Lecture 1: Households

Heterogeneous Agents in Macroeconomics

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Campbell and Mankiw (1989)

Unit-mass of consumers. Fraction $(1-\lambda)$ follow the permanent income rule. Consumption is C, δ is the discount rate. Type-2 agent solves:

$$\max \mathbb{E}_t \sum_{s=0}^{\infty} (1+\delta)^{-s} \mathcal{U}(C_{t+s}), \quad \mathcal{U}' > 0, \quad \mathcal{U}'' < 0$$

Solution:

$$\mathbb{E}_t \mathcal{U}'(C_{t+1}) = \left(\frac{1+\delta}{1+r}\right) \mathcal{U}' C_t$$

Assume $r = \delta$ and linear marginal utility:

$$\mathbb{E}_t C_{t+1} = C_t \qquad \Delta C_t = \eta_t$$

η_t: innovation in permanent income.

Campbell and Mankiw (1989)

Fraction λ of consumers are "rule-of-thumb", consume current income.

Incomes of the two types: $Y_{1,t}$ and $Y_{t,2}$. Total income is $Y_t = Y_{1,t} + Y_{2,t}$. Thus, $Y_{1,t} = \lambda Y_t$ and $Y_{2,t} = (1-\lambda)Y_t$.

Consumption of the two types: $C_{1,t} = Y_{1,t}$ implying $\Delta C_{1,t} = \Delta Y_{1,t} = \lambda Y_t$. And $\Delta C_{2,t} = (1-\lambda)\eta_t$.

Aggregate consumption growth:

$$\Delta C_t = \lambda \Delta Y_t + (1-\lambda)\eta_t$$

Campbell and Mankiw (1989)

Now, allow for time-varying real rate r_t. Log-linearized Euler equation:

$$\Delta C_t = \sigma r_t + \eta_t$$

with σ the elasticity of intertemporal substitution.

With rule-of-thumb consumers:

$$\Delta C_t = \lambda \Delta Y_t + (1 - \lambda)\sigma r_t + \eta_t$$

Campbell and Mankiw (1989) estimate low σ and large λ implying: rejection of the PIH with low EIS, in favor of a model with rule-of-thumb households.

Campbell and Mankiw (1989)

Derivation of the consumption function given risky returns R_t.

Periodic budget constraint:

$$W_{t+1} = R_{t+1}\left(W_t - C_t\right) \tag{1}$$

Forward-iteration plus the tranversality condition:

$$W_t = C_t + \sum_{s=1}^{\infty} C_{t+s}\left(\prod_{j=1}^{s} R_{t+j}\right)^{-1} \tag{2}$$

Divide (1) by W_t and take logs:

$$w_{t+1} - w_t = r_{t+1} + \log\left(1 - \frac{C_t}{W_t}\right) = r_{t+1} + \log\left(1 - \exp\left(c_t - w_t\right)\right) \tag{3}$$

Campbell and Mankiw (1989)

First-order Taylor approximation of the second, non-linear term around $c_t - w_t = c - w$:

$$\log\left(1 - \exp\left(c_t - w_t\right)\right) \approx \underbrace{\log(\rho) - \frac{\rho - 1}{\rho}\log(1-\rho)}_{\kappa} + \frac{\rho - 1}{\rho}(c_t - w_t)$$

with $\rho := 1 - \exp(c - w)$. Plug back into (3):

$$\Delta w_{t+1} \approx \kappa + r_{t+1} + \frac{\rho - 1}{\rho}(c_t - w_t) \tag{4}$$

Solve (4) forward:

$$c_t - w_t = \frac{\kappa\rho}{1-\rho} + \sum_{s=1}^{\infty} \rho^s (r_{t+s} - \Delta c_{t+s}) \tag{5}$$

This is the log-linear intertemporal budget constraint.

Campbell and Mankiw (1989)

Back to the log-linear Euler equation:

$$\mathbb{E}_t \Delta c_{t+1} = \sigma \mathbb{E}_t r_{t+1} \tag{6}$$

Take conditional expectation of (5):

$$c_t - w_t = \mathbb{E}_t \left(\sum_{s=1}^{\infty} \rho^s (r_{t+s} - \Delta c_{t+s}) \right) + \frac{\kappa\rho}{1-\rho} \tag{7}$$

Substitute (6) into (7):

$$c_t - w_t = (1-\sigma)\mathbb{E}_t \sum_{s=1}^{\infty} \rho^s r_{t+s} + \frac{\rho\kappa}{1-\rho} \tag{8}$$

Optimal consumption as a function of wealth, interest rates, and a constant.

