Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems – MTNS 2010 • 5–9 July, 2010 • Budapest, Hungary

Simple Homothetic Tube Model Predictive Control

Saša V. Raković, Basil Kouvaritakis, Rolf Findeisen and Mark Cannon

***Abstract*— This paper considers the robust model predictive control synthesis problem for constrained linear discrete time systems. The manuscript introduces a simple homothetic tube model predictive control synthesis method. The proposed method employs several novel features including: a more general parameterization of the state and control tubes based on homothety and invariance; a more flexible form of the terminal constraint set; and a relaxation of the dynamics of the sets that define the state and control tubes. Under rather mild assumptions it is demonstrated that the proposed method is computationally efficient while it induces strong system theoretic properties.**

I. INTRODUCTION

Both classical and contemporary research have recognized the need for a mathematical framework that deals with control synthesis under constraints and uncertainty. The computational complexity normally associated with dynamic programming [1]–[3] is certainly an issue with the control synthesis problem in the presence of constraints and uncertainty. Tube model predictive control (TMPC) [4]–[14] constitutes a sensible approximate solution methodology. The deployment of tubes in robust model predictive control is made possible through a parameterization of the control policy that allows for the direct handling of uncertainty and its interaction with the system dynamics, constraints and performance. In particular, in the case of linear systems with additive bounded uncertainty and convex constraint sets it is possible to use separable control policies. These, in turn, allow for the separation of the evolution of the nominal system (uncertainty free system) and the evolution of the local uncertain system. Then, the effect of the uncertainty can be accounted for by considering the exact reachable tubes centered around the trajectories of the nominal system and invoking suitably modified constraints on the nominal variables; at the same time it is possible to consider simpler and computationally more tractable performance objectives. Early proposals employing this construction include [15]–[17] and were followed with research investigations along similar lines [18]–[21] which adopt the same paradigm for the case of the parametric (multiplicative) uncertainty and for local robustification of nonlinear MPC. The approaches reported in [15]–[22] fall into the class of TMPC synthesis methods, even though the tube terminology was not invoked. These approaches resulted in a reduction of the computational complexity and guaranteed desirable system theoretic properties.

In this manuscript, we utilize the basic concepts of the general homothetic TMPC (HTMPC), recently developed in [23], and offer a simplified HTMPC synthesis for a more structured setting. As in [23], we make use of the homothetic state and control tubes and the corresponding parameterized, separable, control policy. The homothetic state and control tubes are sequences of sets $\{X_k = z_k \oplus \alpha_k S\}_{k \in \mathbb{N}_N}$ and $\{U_k = v_k \oplus \alpha_k R\}_{k \in \mathbb{N}_{N-1}}$ which are parameterized via fixed, but suitably chosen, basic shape sets S and R and the sequences of the centers and diameters $\{z_k\}_{k \in \mathbb{N}_N}$, $\{\alpha_k\}_{k \in \mathbb{N}_N}$ and $\{v_k\}_{k \in \mathbb{N}_{N-1}}$. The associated control policy $\{\pi_k(\cdot, X_k, U_k)\}_{k \in \mathbb{N}_{N-1}}$ is a sequence of control laws parameterized via a local control function $\nu(\cdot)$ (which is a continuous and positively homogeneous function of the first degree) and a sequence of the centers of the homothetic state and control tubes $\{z_k\}_{k \in \mathbb{N}_N}$ and $\{v_k\}_{k \in \mathbb{N}_{N-1}}$ (so that $\pi_k(y, X_k, U_k) = v_k + \nu(y - z_k)$ for all $y \in X_k$). The tube basic shape sets S and R and the local control function $\nu(\cdot)$ are designed off–line and are required to satisfy a reasonable assumption, while the sequences of the centers and diameters $\{z_k\}_{k \in \mathbb{N}_N}$, $\{\alpha_k\}_{k \in \mathbb{N}_N}$ and $\{v_k\}_{k \in \mathbb{N}_{N-1}}$ are decision variables in the on–line optimization. We employ globally relaxed set–dynamics of the underlying state and control tubes which allows for less conservative on–line constraint handling, performed only locally with respect to the actual state–control tube process. An additional novelty of our proposal is the construction of a suitable terminal constraint set obtained by analyzing the local homothetic state and control tube dynamics. The exact local set–dynamics of the homothetic state and control tubes is simplified by employing vector–valued dynamics describing the evolution of the centers and the diameters of the outer–bounding homothetic state and control tubes. For computational simplicity we employ a linear/affine approximation of the dynamics of the centers and diameters allowing, in turn, for the utilization of the classical set invariance concepts albeit in a suitably lifted space.

From the computational point of view, our proposal carries a modest increase in complexity compared to the existing methods [5], [7], [8], [11] but makes use of relaxed assumptions. Our method guarantees, under mild assumptions, strong system theoretic properties of the controlled uncertain dynamics.

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ISBN 978-963-311-370-7

Paper Structure

Section II provides preliminaries and the problem formulation. Section III discusses simple homothetic tubes and analyses their local behavior. Section IV introduces the homothetic tube optimal control (HTOC) problem and comments on the corresponding topological properties. Section V discusses the homothetic tube model predictive control and establishes the relevant system theoretic properties. Section VI highlights a further simplification of the homothetic tube optimal and model predictive control. Section VII closes the paper with a few concluding remarks.

