}_{41}\mathrm{Nb}$ respectively.) The energy released is

$$E_{\text{fission}} = M(^{235}_{92}\mathrm{U}) - M(^{140}_{58}\mathrm{Ce}) - M(^{93}_{41}\mathrm{Nb}) - 2M(n)$$

This is very close to 200 MeV. We shall assume that all of the energy, including that of the beta decays is included in the power output by the reactor. (In reality some energy will be lost through neutrinos, and some daughters may have long lifetimes.) Then for power P the rate of reactions W is

$$W = \frac{P}{E_{\text{fission}}} = \frac{1\mathrm{GW}}{200\,\mathrm{MeV}} = 3.2\times10^{19}s^{-1}$$

and the rate of consumption of Uranium mass is $M(U)\times W$ which is about 12 mg/s.

c) For most natural uranium

$$\frac{^{235}\mathrm{U}(t)}{^{238}\mathrm{U}(t)} = 0.0072$$

whereas at Oklo

$$\left.\frac{^{235}\mathrm{U}(t)}{^{238}\mathrm{U}(t)}\right|_{\mathrm{Oklo}} = 0.004$$

Calculating backwards,

$$\begin{aligned} ^{235}\mathrm{U}(t) &= {}^{235}\mathrm{U}(0)\exp(-\Gamma_{235}t) \\ ^{238}\mathrm{U}(t) &= {}^{238}\mathrm{U}(0)\exp(-\Gamma_{238}t) \end{aligned}$$

so

$$\frac{^{235}\mathrm{U}(0)}{^{238}\mathrm{U}(0)} = \frac{^{235}\mathrm{U}(t)}{^{238}\mathrm{U}(t)}\exp[(\Gamma_{235}-\Gamma_{238})t]$$

thus that part of the ratio which was *consumed* is

$$\begin{aligned} &(0.0072 - 0.004)\exp[(\Gamma_{235}-\Gamma_{238})t] \\ &= 0.0032\exp\left[\ln(2)\left(t^{-1}_{\frac{1}{2},235} - t^{-1}_{\frac{1}{2},238}\right)t\right] \\ &= 0.017 \end{aligned}$$

From 1 kg of uranium is 17 g used. The energy released is

$$\frac{17\,\mathsf{g}}{235\,\mathsf{g}\,\mathsf{mol}^{-1}} \cdot 6\times10^{23}\,\mathsf{mol}^{-1}\cdot 170\,\mathsf{MeV} \approx 1\,\mathrm{TJ}$$

Look for fission fragments in rock to test the hypothesis.

5.2. Write down the semi empirical mass formula. Which terms are responsible for the existence of a viable chain reaction of thermal-neutron-induced uranium fission? What distinguishes the isotopes of uranium that support such a reaction?

In the construction of a nuclear fission reactor an important role is often played by water, heavy water or graphite. Describe this role and explain why are these materials are suitable.

Why is the fissile material not completely mixed up with the moderator?

Fission barrier from surface term. Energy release from capture of n on $^{235}\mathrm{U}$ sufficient to overcome barrier (pairing term).

Water as coolant and as moderator. Heavy water or graphite as moderator.

The moderator is separated from the uranium to cool neutrons below the temperature of the $^{238}\mathrm{U}(n,\gamma)$ capture reaction before they come back into contact with the uranium. These capture reactions would otherwise reduce the number of neutrons below that needed to maintain a reaction.

5.3. A neutron produced in a fission reaction is emitted with considerable energy. Discuss how the design of the reactor determines the competition between i) neutron absorption by sharp resonances; ii) neutron decay; iii) neutron energy transfer to the reactor media (and thence to turbines); iv) neutron absorption by resonances with high branching ratios to further fission. Most of these processes happen very fast indeed. How is it possible to control the reactor flux with a response time of seconds to minutes, or even longer?

First parts: See textbooks (e.g. Krane Sec 13.5).

Last part: per fission about 2.5 prompt neutrons per fission. Control relies on ≈ 0.018 delayed neutrons per fission with time constants of order seconds (time constant for reactor as a whole is larger, of order minutes). Delay is due to neutron emission after beta decay.

