Solving Heterogeneous Agent Models with Aggregate Uncertainty

Solution Methods for Macroeconomic Models

Petr Sedláček

Solution Methods for Macroeconomic Models

- Monday - Tuesday: Solving models with “representative agents”
  - ~~Linearization in theory and practice: Dynare~~
  - ~~Non-linear solutions methods: value function iteration, projection~~
  - Analyzing models: parameterization/estimation, simulation/IRFs
- Wednesday - Thursday: Solving models with “heterogeneous agents”
  - ~~Models without aggregate uncertainty: basic algorithm~~
  - Models with aggregate uncertainty: key issues and alternatives
- Friday: “Final assignment”
  - Solve/estimate model with heterogeneous firms and aggregate uncertainty

Overview for today

Extend Ayiagari model to include aggregate uncertainty

1. extend Aiyagari model with aggregate uncertainty
2. Krusell-Smith algorithm
3. practical issues (simulation, accuracy)
4. one alternative

Aiyagari model with aggregate uncertainty

Extend Aiyagari model

Assume presence of aggregate uncertainty

- enters production function of representative firm

$$Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha}$$

- otherwise there is no difference from model before

New individual problem

The individual problem is now

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \ln(c_{i,t})$$

s.t.

$$c_{i,t} + k_{i,t+1} = r_t k_{i,t} + w_t \epsilon_{i,t} + (1-\delta) k_{i,t}$$

$$k_{i,t+1} \geq 0$$

- how is this different from before?!

- $r_t = Z_t \alpha K_t^{\alpha-1}$ and $w_t = Z_t(1-\alpha)K_t^{\alpha}$

Moreover, what do agents really care about?

- not just r_t and w_t
- but also all future values of r and w!

What do agents care about?

The need to forecast prices necessitates

- the need for information for forecasting K
- in general, we need to forecast the joint distribution
    - of capital holdings and idiosyncratic productivity levels
- now this is really a tough problem
    - the distribution is infinite-dimensional!

Before moving on

The above discussion makes clear

- that partial equilibrium is much easier
- even with aggregate uncertainty

The individual's problem is easy to solve if

- you know the path of r_t and w_t!
- the above hints at how to solve the GE model

Overview for today

Extend Ayiagari model to include aggregate uncertainty

1. ~~extend Aiyagari model with aggregate uncertainty~~
2. Krusell-Smith algorithm
3. practical issues (simulation, accuracy)
4. one alternative

Krusell-Smith algorithm

Main idea of Krusell-Smith algorithm

Principle is similar to that in Aiyagari model

- guess (evolution of) prices
- individual problem solved for given (evolution of) prices
- simulate economy, aggregate, check implied price paths

The difficulty is that to forecast K

- we really need to forecast the entire joint distribution
    - of capital holdings and idiosyncratic productivity

Forecasting the joint distribution

Let $\mathbb{F}_t$ be the beginning-of-period joint distribution

- of capital holdings and idiosyncratic productivity

Its evolution can then be described as

$$\mathbb{F}_{t+1} = \mu(Z_t, \mathbb{F}_t)$$

- given today's joint distribution
- and today's aggregate shock
- can forecast tomorrows joint distribution
  - recall that idiosyncratic shocks are exogenous

Forecasting the joint distribution

However, this is still a crazy difficult problem!

- distribution is infite-dimensional

Krusell and Smith (1998) propose to instead

- approximate the distribution by
- focusing on a limited set of characteristics

Krusell-Smith algorithm

An iterative procedure

Instead of figuring out $\mu(.,.)$

- assume an approximating "aggregate law of motion"

$$m_{t+1} = \overline{\mu}(Z_t, m_t; \psi_{\overline{\mu}})$$

- m_t is a set of moments of the joint distribution
    - you need to make a stand on m_t beforehand
    - we'll discuss such choices later