Campbell and Mankiw (1989)

$$c_t - w_t = (1-\sigma)\mathbb{E}_t \sum_{s=1}^{\infty} \rho^s r_{t+s} + \frac{\rho\kappa}{1-\rho} \tag{9}$$

Observations:

1. $\sigma = 1$: log-utility. Consumption is a constant fraction of wealth.
2. $\sigma < 1$: income effect dominates. High interest rates raise consumption.
3. $\sigma > 1$: substitution effect dominates. High interest rates lower consumption.
4. Persistent changes in rates have stronger effects on consumption

Campbell and Mankiw (1989)

Endogenize wealth. Back to the budget constraint where we do not impose market clearing:

$$W_{t+1} = R_{t+1}\left(W_t - Y_t\right) \tag{10}$$

Divide by W, take logs, and linearize as before:

$$y_t - w_t = \frac{\kappa\rho}{1-\rho} + \sum_{s=1}^{\infty} \rho^s (r_{t+s} - \Delta y_{t+s}) \tag{11}$$

Substitute into (9) and get:

$$c_t - y_t = \mathbb{E}_t \sum_{s=1}^{\infty} \rho^s \left(\Delta y_{t+s} - \sigma r_{t+s}\right) \tag{12}$$

As σ falls, approach the permanent income model.

Campbell and Mankiw (1989)

Generalized case with rule-of-thumb consumers:

$$c_t - y_t = (1-\lambda)\mathbb{E}_t \sum_{s=1}^{\infty} \rho^s \left(\Delta y_{t+s} - \sigma r_{t+s}\right) \tag{13}$$

Rule-of-thumb households reduce variability of the (log) consumption-income ratio.

In one extreme of $\lambda = 1$, consumption is completely out of income as the world is populated only by "hand-to-mouth" consumers.

In the other, one approaches the PIM for a given σ with consumption reacting more to the discounted stream of future incomes.

The Campbell-Mankiw estimate for $\hat{\lambda}$ is around 0.3-0.5.

Aiyagari (1994)

Unit mass of ex-ante identical consumers who solve:

$$\max \left[\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \mathcal{U}(C_t) \right] \tag{14}$$

s.t.

$$c_t + k_{t+1} = (1 + \tilde{r})k_t + ws_t \tag{15}$$

$$k_{t+1} \in \mathcal{K} \tag{16}$$

$$k_t \geq -\phi \tag{17}$$

$$s_t \quad \text{follows } m\text{-state first-order Markov with transition matrix } \mathcal{P} \tag{18}$$

Aiyagari (1994)

Eq. (14): $\mathcal{U}$ is strictly increasing, concave and differentiable. $\beta \in (0, 1)$ is the subjective discount factor

Eq. (15): k_t is a single asset (capital stock), δ is the depreciation rate, $\tilde{r}$ is the rate of return, w is the market wage rate, s is employment status or labor income.

Eq. (16): the asset is constrained by a grid $\mathcal{K} = [k_1, \ldots, k_n]$. $k_1 = -\phi$. k_n is a non-binding upper bound.

Eq. (17): borrowing constraint. ϕ is the borrowing limit, ad-hoc or natural.

Eq. (18): is the source of uninsured idiosyncratic (labor) income risk.

Aiyagari (1994)

Suppose $m = 2$. Denote $h \in [1, \ldots, n]$. Then, the corresponding Bellman equations of the above problem are, for all h:

$$v(k_h, s_1) = \max_{k' \in \mathcal{K}} \left[\mathcal{U}\left((1+\tilde{r})k_h + ws_1 - k'\right) + \beta \sum_{j=1}^{2} \mathcal{P}(1,j) v(k', s_j) \right]$$

$$v(k_h, s_2) = \max_{k' \in \mathcal{K}} \left[\mathcal{U}\left((1+\tilde{r})k_h + ws_2 - k'\right) + \beta \sum_{j=1}^{2} \mathcal{P}(2,j) v(k', s_j) \right]$$

$v(k, s)$ is the value function: the optimal value of the objective function of the program.

$k' = g(k, s)$ is the associated policy function that maps (k, s) into optimal asset holdings.

AIYAGARI (1994)

Denote the unconditional distribution of (k, s) at time t with $\lambda_t(k, s) = \text{Prob}\,(k_t = k, s_t = s)$.

Markov transition matrix $\mathcal{P}$ and the policy $g(k, s)$ induce a law of motion for the distribution:

$$\lambda_{t+1}(k', s') = \sum_{s} \sum_{k:k'=g(k,s)} \lambda_t\,(k, s)\,\mathcal{P}(s, s') \tag{19}$$

Stationary distribution $\lambda_{t+1} = \lambda_t = \lambda$ solves Eq. (19).

Aiyagari (1994)

Average asset holdings in the economy, given r:

$$E(k)(r) = \sum_{k,s} \lambda(k,s) g(k,s) \tag{20}$$

The duality of time:

1. One individual living recurring episodes over time ...
2. ... or infinitely large cross section living for one period.

Both 1. and 2. induce the same population average in Eq. (20).

The goal of this class of models: equilibrate (r) while maintaining individual optimality and stationarity of the distribution.