5.4. a) Draw a diagram showing how the fusion process

$$p + p \to d + e^+ + \nu$$

is related to the Fermi theory of beta decay.

b) Find the approximate height $\mathcal{B}$ of the Coulomb barrier for pp fusion. To approximately what temperature would one need to heat hydrogen for pp fusion to overcome the Coulomb barrier?

c) If plasma at this temperature is to be magnetically confined what will be a typical Larmor radius (gyro-radius) for the duterium ions in the magnetic field?

d) Given the following reactions and energy release (in MeV)

$$\begin{array}{ll} d + d \to {}^3\mathrm{He} + n & Q = 3.27, \\ d + d \to t + p & Q = 4.04, \\ d + t \to {}^4\mathrm{He} + n & Q = 17.6, \end{array}$$

suggest *two* reasons why the artificial fusion reactors depend largely on the $d + t$ reaction.

e) Tritium has a half-life of about 12 years, and must be generated through reactions with both ${}^6\mathrm{Li}$ and ${}^7\mathrm{Li}$. Write down the form of these reactions, and explain why the ${}^7\mathrm{Li}$ reaction is helpful even though it is endothermic.

[d means ${}^2\mathrm{H}$ and t means ${}^3\mathrm{H}$. You may assume the magnetic field strength is 13.5 Tesla, which is what has been proposed for the ITER Tokamak.]

a) $p \to n + e^+ + \nu$ can be described by the same four-fermion matrix element assumed in Fermi theory of beta decay.

b) The barrier height is

$$\mathcal{B} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_1 + r_2} = \frac{\alpha}{r_1 + r_2}$$

Taking $r_1 = r_2 \approx 1\,\mathrm{fm}$ we find that $B = \frac{1}{137} \frac{197\,\mathrm{MeV\,fm}}{2\,\mathrm{fm}} \approx 700\,\mathrm{keV}$. For a fast rate we would need to heat each such that $\frac{3}{2} k_B T \approx B$ meaning that $T \approx 5.4 \times 10^9\,\mathrm{K}$. To confine material at this temperature one either uses magnetic confinement (e.g. JET) or inertial confinement (e.g. NIF).

c) The Larmor radius is

$$r = \frac{m v_\perp}{qB} \approx \frac{\sqrt{2mE}}{qB}$$

For $B = 13.5\,\mathrm{T}$, $m_{^2\mathrm{H}} \approx 2\,\mathrm{GeV}$, $E \approx 700\,\mathrm{keV}$,

$$r[\mathrm{m}] = \frac{1}{0.3} \frac{\sqrt{2 \cdot 2 \cdot 7 \times 10^{-4}}}{13.5}$$

so is about 1.3 cm.

d) The larger Q value makes dt ignition easier – the Lawson criterion is

$$n\tau > \frac{12k_BT}{\langle v\sigma\rangle Q}$$

$d+t$ also has a lower Coulomb barrier than $p+p$. The nuclear sizes are larger, and the nucleons can arrange themselves such that the protons are as far apart as possible. The Boltzmann-averaged cross section $\langle v\sigma\rangle$ is therefore larger.

e) The reactions are

$$n + {}^6\mathrm{Li} \rightarrow \alpha + t \quad Q = 4.8\,\mathsf{MeV}$$

and

$$n + {}^7\mathrm{Li} \rightarrow \alpha + t + n \quad Q = -2.47\,\mathsf{MeV}$$

The latter is important since we need to produce > 1 tritium for each neutron, because of neutron and tritium losses and decays.

5.5. Show that for a star in a state of hydrostatic equilibrium (with pressure balancing gravity), the pressure gradient is given by

$$\frac{dP}{dr} = -\rho\frac{Gm}{r^2}$$

where m is the mass contained within the sphere of radius r. Hence show that the pressure at the center of a star satisfies

$$P_c = \int_0^M \frac{Gm\,dm}{4\pi r^4} > \int_0^M \frac{Gm\,dm}{4\pi R^4},$$

where R is the radius of the star.

Estimate the pressure and temperature at the the center of the sun.