Basic Nomenclature and Definitions

The sets of non–negative, positive integers and reals are denoted by $\mathbb{N}$, $\mathbb{N}_+$, and $\mathbb{R}_+$, respectively, i.e. $\mathbb{N} := \{0,1,2,\ldots\}$, $\mathbb{N}_+ := \{1,2,\ldots\}$ and $\mathbb{R}_+ := \{x \in \mathbb{R} \ : \ x \geq 0\}$. Given non–negative integers $a \in \mathbb{N}$ and $b \in \mathbb{N}$ such that $a < b$ we denote $\mathbb{N}_{[a:b]} := \{a, a+1, \ldots, b-1, b\}$; We write $\mathbb{N}_b$ for $\mathbb{N}_{[0:b]}$. Given a matrix $M \in \mathbb{R}^{n\times n}$, $\rho(M)$ denotes the largest absolute value of its eigenvalues.

Given two sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, the Minkowski set addition is defined by $X \oplus Y := \{x+y \ : \ x \in X, \ y \in Y\}$, we write $x \oplus X$ instead of $\{x\} \oplus X$. Given a set X and a real matrix M of compatible dimensions (possibly a scalar) the image of X under M is denoted by $MX := \{Mx \ : \ x \in X\}$. Given a set $Z \subset \mathbb{R}^{n+m}$ its projection onto $\mathbb{R}^n$ is denoted by $\mathrm{Projection}_{\mathbb{R}^n}(Z) = \{x \in \mathbb{R}^n \ : \ \exists y \in \mathbb{R}^m \text{ such that } (x,y) \in Z\}$. A function $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is said to be positively homogeneous of the first degree if $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \mathbb{R}_+$. If $f(\cdot)$ is a set–valued function from, say, X into U, namely, its values are subsets of U, then its graph is the set $\mathrm{graph}(f) := \{(x,y) \ : \ x \in X, \ y \in f(x)\}$.

A set $X \subset \mathbb{R}^n$ is said to be a non–trivial set if it is a proper, non–empty, subset of $\mathbb{R}^n$ and it is not a singleton set. A set $X \subset \mathbb{R}^n$ is *a C–set* if it is compact, convex, and contains the origin. A set $X \subset \mathbb{R}^n$ is *a proper C–set, or just a PC–set,* if it is a C–set and contains the origin in its (non–empty) interior. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half–spaces and a *polytope* is the closed and bounded polyhedron. The boundary, closure and interior of a set X are denoted, respectively, by $\mathrm{boundary}(X)$, $\mathrm{closure}(X)$ and $\mathrm{interior}(X)$. Given a set $X \subset \mathbb{R}^n$, $\mathrm{convh}(X)$ denotes its convex hull.

The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $|x|$ while the corresponding unit norm ball is denoted by $\mathcal{B}_2^n$. The functions $h(\mathcal{B}_2^n,\cdot,\cdot)$ and $H(\mathcal{B}_2^n,\cdot,\cdot)$ defined, for any two non–empty compact subsets of $\mathbb{R}^n$ say X and Y, by:

$$h(\mathcal{B}_2^n, X, Y) := \min_{\alpha}\{\alpha \ : \ X \subseteq Y \oplus \alpha\mathcal{B}_2^n, \ \alpha \in \mathbb{R}_+\} \text{ and}$$
$$H(\mathcal{B}_2^n, X, Y) := \max\{h(\mathcal{B}_2^n, X, Y), h(\mathcal{B}_2^n, Y, X)\},$$

denote the Hausdorff semi–distance and the Hausdorff distance (metric).

For typographical convenience, we distinguish row vectors from column vectors only when needed and employ the same symbol for a variable x and its vectorized form in the algebraic expressions.

II. PRELIMINARIES & PROBLEM FORMULATION

We consider linear, time–invariant, discrete time systems, given by:

$$x^+ = Ax + Bu + w, \tag{II.1}$$

where $x \in \mathbb{R}^n$ is the current state, $u \in \mathbb{R}^m$ is the current control, x^+ is the successor state, $w \in \mathbb{R}^n$ is the disturbance taking values in the set $\mathbb{W} \subset \mathbb{R}^n$ and matrices A and B are of compatible dimensions, i.e. $(A,B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$. Thus, if at any time $k \in \mathbb{N}$ the state is x_k, the applied control is u_k and the disturbance is w_k, the state at time $k+1$ satisfies $x_{k+1} = Ax_k + Bu_k + w_k$. The system variables x, u and w are subject to hard constraints:

$$x \in \mathbb{X}, \ u \in \mathbb{U} \text{ and } \ w \in \mathbb{W}. \tag{II.2}$$

We work, throughout this note, under the following standing assumptions and clarifying interpretation:

Assumption 1: The matrix pair $(A,B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is stabilizable.

Assumption 2: The state and control constraint sets $\mathbb{X}$ and $\mathbb{U}$ are PC–polytopic sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and the disturbance constraint set $\mathbb{W}$ is a non–trivial C–polytopic set in $\mathbb{R}^n$.

Interpretation 1: At any time instance $k \in \mathbb{N}$, the state x_k is known when the current control action u_k is determined, while the current disturbance w_k and future disturbances $w_{k+i}, \ i \in \mathbb{N}_+$ are not known and can take any arbitrary values $w_{k+i} \in \mathbb{W}, \ i \in \mathbb{N}$.

