Lecture 5: Granularity

Heterogeneous Agents in Macroeconomics

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GABAIX (2016)

Power law is a scaling relationship between X and Y of the type:

$$Y = \alpha X^{\beta} \tag{1}$$

β is the power law exponent and α a constant.

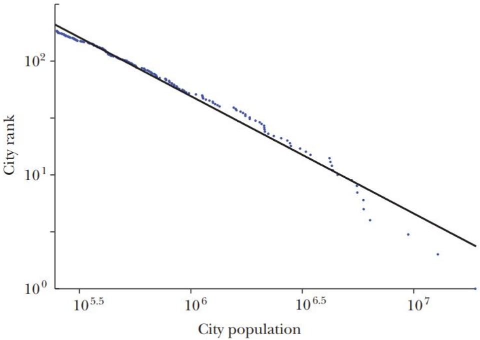
Simple empirical test for power law dynamics in practice with OLS:

$$\log(\text{Rank}) = \alpha - \beta \log(\text{Size}) \tag{2}$$

Power Laws

$$\log(\text{Rank}) = 7.88 - 1.03\log(\text{Size})$$

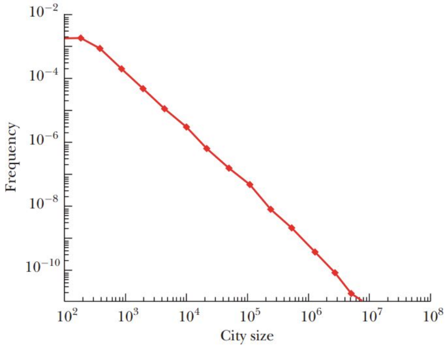
Figure 1
A Plot of City Rank versus Size for all US Cities with Population over 250,000 in 2010



Source: Author, using data from the *Statistical Abstract of the United States* (2012).
Notes: The dots plot the empirical data. The line is a power law fit ($R^2 = 0.98$), regressing ln *Rank* on ln *Size*. The slope is -1.03, close to the ideal Zipf's law, which would have a slope of -1.

Power Laws

Figure 2
Density Function of City Sizes (Agglomerations) for the United Kingdom



Source: Rozenfeld et al. (2011).
Notes: We see a pretty good power law fit starting at about 500 inhabitants. The Pareto exponent is actually statistically non-different from 1 for size $S > 12{,}000$ inhabitants.

Pareto Distribution

Probability that a random variable X is greater than some x is:

$$\text{Pr}(X > x) = \begin{cases} \alpha^{\xi} \times x^{-\xi} & x \geq \alpha \\ 1 & x < \alpha \end{cases}$$

ξ is called the Pareto exponent or the Pareto tail.

A lower ξ means higher inequality and concentration in the distribution.

A Pareto principle states that 80% of acts or consequences originate from 20% of agents. In practice, the Pareto principle roughly corresponds to $\xi \sim 1.2$.

For $\xi \in (1, 2]$ the variance of X is infinite and standard central limit theorems break down.

Zipf's Law

A special case of the above for $\xi = 1$ is the "Zipf's Law".

Suppose there are N firms in an economy with independent shocks.

Aggregate fluctuations are proportional to $\frac{1}{\sqrt{N}}$. As $N \to \infty$, idiosyncratic fluctuations have a negligible effect.

Under Zipf's law, the above is no longer true. Aggregate volatility instead decays according to $\frac{1}{\ln N}$, which is much weaker.

If the distribution of firms is fat-tailed, diversification from firm-level disturbances is smaller as the variance of idiosyncratic shocks does not wash out as N grows.

Gabaix (2011)

Islands economy with N firms, no network effects, and exogenous production.

Firm i produces a quantity S_{it} of the good. Growth rate is:

$$\frac{\Delta S_{i,t+1}}{S_{i,t}} = \frac{S_{i,t+1} - S_{i,t}}{S_{i,t}} = \sigma_i \epsilon_{i,t+1} \tag{3}$$

σ_i is idiosyncratic volatility and $\epsilon_{i,t+1}$ are i.i.d. random shocks.

Total economywide GDP is:

$$Y_t = \sum_{i=1}^{N} S_{i,t} \tag{4}$$

GABAIX (2011)

GDP growth is:

$$\frac{\Delta Y_{t+1}}{Y_t} = \frac{1}{Y_t}\sum_{i=1}^{N} \Delta S_{i,t+1} = \sum_{i=1}^{N} \sigma_i \frac{S_{i,t}}{Y_t}\epsilon_{i,t+1} \quad (5)$$

Since shocks $\epsilon_{i,t+1}$ are uncorrelated, GDP volatility is:

$$\sigma_{GDP} = \left(\text{var}\frac{\Delta Y_{t+1}}{Y_t}\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{N} \sigma_i^2 \times \left(\frac{S_{i,t}}{Y_t}\right)^2\right)^{\frac{1}{2}} \quad (6)$$

Aggregate variance is the weighted sum of idiosyncratic shocks' variances σ_i^2 with weights equal to the squared share of firm-level output.

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Assume that firms all have the same volatility $\sigma_i = \sigma$. Then:

$$\sigma_{GDP} = \sigma \left[\sum_{i=1}^{N} \left(\frac{S_{i,t}}{Y_t} \right)^2 \right]^{\frac{1}{2}} = \sigma \mathcal{H} \tag{7}$$

where $\mathcal{H}$ is the square root of the sales Herfindahl-Hirschman index: a measure of concentration.