For an element at radius r, with volume $dr \times dS$ and hence mass $dm = \rho dr\,dS$ the gravitational and pressure forces must balance

$$\begin{aligned} 0 &= -\frac{Gm\,dm}{r^2} + [P(r) - P(r+dr)]dS \\ &= -\frac{Gm\,dm}{r^2} - \frac{dP}{dr}\frac{dm}{\rho} \end{aligned}$$

giving the required result, which can also be expressed as

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

The central pressure follows from integrating the above. The pressure the pressure at the center of the sun must then satisfy

$$P_C > \frac{GM^2}{8\pi R^4} = 4.5\times 10^{13}\,\mathrm{Pa} = 4.5\times 10^8\,\mathrm{atm}.$$

since $G = 6.67\,\mathsf{N\,m^2\,kg^{-2}}$, $M = 2\times 10^{30}$ kg, $R = 7\times 10^8$ m. The actual pressure is about a factor of 10 larger.

The temperature can be found from the gas law $PV = nk_BT$ so that

$$T = \frac{m_H P}{k_B \rho}$$

Taking the average density of the sun from a spherical approximation,

$$\rho = \frac{2 \times 10^{30}\,\mathsf{kg}}{\frac{4}{3}\pi(7 \times 10^{8}\,\mathsf{m})^3} = 1.8 \times 10^3\,\mathsf{kg\,m^{-3}}.$$

For $m_H = 1.67 \times 10^{-27}$ kg and $k_B = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$ we find $T > 6 \times 10^6$ K. Models suggest a core temperature of about 1.6×10^7 K This is in the right ball-park for fusion, though clearly not in the MeV $\sim 10^{10}$ K regime at which the pp Coulomb barrier becomes unimportant. Nevertheless tunnelling together with the tail of the Maxwell-Boltzmann can do the job.

[For more on stellar structure see e.g. Prailnik or Clayton]

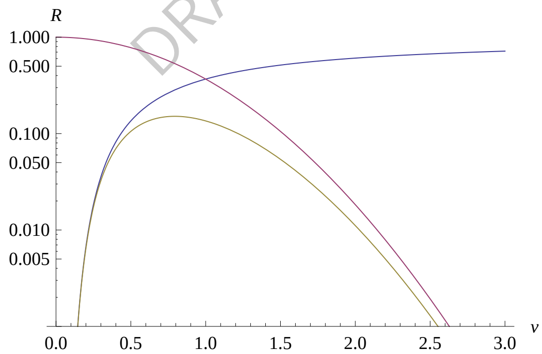
5.6. The rate of thermonuclear fusion reactions is approximately proportional to

$$\exp\left(-\frac{2\pi Z_1 Z_2 \alpha c}{v}\right)\exp\left(-\frac{mv^2}{2kT}\right).$$

Sketch the form of this curve and explain the origin of these two terms.

Find the value of v at which the rate is maximal. At what temperature should one run a Tokamak?

The first term is the tunnelling probability, the as same we calculated for α decay in the limit where $Q = mv^2$ was small. [We've simplified things by only considering one-particle dymanics.] The second term is the Boltzman factor.



The maximum occurs when

$$0 = \frac{d}{dv}\left(-\frac{2\pi Z_1 Z_2 \alpha c}{v} - \frac{mv^2}{2k_B T}\right) = \frac{2\pi Z_1 Z_2 \alpha c}{v^2} - \frac{mv}{k_B T}$$

at

$$v/c = \left(\frac{2\pi Z_1 Z_2 \alpha k_B T}{mc^2}\right)^{\frac{1}{3}}.$$

5.7. a) Assuming that the energy for the sun's luminosity is provided by the conversion of $4H \to {}^4\mathrm{He}$, and that the neutrinos carry off only about 3 percent of the energy liberated how many neutrinos are liberated each second from the sun?

b) What neutrino flux would you expect to find at the Earth?

c) By what sequence of reactions do the above conversions dominantly proceed?

d) Why might the alternative rare process

$$p + e^- + p \to d + \nu_e$$

be of interest when studying solar neutrinos from the earth?