Central focus of this note is the efficient computation of homothetic state and control tubes as well as associated control policies. For any non–empty sets $S \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}^m$, any function $\nu(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ and any positive integer $N \in \mathbb{N}_+$, the homothetic state tube is a sequence of sets $\mathbf{X}_N := \{X_k\}_{k\in\mathbb{N}_N}$ where sets X_k are given, for all $k \in \mathbb{N}_N$, by:

$$X_k := z_k \oplus \alpha_k S \text{ with } (z_k, \alpha_k) \in \mathbb{R}^n \times \mathbb{R}_+. \tag{II.3}$$

Likewise, the homothetic control tube is a sequence of sets $\mathbf{U}_{N-1} := \{U_k\}_{k\in\mathbb{N}_{N-1}}$ where sets U_k are given, for all $k \in \mathbb{N}_{N-1}$, by:

$$U_k := v_k \oplus \alpha_k R \text{ with } (v_k, \alpha_k) \in \mathbb{R}^m \times \mathbb{R}_+, \tag{II.4}$$

and the corresponding ν–parameterized control policy is a sequence of control laws $\Pi_{N-1} := \{\pi_k(\cdot, X_k, U_k)\}_{k\in\mathbb{N}_{N-1}}$ where control laws $\pi_k(\cdot, X_k, U_k)$ are such that, $\forall k \in \mathbb{N}_{N-1}$,:

$$\forall y \in X_k, \ \pi_k(y, X_k, U_k) := v_k + \nu(y - z_k). \tag{II.5}$$

Clearly, for given sets S and R and a given function $\nu(\cdot)$, the homothetic state and control tubes, $\mathbf{X}_N$ and $\mathbf{U}_{N-1}$, and the corresponding ν–parameterized control policy Π_{N-1} are fully determined by the sequences of the centers of the homothetic state and control tubes, $\{z_k \in \mathbb{R}^n\}_{k\in\mathbb{N}_N}$ and $\{v_k \in \mathbb{R}^m\}_{k\in\mathbb{N}_{N-1}}$, and the sequence of the diameters of the homothetic state and control tubes $\{\alpha_k \in \mathbb{R}_+\}_{k\in\mathbb{N}_N}$. Consequently, we introduce a decision variable $\mathbf{d}_N := (z_0, \ldots, z_N, \alpha_0, \ldots, \alpha_N, v_0, \ldots, v_{N-1}) \in$

$\mathbb{R}^{N(n+m+1)+n+1}$. Given any $x \in \mathbb{X}$, we consider the following set of constraints on the decision variable $\mathbf{d}_N$ or equivalently on the homothetic state and control tubes, $\mathbf{X}_N$ and $\mathbf{U}_{N-1}$, and the corresponding ν–parameterized control policy Π_{N-1}:

$$x \in z_0 \oplus \alpha_0 S, \tag{II.6a}$$
$$\forall k \in \mathbb{N}_{N-1},\ \alpha_k \geq 0, \tag{II.6b}$$
$$\forall k \in \mathbb{N}_{N-1},\ z_k \oplus \alpha_k S \subseteq \mathbb{X}, \tag{II.6c}$$
$$\forall k \in \mathbb{N}_{N-1},\ v_k \oplus \alpha_k R \subseteq \mathbb{U}, \tag{II.6d}$$
$$\forall k \in \mathbb{N}_{N-1},\ \forall y \in z_k \oplus \alpha_k S,$$
$$Ay + B(v_k + \nu(y - z_k)) \oplus \mathbb{W} \subseteq z_{k+1} \oplus \alpha_{k+1} S, \tag{II.6e}$$
$$\forall k \in \mathbb{N}_{N-1},\ \forall y \in z_k \oplus \alpha_k S,$$
$$\nu(y - z_k) \in \alpha_k R,\ \text{and}, \tag{II.6f}$$
$$(z_N, \alpha_N) \in G_f, \tag{II.6g}$$

where $G_f \subseteq \mathbb{R}^{n+1}$ is an appropriate terminal constraint set. We introduce the set–valued map $\mathcal{D}_N(\cdot)$ given, for all $x \in \mathbb{X}$, by:

$$\mathcal{D}_N(x) := \{\mathbf{d}_N \ : \ \text{(II.6) holds}\}. \tag{II.7}$$

For any given $x \in \mathbb{X}$, the set $\mathcal{D}_N(x)$ represents the set of admissible decision variables $\mathbf{d}_N$ (yielding, in turn, the set of admissible homothetic state and control tubes, $\mathbf{X}_N$ and $\mathbf{U}_{N-1}$, and corresponding ν–parameterized control policies Π_{N-1} via (II.3)–(II.5)). For any $\mathbf{d}_N \in \mathbb{R}^{N(n+m+1)+n+1}$, let a cost function $V_N(\cdot) : \mathbb{R}^{N(n+m+1)+n+1} \to \mathbb{R}_+$ be given by:

$$V_N(\mathbf{d}_N) := \sum_{k \in \mathbb{N}_{N-1}} \ell(z_k, \alpha_k, v_k) + V_f(z_N, \alpha_N), \tag{II.8}$$

where $\ell(\cdot,\cdot,\cdot) : \mathbb{R}^{n+m+1} \to \mathbb{R}_+$ and $V_f(\cdot,\cdot) : \mathbb{R}^{n+1} \to \mathbb{R}_+$ are the homothetic state–control tube pair stage and terminal cost functions. The finite horizon HTOC problem $\mathbb{P}_N(x)$ of our interest is specified, for any $x \in \mathbb{X}$, by:

$$V_N^0(x) := \inf_{\mathbf{d}_N}\{V_N(\mathbf{d}_N) \ : \ \mathbf{d}_N \in \mathcal{D}_N(x)\},\ \text{and},$$
$$\mathbf{d}_N^0(x) := \arg\inf_{\mathbf{d}_N}\{V_N(\mathbf{d}_N) \ : \ \mathbf{d}_N \in \mathcal{D}_N(x)\}. \tag{II.9}$$

The main objectives of this manuscript are to:

(i) discuss the solvability of the finite horizon HTOC problem $\mathbb{P}_N(x),\ x \in \mathbb{X}$ for fixed, but suitably chosen sets S and R and corresponding control function $\nu(\cdot)$ as well as terminal constraint set G_f and stage and terminal cost functions $\ell(\cdot,\cdot,\cdot)$ and $V_f(\cdot,\cdot)$; and

(ii) analyze the repetitive on–line application of the HTOC problem $\mathbb{P}_N(x),\ x \in \mathbb{X}$ and outline the relevant system theoretic properties of the corresponding simple HTMPC.