Krusell-Smith algorithm

1. guess a value for $\psi_{\overline{\mu}}$
   - implies values for K_t^D and thus r_t and w_t
2. solve individual problem with given aggregate law of motion
3. simulate economy and calculate moments of joint distribution
4. estimate $\hat{\psi}_{\overline{\mu}}$ implied by simulation
5. compare to previous guess
   - if $\hat{\psi}_{\overline{\mu}} = \psi_{\overline{\mu}} \rightarrow$ stop
   - if $\hat{\psi}_{\overline{\mu}} \neq \psi_{\overline{\mu}} \rightarrow$ update and go to 2

What to keep in mind

Algorithm based on idea of "approximate aggregation"

- evolution of prices described well using
  - exogenous shocks
  - a limited number of moments of current distribution
- does not mean that
  - behavior aggregates to rep-agent model
  - or that individual variables behave as aggregates!

Taking stock

Krusell-Smith algorithm

- same principle as when solving model without aggregate uncertainty
- key difficulty lies in having to forecast evolution of prices
- need to track entire joint distribution of state variables

Overview for today

Extend Ayiagari model to include aggregate uncertainty

1. ~~extend Aiyagari model with aggregate uncertainty~~
2. ~~Krusell-Smith algorithm~~
3. practical issues (simulation, accuracy)
4. one alternative

Practical issues

Krusell-Smith algorithm

1. guess aggregate law of motion ($\psi_{\overline{\mu}}$)
   - implies values for K_t^D and thus r_t and w_t
2. solve individual problem with given aggregate law of motion
3. simulate economy and calculate moments of joint distribution
4. estimate $\hat{\psi}_{\overline{\mu}}$ implied by simulation
5. compare to previous guess
   - if $\hat{\psi}_{\overline{\mu}} = \psi_{\overline{\mu}} \rightarrow$ stop
   - if $\hat{\psi}_{\overline{\mu}} \neq \psi_{\overline{\mu}} \rightarrow$ update and go to 2

Choices to be made

There are several choices you must make

- how to solve individual problem?
- which moments to pick (and of what)?
- how to update the aggregate law of motion?
- how to simulate the economy?
- when to stop iterating?
- how to check for accuracy?
- non-trivial market clearing and imposing equilibrium?

Practical issues

Individual problem

How to solve individual problem

You can use your favorite solution method, but

- number of state variables has increased

How to solve individual problem

What trade-offs are you facing?

Practical issues

Aggregate law of motion

Which moments to pick?

Krusell-Smith used only mean of capital

- there is no rule about this $\rightarrow$ depends on application
- in their setup, means give sufficiently accurate results
    - if policy rules are *exactly linear* in levels
    - only mean necessary for computing next period's mean
        - distribution of wealth doesn't matter
    - in their setup, policy rules close to linear in levels

Which moments to pick?

Essentially trial and error

- use “bottom-up” approach
    - start with means
    - solve model (i.e. until aggregate law of motion converges)
    - check accuracy
    - if you’re lucky, you’re done
    - if not, adjust (number of) moments

Which moments to pick?

"Non-traditional" moments can be informative

- e.g. mass of agents around important cutoff

Past values of aggregate shocks

- easier to implement
- often quite informative about distribution
- intuition?

Aggregate law of motion for what?

Above we've used moments of joint distribution

- this implied path for K_t^D and in turn w_t and r_t

But it is really prices that agents care about

- specify aggregate law of motion directly for prices
- but, still need laws of motion for moments of interest!

How to update coefficients?

Let $\psi^{q}_{\overline{\mu}}$ be the coefficient guess in the qth iteration

- after solving and simulating the model economy
- obtain a time-series of the moments of interest m_t
- use this time-series to estimate coefficients
  - regress m_{t+1} on Z_t and m_t
  - obtain $\hat{\psi}_{\overline{\mu}}$
- update coefficient guess according to

$$\psi^{q+1}_{\overline{\mu}} = \lambda \hat{\psi}_{\overline{\mu}} + (1 - \lambda)\psi^{q}_{\overline{\mu}}$$