[$4M(^1\mathrm{H}) - M(^4\mathrm{He}) = 26.73$ MeV. The earth is on average about 1.50×10^{11} m from the sun, and is subject to a radiation flux of about $1.3\,\mathrm{kW\,m^{-2}}$.]

The solar power P is $4\pi r_{\mathrm{earth}}^2 J$ where J is the energy flux at the earth.

$$P = 4\pi \cdot (1.5 \times 10^{11}\,\mathrm{m})^2 \cdot (1.3 \times 10^3\,\mathrm{W\,m^{-2}}) = 3.7 \times 10^{26}\,\mathrm{W}.$$

In principle we now need to multiply this by $1/(1-f_\nu)$ to work out what the total power (since above we have calculated the visible power output excluding the neutrinos). It is clear from the question that the fraction of the power found in neutrinos f_ν is small, so this correction can be neglected at the level of precision we are using.

There are two neutrinos liberated in the process

$$4\,^1\mathrm{H} \to {}^4\mathrm{He} + 2\,e^+ + 2\,\nu_e + 26.73\,\mathrm{MeV}.$$

The neutrino production rate is then

$$\Gamma_\nu = 2 \times \frac{(3.7 \times 10^{26}\,\mathrm{W})}{26.73\,\mathrm{MeV} \times (1.6 \times 10^{-19} \times 10^6\,\mathrm{J/MeV})} = 1.7 \times 10^{38}\,\mathrm{s^{-1}}$$

The solar neutrino flux at earth is then

$$F_\nu = \frac{\Gamma_\nu}{4\pi r_{\mathrm{earth}}^2} = 6.1 \times 10^{14}\,\mathrm{m^{-2}\,s^{-1}}.$$

The $\mathbf{p-p}$ **chain** relies on initial production (via weak interaction) of deuteron

$$p + p \to d + e^+ + \nu$$

then further proton absorption

$$d + p \to {}^3\mathrm{He} + \gamma,$$

then fusion of two Helium nuclei and emission of a pair of protons. . .

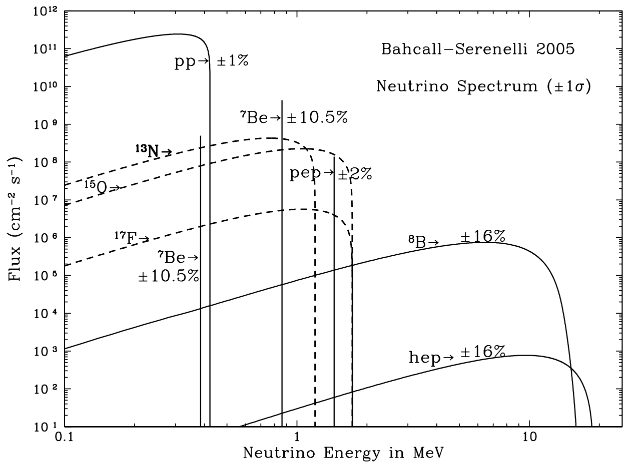
$${}^3\mathrm{He} + {}^3\mathrm{He} \to {}^4\mathrm{He} + 2p.$$

The more direct alternatives e.g.

$$p + p \ \to {}^2\mathrm{He}$$

are not useful since ${}^2\mathrm{He}$ is unstable against disintegration.

The pep process is interesting since the emitted neutrinos are monoenergetic.



Solar neutrino predictions from J.N. Bahcall and A.M. Serenelli *New Solar Opacities, Abundances, Helioseismology, and Neutrino Fluxes* ApJ, **621**, L85 (2005).

5.8. Give brief accounts of the methods of synthesis of:

a) $^{12}\mathrm{C}$

b) $^{28}\mathrm{Si}$

c) $^{56}\mathrm{Fe}$

d) $^{238}\mathrm{U}$

a) $^{12}\mathrm{C}$ generated via Helium burning at $T \sim 10^8$ K (*cf pp* at $\sim 4\times10^6$ K and CNO at $\sim 1.5\times10^7$ K). Though unstable with lifetime 2.6×10^{-16} s, $^{8}\mathrm{Be}$ exists at small concentration through the reaction

$$^{4}\mathrm{He} + {}^{4}\mathrm{He} \leftrightarrow {}^{8}\mathrm{Be}.$$

Collision with another α particle leads to

$$^{8}\mathrm{Be} \rightarrow {}^{12}\mathrm{C},$$

a process which is enhanced by a $^{12}\mathrm{C}^*$ resonance at ~ 7.6 MeV.