Aiyagari (1994)

Production:

$$F(K, N) = AK^{\alpha}N^{1-\alpha}, \quad \alpha \in (0, 1) \tag{21}$$

N is exogenous. K is endogenous. Rental rates on capital and labor:

$$w = \partial F(K, N) / \partial N \tag{22}$$

$$r = \partial F(K, N) / \partial K \tag{23}$$

Aggregate capital evolves according to:

$$K_{t+1} = (1 - \delta)K_t + I_t$$

Equilibrium risk-free return:

$$\tilde{r} = r - \delta \tag{24}$$

Aiyagari (1994)

More on borrowing limits. Solve Eq. (15) forward:

$$k_t \geq \frac{1}{1+r-\delta} \sum_{j=0}^{\infty} (1+r-\delta)^{-j} (c_t - w s_{t+j}) \tag{25}$$

Suppose $\underline{s} = \min(s)$ is the worst possible state. Then, assuming $c_t \geq 0$:

$$k_t \geq -\frac{\underline{s} w}{r-\delta} \tag{26}$$

Eq. (26) defines the natural debt limit. Generally:

$$k_t \geq -\phi, \quad \phi = \min\left(\frac{\underline{s} w}{r-\delta}, b\right) \tag{27}$$

where $b \geq 0$ is an ad-hoc debt limit.

Aiyagari (1994)

Equilibrium of the Aiyagari (1994) model:

A stationary equilibrium is a value function $v(k, s)$ with the associated policy function $g(k, s)$, a probability distribution $\lambda(k, s)$, and endogenous aggregate states (K, r, w) such that:

1. $g(k, s)$ solves the household problem in (14)-(18).
2. Prices (w, r) are solved according to (22) and (23).
3. Distribution $\lambda(k, s)$ is stationary and satisfies (19).
4. Aggregate capital K is implied by optimal $g(k, s)$ and $\lambda(k, s)$ as in (20).

Aiyagari (1994)

Equilibrium is defined by a fixed point mapping of $K \in \mathbb{R}$ into $\mathbb{R}$. Denote iteration count j:

1. Start with some $K_{j=0}$. Compute prices (r, w). Given prices, solve the household problem and obtain $g_j(k, s)$.
2. Compute the stationary distribution $\lambda_j(k, s)$.
3. Given $g_j(k, s)$ and $\lambda_j(k, s)$ compute the new aggregate capital stock:

$$K^* = \sum_{k,s} \lambda_j(k, s) g_j(k, s)$$

4. Compute the error $\epsilon = |K^* - K_j|$. Update the initial guess given some $\nu \in (0, 1)$:

$$K_{j+1} = \nu K_j + (1 - \nu) K^*$$

5. Iterate until convergence, i.e. ϵ is very small.

Aiyagari (1994)

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TABLE II

A. Net return to capital in %/aggregate saving rate in % ($\sigma = 0.2$)

$\rho \backslash \mu$	1	3	5
0	4.1666/23.67	4.1456/23.71	4.0858/23.83
0.3	4.1365/23.73	4.0432/23.91	3.9054/24.19
0.6	4.0912/23.82	3.8767/24.25	3.5857/24.86
0.9	3.9305/24.14	3.2903/25.51	2.5260/27.36

B. Net return to capital in %/aggregate saving rate in % ($\sigma = 0.4$)

$\rho \backslash \mu$	1	3	5
0	4.0649/23.87	3.7816/24.44	3.4177/25.22
0.3	3.9554/24.09	3.4188/25.22	2.8032/26.66
0.6	3.7567/24.50	2.7835/26.71	1.8070/29.37
0.9	3.3054/25.47	1.2894/31.00	−0.3456/37.63

σ: variance of labor income risk.
ρ: serial correlation of labor income risk.
μ: relative risk aversion.

Aiyagari (1994)

Aiyagari (1994) shows that market incompleteness and uninsured idiosyncratic labor income risk are a source of a precautionary savings motive.

Households save in the only asset they can - the risk-free capital stock k - more than they would otherwise.

As a result, the equilibrium aggregate capital stock of the economy is greater than in the complete-markets benchmark.

Furthermore, the risk-free interest rate is lower.

Stronger market incompleteness increases the buffer stock of capital, further depressing the rate.

Takeaway

There are (at least) two fundamental ways of introducing heterogeneity.

1. Ex-ante heterogeneity: agents differ by nature from the beginning of time.

Such differences could be in anything structural: hand-to-mouth vs permanent income, risk aversion, productivity, etc.

2. Ex-post heterogeneity: agents are the same but - due to incomplete markets and idiosyncratic shocks - are different after the fact.

Such differences could be due to labor income risk, asset return risk, health outcomes, etc.

Takeaway

With ex-ante heterogeneity, you are essentially solving the same problem but twice - each time conditional on a different structural parameter/assumption.

With ex-post heterogeneity, a key insight is the precautionary saving behavior of agents that face market incompleteness.

Generally, in the two classes of models the *level* of macroeconomic aggregates in equilibrium differs from the representative-agent benchmark.

However, it's not always obvious if *sensitivity* to aggregate shocks differs.