  - choosing λ is again tricky
  - strike compromise between speed and convergence

Practical issues

Simulation

Two ways how to simulate

Given individual policy rules

- simulate a large cross-section of agents
  - use Monte-Carlo integration to get moments
- use a grid method not requiring stochastic simulation
  - not introducing any sampling noise

Simulating a cross-section

- very intuitive and simple to implement
- however, quite computationally costly
  - need large cross-section
  - need long cross-section

Policy rules $k_{i,t+1} = k(k_{i,t}, \epsilon_{i,t}, S_t)$ are given

- where S_t is aggregate state (including tracked moments)

Create fine grid of nodes ($\kappa_{i,j}$) for joint distribution

- $p_{i,j,t}$ is the beginning-of-period mass of agents with
  - $k_{i,t}$ capital holdings and
  - $\epsilon_{j,t}$ level of idiosyncratic productivity

Grid method

Assume an initial joint distribution

- first, focus on capital choice

  1. for each node, figure out end-of-period capital choice

$$k_{i,j,t+1} = k(k_{i,t}, \epsilon_{j,t}, S_t)$$

  2. assign beginning-of-period mass ($p_{i,j,t}$) to nodes

     - issue: no mass in between nodes
     - split mass proportionally between neighboring nodes

Grid method

Split beginning-of-period mass proportionally

$$\omega_{i,j,t} = \frac{k_{i,j,t+1} - \kappa_{i-1,j}}{\kappa_{i,j} - \kappa_{i-1,j}}$$

$$\overline{p}_{i-1,j,t} = \overline{p}_{i-1,j,t} + p_{i,j,t}(1 - \omega_{i,j,t})$$

$$\overline{p}_{i,j,t} = \overline{p}_{i,j,t} + p_{i,j,t}\omega_{i,j,t}$$

- $\overline{p}_{i,j,t}$ is end-of-period mass
- careful at end points
- careful, one grid point can “be filled” by many others!

Grid method

- second, focus on idiosyncratic productivity
    - use transition law to figure out
    - beginning-of-period joint distribution in next period

1. for each node, figure out next-periods mean productivity value

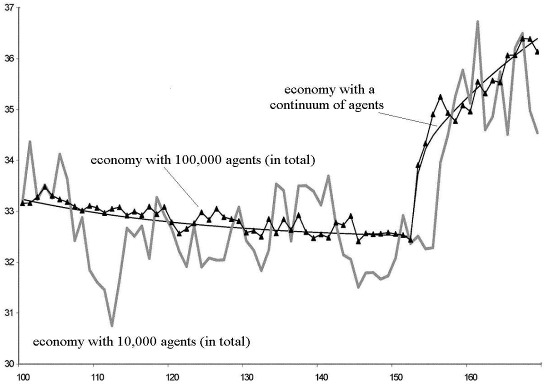
$$\epsilon_{j,t+1} = \rho \epsilon_{j,t}$$

2. assign mass end-of-period mass $(\overline{p}_{i,j,t})$ to nodes
    - according to distribution of idiosyncratic shocks, e.g. $N(0, \sigma_\epsilon^2)$
    - again split mass proportionally between nodes

Trade-offs with the above

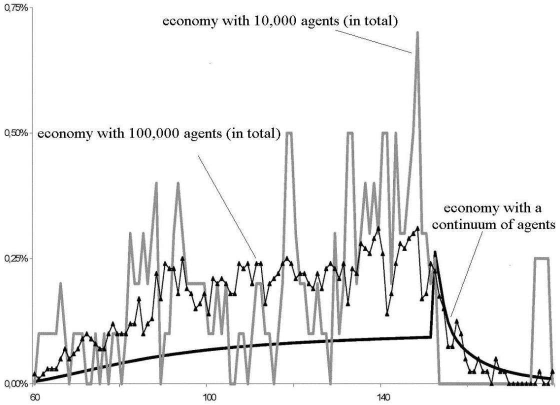
- Monte-Carlo simulation
  - can be computationally costly
  - introduces sampling noise
- Grid method
  - constructing the grid is not easy
    - may need many nodes
    - sometimes need precession in parts of state-space

When is simulation choice important (Algan et al., 2010)?