III. SIMPLE HOMOTHETIC TUBES

Our first step is to discuss sensible choice of the sets S and R and corresponding control function $\nu(\cdot)$ as well as stage and terminal cost functions $\ell(\cdot,\cdot,\cdot)$ and $V_f(\cdot,\cdot)$; the outlined choice allows for a significant simplification of the finite horizon HTOC problem $\mathbb{P}_N(x),\ x \in \mathbb{X}$.

A. Homothetic Tubes : Constituting Components

The first mild but simplifying assumption is concerned with the desirable properties of the local control function $\nu(\cdot)$, state and control tube cross–section shape sets S and R and an additional simplifying component, namely the nominal successor state set S^+:

Assumption 3: (i) The control function $\nu(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous, positively homogeneous function of the first degree.

(ii) The state tube cross–section shape set S is a non–trivial C–polytopic set in $\mathbb{R}^n$ such that:

$$\{As + B\nu(s) \ : \ s \in S\} \oplus \mathbb{W} \subseteq S. \tag{III.1}$$

(iii) The control tube cross–section shape set R is a C–set in $\mathbb{R}^m$ given by:

$$R := \text{convh}(\{\nu(s) \ : \ s \in S\}). \tag{III.2}$$

(iv) The nominal successor state set S^+ is a C–set in $\mathbb{R}^n$ given by:

$$S^+ := \text{convh}(\{As + B\nu(s) \ : \ s \in S\}). \tag{III.3}$$

We observe that, under Assumption 3, for any given $\alpha \in \mathbb{R}_+$ and all $s \in \alpha S$ it holds that $\nu(s) \in \alpha R$ and $As + B\nu(s) \in \alpha S^+$ due to homogeneity of $\nu(\cdot)$ and the C–property of the relevant sets. Clearly $Ay + B(v + \nu(y - z)) = Az + Bv + A(y-z) + B\nu(y-z)$ and consequently, under Assumption 3, the set of constraints in (II.6e) is equivalently expressed, for all $k \in \mathbb{N}_{N-1}$, as:

$$Az_k + Bv_k \oplus \alpha_k S^+ \oplus \mathbb{W} \subseteq z_{k+1} \oplus \alpha_{k+1} S, \tag{III.4}$$

while, in addition, the set of constraints in (II.6f) is satisfied by construction. We point out that the constraints (III.4) specify the global homothetic state tube set–dynamics and do not necessarily require that $\forall k \in \mathbb{N}_{N-1},\ z_{k+1} = Az_k + Bv_k$ and $\alpha_{k+1} = \alpha_k = 1$ as in earlier proposals [7].

A further simplification is obtained by considering strictly convex and quadratic stage and terminal cost functions $\ell(\cdot,\cdot,\cdot)$ and $V_f(\cdot,\cdot)$ given, for all $(z, \alpha, v) \in \mathbb{R}^{n+m+1}$, by:

$$\ell(z, \alpha, v) = z'Q_z z + q_\alpha(\alpha - \bar{\alpha})^2 + v'Q_v v, \tag{III.5}$$

and, for all $(z, \alpha) \in \mathbb{R}^{n+1}$,

$$V_f(z, \alpha) = z'P_z z + p_\alpha(\alpha - \bar{\alpha})^2, \tag{III.6}$$

where $Q_z \in \mathbb{R}^{n \times n},\ Q_z = Q_z' > 0,\ Q_v \in \mathbb{R}^{m \times m},\ Q_v = Q_v' > 0,\ P_z \in \mathbb{R}^{n \times n},\ P_z = P_z' > 0,\ q_\alpha \in \mathbb{R}_+$ and $p_\alpha \in \mathbb{R}_+$. The scalar $\bar{\alpha} \in \mathbb{R}_+$ is specified below in Section III-B.

B. Homothetic Tubes : Simplified Local Tube Dynamics

Our second step is to provide a suitable choice of the terminal constraint set G_f and invoke additional assumptions on stage and terminal cost functions $\ell(\cdot,\cdot,\cdot)$ and $V_f(\cdot,\cdot)$ guaranteeing desirable invariance and stabilizing properties.