Suppose all firms share identical size equal to $\frac{1}{N}$ of GDP and identical σ. Then:

$$\sigma_{GDP} = \frac{\sigma}{\sqrt{N}} \tag{8}$$

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Now, calibrate. Suppose $\sigma = 0.1$. N=1 million. Then:

$$\sigma_{GDP} = \frac{\sigma}{\sqrt{N}} = \frac{0.1}{1,000} = 0.01\%$$

Aggregate volatility is roughly 1%. Idiosyncratic volatility of shocks that are drawn from a distribution with finite variance is way too small.

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Proposition 1

Consider an islands economy with N firms whose sizes S_i are drawn from a distribution with finite variance. Suppose they all have the same idiosyncratic volatility σ. Then, the economy's GDP volatility follows, as $N \to \infty$:

$$\sigma_{GDP} \sim \frac{\mathbb{E}[S^2]^{1/2}}{\mathbb{E}[S]} \frac{\sigma}{\sqrt{N}} \tag{9}$$

Proof - in class.

Volatility of GDP decays at the rate of $\frac{1}{\sqrt{N}}$ as N grows. For very large N, the contribution of idiosyncratic vol. to aggregate vol. becomes insignificant.

Gabaix (2011)

Proposition 2

Consider a series of island economies indexed by $N \geq 1$. *Economy* N *has* N *firms whose growth rate volatility is* σ *and whose sizes* $S_1, \ldots, S_N$ *are drawn from a power law density:*

$$\mathbb{P}(S > x) = \alpha x^{-\xi} \tag{10}$$

For $x > \alpha^{\frac{1}{\xi}}$, *with exponent* $\xi \geq 1$. *Then, as* $N \to \infty$, *GDP volatility follows:*

$$\sigma_{GDP} \sim \begin{cases} \frac{v}{\ln N}\sigma, & \text{for} \quad \xi = 1 \\ \frac{v}{N^{1-\frac{1}{\xi}}}\sigma, & \text{for} \quad 1 < \xi < 2 \\ \frac{v}{N^{1/2}}\sigma, & \text{for} \quad \xi \geq 2 \end{cases}$$

Where v is a random variable. When $v \leq 2$, v is the square root of a Lévy distribution with exponent $\xi/2$. When $\xi > 2$, v is a constant. Proof - in class.

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Notation $\sigma_{\text{GDP}} \sim \frac{v}{N^{1-\frac{1}{\xi}}}\sigma$ means that $\sigma_{\text{GDP}} \times N^{1-\frac{1}{\xi}} \xrightarrow{d} v\sigma$, i.e. convergence in distribution as $N \to \infty$.

The firm size distribution has thin tails, i.e. finite variance, iff $\xi > 2$.

If the distribution has thin tails, σ_{GDP} decays according to $\frac{1}{\sqrt{N}}$ as in the standard model.

For $\xi < 2$, then σ_{GDP} decays much slower.

The Granular Residual

Suppose firm productivity evolves as:

$$g_{i,t} \equiv z_{it} - z_{i,t-1} = \beta' X_{it} + \epsilon_{it} \tag{11}$$

The theoretical granular residual is defined as:

$$\Gamma_t^* := \sum_{i=1}^{K} \frac{S_{i,t-1}}{Y_{t-1}} \epsilon_{i,t} \tag{12}$$

This is a size-weighted sum of idiosyncratic shocks for K (usually largest) firms. The estimated granular residual is:

$$\Gamma_t := \sum_{i=1}^{K} \frac{S_{i,t-1}}{Y_{t-1}} \hat{\epsilon}_{i,t} \tag{13}$$

Identification is achieved if $\Gamma_t = \Gamma_t^*$.

Gabaix and Koijen (2023)

Granular instrumental variables.

Let an outcome be Y_t and the main regressor be $\bar{X}_t = \sum_i^N s_{i,t}x_{i,t}$, which is a size-weighted average of truly idiosyncratic shocks.

Suppose a generic relationship of the form:

$$Y_t = \alpha + \beta\bar{X}_t + \nu_t \tag{14}$$

$$x_{it} = \eta_t + u_{i,t} \tag{15}$$

Identification fails whenever η_t is correlated with ν_t, and thus $\bar{X}_t$ is correlated with ν_t. η_t stands for any confounding factor that deems $\bar{X}_t$ endogenous.

Gabaix and Koijen (2023)

Granular IV is the size-weighted average regressor minus its equal-weighted average:

$$z_t := \sum_i^N s_{i,t} x_{i,t} - \frac{1}{N} \sum_i^N x_{i,t} \quad (16)$$

Given that:

$$\sum_i^N s_{i,t} x_{i,t} = \eta_t + \sum_i^N s_{i,t} u_{i,t} \quad \text{and} \quad \frac{1}{N} \sum_i^N x_{i,t} = \eta_t + \frac{1}{N} \sum_i^N u_{i,t} \quad (17)$$

The GIV is:

$$z_t = \sum_i^N s_{i,t} u_{i,t} - \frac{1}{N} \sum_i^N u_{i,t} \quad (18)$$

The exogeneity condition $\mathbb{E}(u_t \nu_t) = 0$ guarantees:

$$\mathbb{E}(z_t \nu_t) = 0 \quad (19)$$

TAKEAWAY

If the distribution of an agent's characteristic is fat-tailed, then standard central limit theorems fail.

Idiosyncratic shocks may not not wash out in the aggregate and a small number of agents can drive aggregate fluctuations.

The granular IV identifies causal empirical relationships by constructing characteristic-weighted sums of idiosyncratic shocks.

Granularity can be viewed as a specific, extreme form of heterogeneity.