Notes: This graph plots the simulated aggregate capital stock of the unemployed using either a finite number (10,000) or a continuum of agents. It displays a subset of the observations shown in Figure 1.

When is simulation choice important (Algan et al., 2010)?



Notes: This graph plots the simulated fraction of unemployed agent at the borrowing constraint using either a finite number (10,000) or a continuum of agents.

Practical issues

Accuracy

When to stop?

When to stop what?

- when to stop iterating on aggregate law of motion?
- when to stop looking for accurate law of motion?
    - i.e. even if it has converged!

Updating procedure

Let $\psi_{\overline{\mu}}^{q}$ be the coefficient guess in the qth iteration

- after solving and simulating the model economy
- obtain a time-series of the moments of interest m_t
- use this time-series to estimate coefficients
  - regress m_{t+1} on Z_t and m_t
  - obtain $\hat{\psi}_{\overline{\mu}}$
- update coefficient guess according to

$$\psi_{\overline{\mu}}^{q+1} = \lambda \hat{\psi}_{\overline{\mu}} + (1 - \lambda) \psi_{\overline{\mu}}^{q}$$

When to stop updating?

Choice of λ

- strike compromise between speed and chance of convergence
- must specify a measure of (update) distance
  - e.g. max-abs-difference between updates

$$e^q = \max(\text{abs}(\psi_{\mu}^{q} - \psi_{\mu}^{q-1}))$$

  - or often more conveniently in percentage terms

$$e^q = \max(\text{abs}((\psi_{\mu}^{q} - \psi_{\mu}^{q-1})/\psi_{\mu}^{q-1}))$$

- stop when e^q falls below a certain threshold (e.g. 0.01%)

When to stop updating?

What does convergence of the aggregate law of motion mean?

Accuracy checks

What dimensions of inaccuracy are there in our setup?

After we stop updating law of motion

We have

- individual policy rules
- initial joint distribution
- a simulation method
- a candidate aggregate law of motion

$$m_{t+1} = \overline{\mu}(Z_t, m_t; \psi_{\overline{\mu}}) + u_{t+1} \tag{1}$$

  - the above needs to be chekced for accuracy
  - true law of motion has $u_{t+1} = 0 \ \ \forall t$

Popular way to check for accuracy

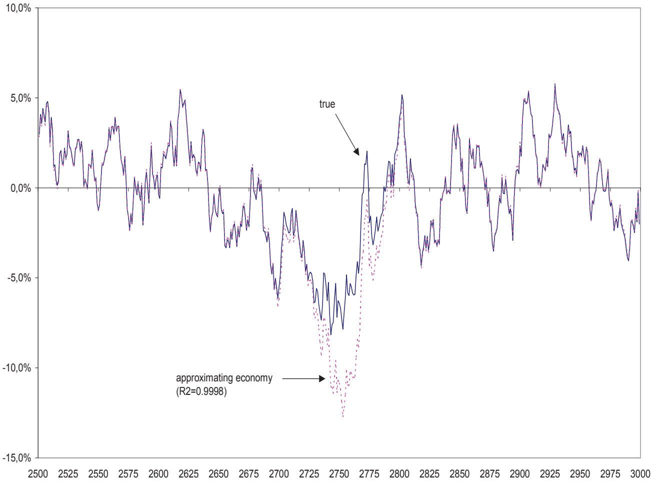
Use only individual policy rules

- simulate economy and obtain a time-series for m_t
- use these time-series to run regression (1)
- evaluate goodness of fit by looking at R^2
  - and sometimes standard error of regression