Before proceeding, let for any $(z, v, \alpha) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+$,

$$X(z, \alpha) := z \oplus \alpha S \ \text{and} \ U(v, \alpha) := v \oplus \alpha R. \tag{III.7}$$

We consider the local control function, $\pi_f(\cdot, X(z,\alpha), U(Kz,\alpha))$, given, for all $x \in X(z,\alpha) := z \oplus \alpha S$ with $(z,\alpha) \in \mathbb{R}^n \times \mathbb{R}_+$, by:

$$\pi_f(x, X(z,\alpha), U(Kz,\alpha)) = Kz + \nu(x - z), \qquad \text{(III.8)}$$

where $K \in \mathbb{R}^{m\times n}$. We note that, for any given $(z,\alpha) \in \mathbb{R}^n \times \mathbb{R}_+$ and all $x \in X(z,\alpha) = z \oplus \alpha S$, it holds that $\pi_f(x, X(z,\alpha), U(Kz,\alpha)) = Kz + \nu(x-z) \in U(Kz,\alpha)$. Our primary aim, in this subsection, is to determine the terminal set G_f such that for all $(z,\alpha) \in G_f$ it holds that:

$$X(z,\alpha) = z \oplus \alpha S \subseteq \mathbb{X}, \qquad \text{(III.9a)}$$
$$U(Kz,\alpha) = Kz \oplus \alpha R \subseteq \mathbb{U}, \text{ and,} \qquad \text{(III.9b)}$$
$$(A+BK)z \oplus \alpha S^+ \oplus \mathbb{W} \subseteq z^+ \oplus \alpha^+ S \qquad \text{(III.9c)}$$

for some $(z^+, \alpha^+) \in G_f$. Motivated by simplicity and computational considerations, we enforce the condition (III.9c) by setting:

$$z^+ = (A+BK)z \text{ and } \alpha^+ = \lambda\alpha + \mu, \qquad \text{(III.10)}$$

where:

$$\mu := \min_{\eta}\{\eta \ : \ \mathbb{W} \subseteq \eta S, \ \eta \geq 0\}, \text{ and,} \qquad \text{(III.11a)}$$
$$\lambda := \min_{\eta}\{\eta \ : \ S^+ \subseteq \eta S, \ \eta \geq 0\}. \qquad \text{(III.11b)}$$

Note that, under Assumption 3, it is guaranteed that $\mu \in [0,1]$ and $\lambda \in [0,1]$. In order to ensure the satisfaction of the conditions (III.9a) and (III.9b) we introduce the set of constraints on the (z,α) variable:

$$(z,\alpha) \in G \text{ where} \qquad \text{(III.12a)}$$
$$G := \{(z,\alpha) \ : \ z \oplus \alpha S \subseteq \mathbb{X}, \ Kz \oplus \alpha R \subseteq \mathbb{U} \text{ and } \alpha \geq 0\}. \qquad \text{(III.12b)}$$

It is important to note that, under Assumptions 2 and 3, the set G is a C–polytopic set in $\mathbb{R}^n \times \mathbb{R}_+$. We underline that the ordinary vector–valued dynamics specified in (III.10) define completely the local, outer–bounding, homothetic set–dynamics (see (III.9c)) while the conditions (III.9a) and (III.9b) impose constraints on the (z,α) dynamics via (III.12). Consequently, the determination of an appropriate terminal constraint set G_f, ensuring that the relations (III.9) are satisfied, is reduced to the problem of the determination of a positively invariant set for the dynamics (III.10) subject to constraints (III.12).

We are now in position to invoke our second mild and simplifying assumption:

Assumption 4: (i) The matrix $K \in \mathbb{R}^{m\times n}$ and scalar λ in (III.10) and are such that $\rho(A+BK) < 1$ and $\lambda \in [0,1)$.

(ii) The scalar $\bar{\alpha} \in \mathbb{R}_+$ is given by:

$$\bar{\alpha} = (1-\lambda)^{-1}\mu, \qquad \text{(III.13)}$$

where λ and μ are given as in (III.10), and is such that $(0,\bar{\alpha}) \in \text{interior}(G)$ where the set G is specified in (III.12b).

(iii) The terminal constraint set G_f is a non–trivial C–polytopic set in $\mathbb{R}^n \times \mathbb{R}_+$ and is a positively invariant set for the dynamics (III.10) subject to constraints (III.12), i.e. it is such that $G_f \subseteq G$ and:

$$\forall (z,\alpha) \in G_f, \ ((A+BK)z, \lambda\alpha + \mu) \in G_f. \qquad \text{(III.14)}$$

(iv) The stage and terminal cost functions $\ell(\cdot,\cdot,\cdot)$ and $V_f(\cdot,\cdot)$ are given as in (III.5) and (III.6) and:

$$(A+BK)'P_z(A+BK) - P_z \leq -(Q_z + K'Q_vK),$$
$$\text{and } p_\alpha \geq (1-\lambda^2)^{-1}q_\alpha. \qquad \text{(III.15)}$$

Now, we comment briefly on the consequences of the invoked assumptions. As already indicated, under Assumptions 2 and 3, the set G in (III.12b) is a C–polytopic set in $\mathbb{R}^n \times \mathbb{R}_+$. The conditions (i) and (ii) in Assumption 4 guarantee then the existence of the terminal constraint set G_f postulated in the condition (iii) in Assumption 4. In fact, under above mentioned assumptions, a direct use of results in [24], [25] implies that the maximal positively invariant set for the dynamics (III.10) subject to constraints (III.12) is finitely determined and is a non–trivial C–polytopic set in $\mathbb{R}^n \times \mathbb{R}_+$. The conditions (i) and (ii) in Assumption 4 also imply that:

$$\bar{S} \subseteq \mathbb{X}, \ \bar{R} \subseteq \mathbb{U} \text{ and } \bar{S}^+ \oplus \mathbb{W} \subseteq \bar{S}, \text{ where,}$$
$$\bar{S} := \bar{\alpha}S, \ \bar{R} := \bar{\alpha}R, \text{ and, } \bar{S}^+ := \bar{\alpha}S^+. \qquad \text{(III.16)}$$