b) $^{28}\mathrm{Si}$ generated during carbon and oxygen burning at $T \sim 10^9$ K

$$^{12}\mathrm{C} + \alpha \rightarrow {}^{16}0$$
$$^{16}\mathrm{O} + {}^{16}0 \rightarrow {}^{28}\mathrm{Si} + {}^{4}\mathrm{He}.$$

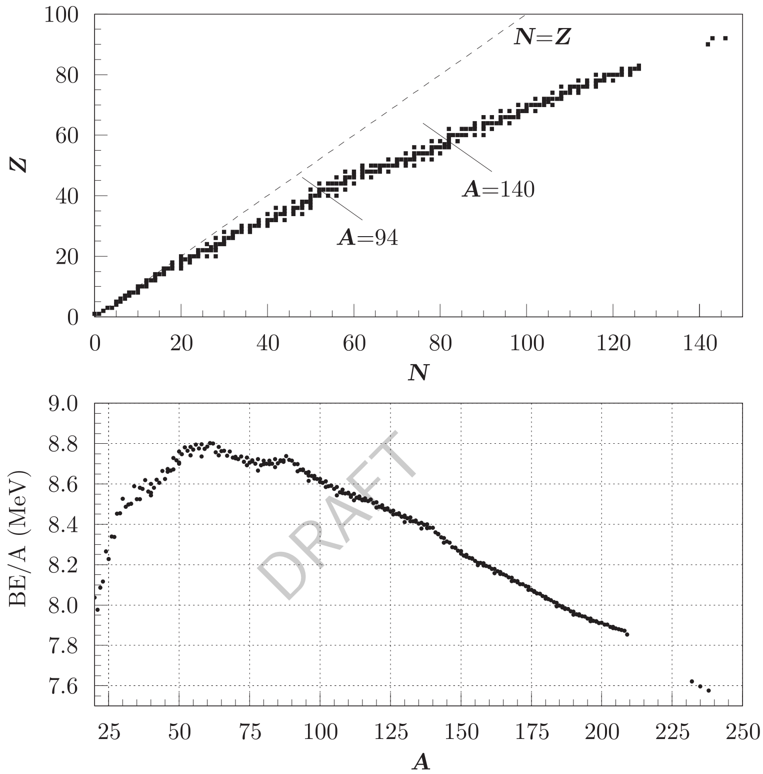
c) The Coulomb barrier is too large for

$$^{28}\mathrm{Si} + {}^{28}\mathrm{Si} \rightarrow {}^{56}\mathrm{Fe}.$$

The reaction proceeds at $\sim 3\times10^9$ K via the 'equilibrium' process. Photons of ~ 1 MeV knock nucleons out of the nuclei creating a soup of nuclei, nuclepons, some alphas and photons. The nucleons are slowly gathered up by nuclei until the isotopes in the peak of the B/A curve are obtained.

d) Elements heavier than the peak value of B/A ($> {}^{56}\mathrm{Fe}$) require energy to be created, and are formed during gravitational collapse of the stellar core when under highly non-equilibrium conditions by the capture of liberated neutrons in conjunction with beta decays.

5.9. The two figures show properties of the 'valley of stability' of nuclei in the N-Z plane and the binding energy per nucleon versus mass number A, for nuclei with lifetimes greater than 10^8 years.



Using the data in the figures, estimate the energy released in the thermal-neutron induced fission of ^{235}U, given that the daughter nuclei tend to cluster asymmetrically around $A = 140$ and 94. Where are the daughter nuclei in relation to the valley of stability and what happens to them subsequently? Compare this with the $^{238}\mathrm{U}$ decay chain, which comprises eight α decays and six β decays to $^{206}\mathrm{Pb}$ with a total release of 48.6 MeV and a lifetime of $2 \times 10^{17}\,\mathrm{s}$.