Issues with popular accuracy tests

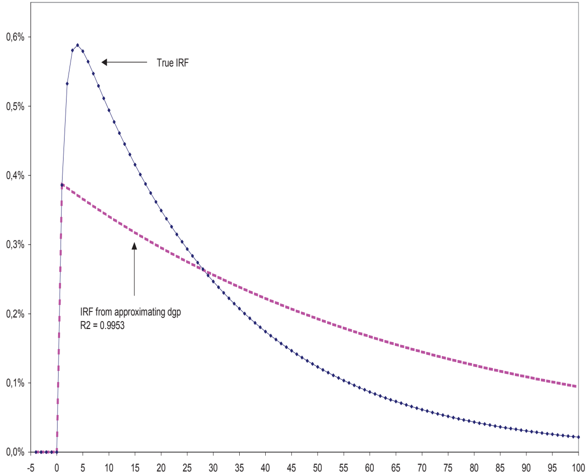
- R^2 and $\hat{\sigma}_u$ are averages
  - can mask potentially large, counteracting, differences
- not clear what is a low R^2
  - check out den Haan's work for fun examples!
- main problem is conceptual!
  - each period, the truth is used to update the approximation!
  - i.e. we're cutting the law of motion too much slack

Great R^2 but poor performance



Source: den Haan (2010)

Great R^2 but poor performance



Source: den Haan (2010)

Better accuracy tests (den Haan, 2010)

Generate two sequences of desired moments m_t

And compare the two sequences

- accuracy plot!
- compute $\max(\text{abs})$ difference and see where it occurs
  - important part of the state-space?
- what is the average/minimum error?
- look at IRFs under the two laws of motion etc.

Practical issues

Imposing equilibrium

Approaching equilibrium?

We've seen that the iterative algorithm approaches to (the) equilibrium

- is this always going to be the case?

Imagine the same economy, but with bonds in zero net supply

- how to solve for the aggregate bond price q_t?

Solving for the bond price

We could try to use an aggregate law of motion for q_t

1. guess coefficients of $q_t = q(S_t; \psi_q)$
2. solve individual problem
3. simulate economy
4. update coefficients ψ_q accordingly

So where's the problem?!

Imposing equilibrium

There is nothing ensuring zero net supply across iterations!

- departures from equilibrium are likely to accumulate!

Instead of q_t we can approximate something else to impose equilibrium

- $d(s_{i,t}) = b'(s_{i,t}) + q_t$
    - q_t is aggregate bond price
    - $b'(s_{i,t})$ is individual bond choice

How does this help?!

- $q_t = \sum_i d(s_{i,t})$ in *each* period
- i.e. are imposed to clear!

Taking Stock

Challenges when solving heterogeneous agent models with aggregate uncertainty

- solving individual problem
- which moments to track
- how to update guesses
- how to simulate economy
- when to stop iterating
- how to check accuracy

Overview for today

Extend Ayiagari model to include aggregate uncertainty

1. ~~extend Aiyagari model with aggregate uncertainty~~
2. ~~Krusell-Smith algorithm~~
3. ~~practical issues (simulation, accuracy)~~
4. one alternative

Alternative Solution Method

Main idea

Krusell-Smith algorithm is computationally costly

- simulation step takes time
- iterative updating takes time
- accuracy checks and moment selection take time

Why do we need to do all the above?

- need to know the joint distribution of capital and productivity
  - distribution of infinitely-lived agents
  - hit by persistent idiosyncratic shocks
- can we get around the above?

Alternative Solution Method

Main idea

Basic idea

Agent heterogeneity?

- what if we had (many) ex-ante heterogeneous agents (I)?

Infinitely-lived agents?

- what if we consider an OLG framework instead (T)?

What have these chagnes given us?

- ex-ante heterogeneity: finite number of types of agents
- finite life-time: finite number of periods to track each type
- $\rightarrow T \times I$ types of agents

Basic idea

This is great, because $T \times I$ is finite!

- moreover, the $T \times I$ types describe the *entire* distribution!

What about the curse of dimensionality?

- $\rightarrow$ do perturbation

No need for iterative procedure

- entire distribution described by transitions between types
    - i.e. aging of agents

Alternative Solution Method

Details

Details

Specific application: firm dynamics model

- there are I *types* of firms
  - productivity/demand heterogeneity (different long-run sizes)
- firms live for (at most) A periods
  - life-cycle heterogeneity within firm types

Details cont.