Note that sets $\bar{S}$ and $\bar{R}$ play a role of *the equilibrium set pair* for the homothetic state and control tubes in analogy with the equilibrium pair $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$ relevant for the deterministic case (i.e. in the absence of the uncertainty). The condition (iv) in Assumption 4 implies that there exist scalars $c_1 \in (0,\infty)$, $c_2 \in (0,\infty)$, $c_3 \in (0,\infty)$ and $c_4 \in (0,\infty)$ such that, for all $(z,\alpha) \in G_f$ it holds that $c_1|(z,\alpha) - (0,\bar{\alpha})| \leq \ell(z,\alpha,Kz) \leq c_2|(z,\alpha) - (0,\bar{\alpha})|$, $c_3|(z,\alpha) - (0,\bar{\alpha})| \leq V_f(z,\alpha) \leq c_4|(z,\alpha) - (0,\bar{\alpha})|$ and $V_f((A+BK)z, \lambda\alpha+\mu) - V_f(z,\alpha) \leq -\ell(z,\alpha,Kz)$. In other words, the terminal cost function $V_f(\cdot,\cdot)$ is a local Lyapunov function for the dynamics (III.10) relative to the equilibrium point $(0,\bar{\alpha})$ and with the basin of attraction being equal to the positively invariant set G_f. Summarizing the discussion above we have:

Proposition 1: Suppose Assumptions 1–4 hold and consider any sequence $\{(z_k,\alpha_k)\}_{k\in\mathbb{N}}$ generated, for all $k \in \mathbb{N}$, by $z_{k+1} = (A+BK)z_k$ and $\alpha_{k+1} = \lambda\alpha_k + \mu$ for any arbitrary $(z_0,\alpha_0) \in G_f$. Then:

(i) For all $k \in \mathbb{N}$, $(z_k,\alpha_k) \in G_f$,

(ii) For all $k \in \mathbb{N}$, $|(z_k,\alpha_k) - (0,\bar{\alpha})| \leq a^k b|(z_0,\alpha_0) - (0,\bar{\alpha})|$ for some $(a,b) \in [0,1) \times [0,\infty)$, and

(iii) The pair $(0,\bar{\alpha}) \in \mathbb{R}^n \times \mathbb{R}_+$ is an exponentially stable attractor for dynamics (III.10) with the basin of attraction being equal to the set G_f.

We close this section by summarizing relevant consequences of our construction and Proposition 1:

Remark 1: Under the conditions of Proposition 1, the set sequences $\{X(z_k,\alpha_k) = z_k \oplus \alpha_k S\}_{k\in\mathbb{N}}$ and $\{U(Kz_k,\alpha_k) =$

$Kz_k \oplus \alpha_k R\}_{k\in\mathbb{N}}$ induced from the corresponding sequence $\{(z_k, \alpha_k)\}_{k\in\mathbb{N}}$ (which is postulated in Proposition 1) are such that, for all $k \in \mathbb{N}$, it holds that:

$$X(z_k, \alpha_k) \subseteq \mathbb{X} \text{ and } U(Kz_k, \alpha_k) \subseteq \mathbb{U},$$

and, in addition, for all $y \in X(z_k, \alpha_k)$ it holds that:

$$Ay + B(Kz_k + \nu(y - z_k)) \oplus \mathbb{W} \subseteq X(z_{k+1}, \alpha_{k+1}), \text{ and,}$$
$$Kz_k + \nu(y - z_k) \in U(Kz_k, \alpha_k).$$

Furthermore, the set sequences $\{X(z_k, \alpha_k)\}_{k\in\mathbb{N}}$ and $\{U(Kz_k, \alpha_k)\}_{k\in\mathbb{N}}$ converge exponentially fast, with respect to the Hausdorff distance, to the sets $\bar{S} = \bar{\alpha}S$ and $\bar{R} = \bar{\alpha}R$ respectively. The corresponding Hausdorff distances between sets $X(z_k, \alpha_k)$ and $\bar{S}$ as well as between sets $U(Kz_k, \alpha_k)$ and $\bar{R}$ are smaller or equal to $a^k b|(z_0, \alpha_0) - (0, \bar{\alpha})|$ and $a^k c|(Kz_0, \alpha_0) - (0, \bar{\alpha})|$, respectively, for some scalars $(a, b, c) \in [0, 1) \times [0, \infty) \times [0, \infty)$.

Remark 2: The set of states $x \in \mathbb{X}$ for which there exists at least one pair of the set sequences $\{X(z_k, \alpha_k)\}_{k\in\mathbb{N}}$ and $\{U(Kz_k, \alpha_k)\}_{k\in\mathbb{N}}$ enjoying the properties discussed in Remark 1 and satisfying, in addition, that $x_0 \in X(z_0, \alpha_0)$ is implicitly characterized via the conditions $x \in z \oplus \alpha S$ and $(z, \alpha) \in G_f$. More precisely, this is the set $\mathcal{X}_0$ referred to as the 0–step homothetic tubes controllability set and given by:

$$\mathcal{X}_0 = \mathrm{Projection}_{\mathbb{R}^n}\{(x, z, \alpha) \ : x \in z \oplus \alpha S, \; (z, \alpha) \in G_f\}. \quad \text{(III.17)}$$