There is a flow of heat from the Earth's interior amounting to a total of the order of $35\,\mathrm{TW}$. Much of this may be accounted for by decay of radioactive elements. Using the model above for a typical fission, estimate the rate of fissions needed to produce such a heat flow and the associated flux of neutrinos.

In a certain model of the Earth it is postulated that there is a self-sustaining fission reactor at the Earth's centre, fuelled by $^{235}\mathrm{U}$, contributing as much as $5\,\mathrm{TW}$ to the overall heat flow. If the 'geo-neutrino' flux could be measured, how might the 'core reactor' model be tested?

^{235}U has $B/A \approx 7.6$ MeV so total binding energy of $235 \times 7.6 = 1786$ MeV. The fission fragments have binding energies of about $94 \times 8.65 + 140 \times 8.4 = 1989$ MeV, so the energy release is 203 MeV.

The daughter nuclei are below the valley of stability (N too large), so will emit neutrons and beta decay until it reaches the valley.

$$n \to p e^- \bar{\nu}_e$$

Drawing a straight line on the Z vs. N plot, the $A \approx 94$ daughter is about 4 beta decays from the valley. The $A \approx 140$ daughter is also about 8 beta decays from the valley.

The total neutrino rate required from the 238 decay chain is

$$W = 6 \times \frac{35 \times 10^{12}\,\mathrm{W}}{(48.6 \times 10^{6}\,\mathrm{eV})(1.6 \times 10^{-19}\,\mathrm{J\,eV^{-1}})} = 2.7 \times 10^{25}\,\mathrm{s^{-1}}$$

The ratios of neutrinos to energy is

$$\begin{aligned} \left.\frac{n_\nu}{E}\right|_{235,\text{fission}} &= \frac{8}{203\,\mathrm{MeV}} \\ \left.\frac{n_\nu}{E}\right|_{238,\text{decay}} &= \frac{6}{48.6\,\mathrm{MeV}} \end{aligned}$$

so the proposed a self-sustaining 235 reactor would decrease the neutrino flux to

$$\begin{aligned} &\frac{6 \times 30\,\mathrm{TW}}{48.6\,\mathrm{MeV}} + \frac{8 \times 5\,\mathrm{TW}}{203\,\mathrm{MeV}} \\ &= (2.3 + 0.1) \times 10^{25}\,\mathrm{s^{-1}} \end{aligned}$$

5.10. The CNO cycle proceeds in the following steps

			Q / MeV	rate	lifetime
$^{12}\mathrm{C} + p$	$\to$	$^{13}\mathrm{N} + \gamma$	1.944	r_{12}	
$^{13}\mathrm{N}$	$\to$	$^{13}\mathrm{C} + e^+ + \nu$	2.221		τ_N
$^{13}\mathrm{C} + p$	$\to$	$^{14}\mathrm{N} + \gamma$	7.550	r_{13}	
$^{14}\mathrm{N} + p$	$\to$	$^{15}\mathrm{O} + \gamma$	7.293	r_{14}	
$^{15}\mathrm{O}$	$\to$	$^{15}\mathrm{N} + e^+ + \nu$	2.761		τ_O
$^{15}\mathrm{N} + p$	$\to$	$^{12}\mathrm{C} + {}^{4}\mathrm{He}$	4.965	'fast'	

How much energy is released per cycle? Estimate the fraction of that energy in neutrinos.

The beta decay time constants are of the order of minutes. The shortest proton capture time is for ^{15}N which is of the order of years, whereas the other capture timescales are significantly longer. Write the coupled linear differential equations of $(^{12}\mathrm{C}, {}^{13}\mathrm{C}, {}^{14}\mathrm{N})$ in the form

$$\frac{d}{dt}\mathbf{U} = \mathsf{M}\mathbf{U}$$

where M depends on the r_x but not the τ_x.

Show that this set of coupled differential equations admits a solution

$$\mathbf{U}(t) = \sum_{i=1,3} a_i e^{\lambda_i t} \mathbf{u}_i$$

with

$$\lambda_1 = 0 \quad \lambda_{2,3} = \frac{1}{2}(-\Sigma \pm \Delta)$$

and find Σ and Δ. Why must the elements of $\mathbf{u_{2,3}}$ sum to zero?