Each age-type has a different value function!

$$V_{i,a}(S_t) = \pi_{i,a}(S_t) + \beta(1-\delta_a)\mathbb{E}V_{i,a+1}(S_{t+1})$$

With

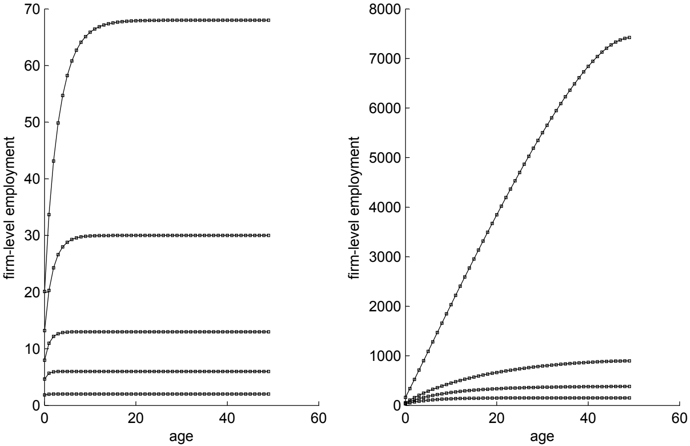
- δ is age-specific death rate
- S_t being the aggregate state
  - aggregate shocks, but also entire distribution of firms!

Keeping track of entire distribution is easy!

$$\omega_{i,a} = (1-\delta)\omega_{i,a-1} \quad a \in (1, A]$$

$$\omega_{i,0} = \text{free entry condition}$$

Resulting economy has lots of heterogeneity



Source: Sedláček and Sterk (2016)

Resulting economy has lots of heterogeneity

Table 1: Employment share distribution by size and age

	data			model		
	small	medium	large	small	medium	large
0 to 5 years	50	41	8	52	40	8
6 to 10 years	36	46	18	36	48	16
11 to 15 years	29	45	26	27	46	27
16 to 20 years	24	42	34	22	44	34
21 to 25 years	18	39	43	18	39	43

Notes: Employment shares in percentages of small (1-19 employees), medium-sized (20-499) and large (500 and over) firms, by age. Data (averages) and models (steady states).

Alternative Solution Method

Implementation

Quantitative implementation

- results in more than 900 state variables
    - include the entire joint distribution of
    - firm employment and their masses
- solved with first-order perturbation
    - along steady state growth path
- solution takes several seconds
    - fast enough that we can estimate parts of the model

Macro-language in Dynare

- can define loops over certain criteria (e.g. types)
- ideal for heterogeneous-agent setups
  - structure of first-order conditions the same across types
  - they differ in (some) parameter values
- other examples of its use include
  - multi-country models

Alternative Solution Method

Taking Stock

Taking stock

Solving models with ex-ante heterogeneity

- can be computationally easier (if linearizing)
- contains the entire distribution of state-variables
- no need to revert to iterative procedures like Krusell-Smith
- not applicable to some models + question of “true” heterogeneity

Overview for today

Extend Ayiagari model to include aggregate uncertainty

1. ~~extend Aiyagari model with aggregate uncertainty~~
2. ~~Krusell-Smith algorithm~~
3. ~~practical issues (simulation, accuracy)~~
4. ~~one alternative~~

Solution Methods for Macroeconomic Models

- Monday - Tuesday: Solving models with "representative agents"
  - ~~Linearization in theory and practice: Dynare~~
  - ~~Non-linear solutions methods: value function iteration, projection~~
  - Analyzing models: parameterization/estimation, simulation/IRFs
- Wednesday - Thursday: Solving models with "heterogeneous agents"
  - ~~Models without aggregate uncertainty: basic algorithm~~
  - ~~Models with aggregate uncertainty: key issues and alternatives~~
- Friday: "Final assignment"
  - Solve/estimate model with heterogeneous firms and aggregate uncertainty


That's all Folks!