Under our assumptions, the set $\{(x, z, \alpha) \ : \ x \in z \oplus \alpha S, \ (z, \alpha) \in G_f\}$ is a non–trivial C–polytopic set in $\mathbb{R}^{2n} \times \mathbb{R}_+$ and, consequently, the 0–step homothetic tubes controllability set $\mathcal{X}_0$ is itself a non–trivial C–polytopic set in $\mathbb{R}^n$ such that $\bar{S} \subseteq \mathcal{X}_0$. We note that, by construction, for all $x_0 \in \mathcal{X}_0$ there exists at least one pair $(z_0(x_0), \alpha_0(x_0))$ such that $x_0 \in z_0(x_0) \oplus \alpha_0(x_0)S$ and $(z_0(x_0), \alpha_0(x_0)) \in G_f$. In turn, the corresponding sequence $\{(z_k(x_0), \alpha_k(x_0))\}_{k\in\mathbb{N}}$ generated, for all $k \in \mathbb{N}$, by $z_{k+1}(x_0) = (A + BK)z_k(x_0)$ and $\alpha_{k+1}(x_0) = \lambda\alpha_k(x_0) + \mu$ induces the local state and control homothetic tubes $\{X(z_k(x_0), \alpha_k(x_0))\}_{k\in\mathbb{N}}$ and $\{U(Kz_k(x_0), \alpha_k(x_0))\}_{k\in\mathbb{N}}$ enjoying the properties discussed in Remark 1. Furthermore, in this case, the actual state and control sequences $\{x_k\}_{k\in\mathbb{N}}$ and $\{u_k\}_{k\in\mathbb{N}}$ generated via $x_{k+1} = Ax_k + B(Kz_k(x_0) + \nu(x_k - z_k(x_0))) + w_k$ and $u_k = Kz_k(x_0) + \nu(x_k - z_k(x_0))$ for any arbitrary disturbance sequence $\{w_k \in \mathbb{W}\}_{k\in\mathbb{N}}$ satisfy:

$$x_k \in X(z_k(x_0), \alpha_k(x_0)) \text{ and } u_k \in U(Kz_k(x_0), \alpha_k(x_0))$$

and, in addition, converge to the sets $\bar{S}$ and $\bar{R}$ exponentially fast as $k \to \infty$ in the sense that the sequences $\{h(\mathcal{B}_2^m, \{x_k\}, \bar{S})\}_{k\in\mathbb{N}}$ and $\{h(\mathcal{B}_2^m, \{u_k\}, \bar{R})\}_{k\in\mathbb{N}}$ converge exponentially fast to 0 as $k \to \infty$.

IV. HOMOTHETIC TUBE OPTIMAL CONTROL

We utilize the analysis of Section III and deploy corresponding simplifications in order to discuss the relevant properties of the finite horizon HTOC problem $\mathbb{P}_N(x), \ x \in \mathbb{X}$ specified in (II.9). Our simplifying assumptions allows us to establish structural properties of the graph of the set–valued map $\mathcal{D}_N(\cdot)$ and the cost function $V_N(\cdot)$ specified in (II.7) and (II.8) respectively and, in turn, to ascertain relevant topological properties of the value function $V_N^0(\cdot)$ and its optimizer $\mathbf{d}_N^0(\cdot)$ given in (II.9). For any positive integer $N \in \mathbb{N}$, the set $\mathcal{X}_N$ given by:

$$\mathcal{X}_N := \{x \ : \ \mathcal{D}_N(x) \neq \emptyset\} \quad \text{(IV.1)}$$

is referred to as the N–step homothetic tubes controllability set and is, in fact, the effective domain of the value function $V_N^0(\cdot)$ and its optimizer $\mathbf{d}_N^0(\cdot)$. Let, for any integer $N \in \mathbb{N}$, $\bar{\mathbf{d}}_N := (0, \ldots, 0, \bar{\alpha}, \ldots, \bar{\alpha}, 0, \ldots, 0) \in \mathbb{R}^{N(n+m+1)+n+1}$. The condition (iv) in Assumption 4 implies the following relevant properties of the cost function $V_N(\cdot)$:

Lemma 1: Suppose Assumption 4 (iv) holds. Then the cost function $V_N(\cdot) \ : \ \mathbb{R}^{N(n+m+1)+n+1} \to \mathbb{R}_+$ given by (II.8), (III.5) and (III.6) is strictly convex and quadratic function such that for all $\mathbf{d}_N \in \mathbb{R}^{N(n+m+1)+n+1}$ it holds that:

$$c_5|\mathbf{d}_N - \bar{\mathbf{d}}_N| \le V_N(\mathbf{d}_N) \le c_6|\mathbf{d}_N - \bar{\mathbf{d}}_N| \quad \text{(IV.2)}$$

for some scalars $c_5 \in (0, \infty)$ and $c_6 \in (0, \infty)$.

Let, for any positive integer $N \in \mathbb{N}$, the set D_N be given by:

$$D_N := \{(x, \mathbf{d}_N) \ : \ \text{(II.6) holds}\} \quad \text{(IV.3)}$$

We recall that, under Assumption 3, the constraints in (II.6f) are satisfied by construction while the constraints in (II.6e) are equivalently expressed via the constraints in (III.4). With this in mind, under Assumptions 2, 3 and 4, a direct use of basic algebra of support functions [26], [27] yields the fact that the set D_N is a non–trivial C–polytopic set in $\mathbb{R}^{n+N(n+m+1)+n+1}$ (i.e. in $(x, \mathbf{d}_N)$–space). A concrete algebraic details are easily obtained by utilizing analysis provided in [23, Section 5].

Lemma 2: Suppose Assumptions 1–4 hold. Then the set D_N given by (IV.3) is a non–trivial C–polytopic set in $\mathbb{R}^{n+N(n+m+1)+n+1}$.