Express the relative equilibrium abundances of $^{12}\mathrm{C}$, $^{13}\mathrm{C}$ and $^{14}\mathrm{N}$ in terms of the r_x, and show that the equilibrium fractions of the beta decaying isotopes satisfy equations of the form

$$B = \tau_B r_A A.$$

The net effect of one cycle is

$$^{12}\mathrm{C} + 4p \rightarrow {}^{12}\mathrm{C} + {}^{4}\mathrm{He} + 2e^{+} + 2\nu_e + 26.7\,\text{MeV}$$

Cross-check: $Q_{p-p} = 4M(\mathrm{H}) - M(^4\mathrm{He}) - 26.73\,\text{MeV}$.

Annihilation of the positrons releases a further 2 MeV. *[Is this already include in Q values? Cross-check suggests yes. If so my neutrino energy values below are too large.]* We might expect the to neutrinos carry off about half the energy from each beta decay, so about 2.5 MeV. So about 10% of the CNO energy would escape as neutrinos. *[Clayton suggests a little less]*

The 'fast' reactions (including the two β decays) rapidly equilibrate, after which the other three species evolve according to

$$\frac{d}{dt}\begin{pmatrix} ^{12}\mathrm{C} \\ ^{13}\mathrm{C} \\ ^{14}\mathrm{N} \end{pmatrix} = \begin{pmatrix} -r_{12} & 0 & r_{14} \\ r_{12} & -r_{13} & 0 \\ 0 & r_{13} & -r_{14} \end{pmatrix} \begin{pmatrix} ^{12}\mathrm{C} \\ ^{13}\mathrm{C} \\ ^{14}\mathrm{N} \end{pmatrix}.$$

If we try a solution

$$\mathbf{U} = \mathbf{u}_i e^{\lambda_i t}$$

then

$$\lambda_i \mathbf{u}_i = \mathsf{M}\mathbf{u}_i.$$

Label eigenvalues λ_i and eigenvectors $\mathbf{u}_i$. Since the differential equation is linear and homogeneous, a linear superposition of solutions of the form $\mathbf{u}_i e^{\lambda_i t}$ will also be a solution. The a_i are determined from the initial abundances.

The eigenvalues follow directly from the secular equation

$$\begin{aligned} 0 &= (-r_{12} - \lambda)[(-r_{13} - \lambda)(-r_{14} - \lambda)] - 0 + r_{14}[(-r_{12})(-r_{13})] \\ &= -\lambda^3 - \lambda^2(r_{12} + r_{13} + r_{14}) - \lambda(r_{12}r_{13} + r_{12}r_{14} + r_{13}r_{14}) \end{aligned}$$

So are $\left\{0, \frac{1}{2}(-\Sigma \pm \Delta)\right\}$ with

$$\Sigma = r_{12} + r_{13} + r_{14},$$

$$\Delta = \left[\Sigma^2 - 4\left(r_{12}r_{13} + r_{12}r_{14} + r_{13}r_{14}\right)\right]^{\frac{1}{2}}$$

The equilibrium abundances are those for the eigenvector associated with the null eigenvalue, which is

$$u_1 = \begin{bmatrix} 1/r_{12} \\ 1/r_{13} \\ 1/r_{14} \end{bmatrix}$$

The eigenvectors $\mathbf{u}_{2,3}$ give the directions in which the system evolves towards equilibrium. Each has components which sum to zero as required by conservation of baryon number.

From generalisation of the null eigenvector above (check), or directly from the equilibrium condition

$$0 = \frac{d}{dt}\left({}^{13}\mathrm{N}\right) = r_{12}\left({}^{12}\mathrm{C}\right) - \frac{1}{\tau_N}\left({}^{13}\mathrm{N}\right)$$

we find

$$^{13}\mathrm{N} = \tau_N r_{12}\,{}^{12}\mathrm{C}$$

with a similar result for $^{15}\mathrm{O}$ (and indeed for $^{15}\mathrm{N}$).

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