Since, by construction, it holds that $\mathcal{X}_N := \mathrm{Projection}_{\mathbb{R}^n} D_N$ it follows that the set $\mathcal{X}_N$ is a non–trivial C–polytopic set in $\mathbb{R}^n$ which, under our assumptions, is such that $\bar{S} \subseteq \mathcal{X}_0 \subseteq \mathcal{X}_N \subseteq \mathbb{X}$. Furthermore, we also have that for all $x \in \mathcal{X}_N$, $\mathcal{D}_N(x) = \{\mathbf{d}_N \ : \ (x, \mathbf{d}_N) \in D_N\}$ and consequently for any fixed $x \in \mathcal{X}_N$ the set $\mathcal{D}_N(x)$ is a polytopic set. Hence, under our assumptions, the finite horizon HTOC problem $\mathbb{P}_N(x)$ given in (II.9) is, for any fixed $x \in \mathcal{X}_N$ a strictly convex quadratic programming problem. The basic results in parametric mathematical programming [28] allows us to establish relevant topological properties of the value function $V_N^0(\cdot)$ and its optimizer $\mathbf{d}_N^0(\cdot)$ given in (II.9):

Proposition 2: Suppose Assumptions 1–4 hold. Then:

(i) The N–step homothetic tubes controllability set $\mathcal{X}_N$ is a non–trivial C–polytopic set in $\mathbb{R}^n$ such that $\bar{S} \subseteq \mathcal{X}_N \subseteq \mathbb{X}$,

(ii) The value function $V_N^0(\cdot) : \mathcal{X}_N \to \mathbb{R}_+$ is a convex, piecewise quadratic and continuous function such that $\forall x \in \bar{S},\ V_N^0(x) = 0$, and

(iii) The function $\mathbf{d}_N^0(\cdot) : \mathcal{X}_N \to \mathbb{R}^{N(n+m+1)+n+1}$ is a single–valued, piecewise affine and continuous function such that $\forall x \in \bar{S},\ \mathbf{d}_N^0(x) = \bar{\mathbf{d}}_N$.

The solution of the finite horizon HTOC problem $\mathbb{P}_N(x),\ x \in \mathcal{X}_N$ allows for the construction of the optimal simple homothetic state and control tubes $\mathbf{X}_N^0(x)$ and $\mathbf{U}_{N-1}^0(x)$ as well as the optimal ν–parametrized control policy Π_{N-1}^0. Namely, the corresponding optimizer $\mathbf{d}_N^0(x) := (z_0^0(x), \ldots, z_N^0(x), \alpha_0^0(x), \ldots, \alpha_N^0(x), v_0^0(x), \ldots, v_{N-1}^0(x))$ induces the optimal simple homothetic optimal state and control tubes $\mathbf{X}_N^0(x) := \{X_k^0(x) = z_k^0(x) \oplus \alpha_k^0(x)S\}_{k\in\mathbb{N}_N}$ and $\mathbf{U}_{N-1}^0(x) := \{U_k^0(x) = v_k^0(x) \oplus \alpha_k^0(x)R\}_{k\in\mathbb{N}_{N-1}}$ as well as the optimal ν–parametrized control policy $\Pi_{N-1}^0 := \{\pi_k^0(\cdot, X_k^0(x), U_k^0(x))\}_{k\in\mathbb{N}_{N-1}}$ such that, for all $k \in \mathbb{N}_{N-1}$ and for all $y \in X_k^0(x)$, $\pi_k^0(y, X_k^0(x), U_k^0(x)) = v_k^0(x) + \nu(y - z_k^0(x))$. The optimal simple homothetic state and control tubes $\mathbf{X}_N^0(x)$ and $\mathbf{U}_{N-1}^0(x)$ and the corresponding optimal ν–parametrized control policy Π_{N-1}^0 satisfy the constraints specified in (II.6). Proposition 1 and Remark 1 imply directly that the corresponding optimizer $\mathbf{d}_N^0(x)$ can be employed to construct a feasible decision variable $\hat{\mathbf{d}}_N(y) := (\hat{z}_0^0(y), \ldots, \hat{z}_N^0(y), \hat{\alpha}_0^0(y), \ldots, \hat{\alpha}_N^0(y), \hat{v}_0^0(y), \ldots, \hat{v}_{N-1}^0(y))$ for any $y \in X_1^0(x)$ by setting for all $k \in \mathbb{N}_{N-1}$:

$$\begin{aligned} &\hat{z}_k(y) = z_{k+1}^0(x) \text{ and } \hat{z}_N(y) = (A + BK)z_N^0(x), \\ &\hat{\alpha}_k(y) = \alpha_{k+1}^0(x) \text{ and } \hat{\alpha}_N(y) = \lambda\alpha_N^0(x) + \mu, \\ &\text{and, for all } k \in \mathbb{N}_{N-2}, \\ &\hat{v}_k(y) = v_{k+1}^0(x) \text{ and } \hat{v}_{N-1}(y) = Kz_N^0(x). \end{aligned} \tag{IV.4}$$

Indeed, the following result guarantees robust recursive feasibility of the finite horizon HTOC problem $\mathbb{P}_N(x),\ x \in \mathcal{X}_N$:

Proposition 3: Suppose Assumptions 1–4 hold. Then, for all $x \in \mathcal{X}_N$ and all $y \in X_1^0(x) := z_1^0(x) \oplus \alpha_1^0(x)S$, it holds that

$$\hat{\mathbf{d}}_N(y) \in \mathcal{D}_N(y), \tag{IV.5}$$

where $\hat{\mathbf{d}}_N(y)$ is specified in (IV.4), and, consequently, for all $x \in \mathcal{X}_N$ it holds that $X_1^0(x) \subseteq \mathcal{X}_N$.

V. HOMOTHETIC TUBE MPC

We now examine the repetitive on–line application of the solution of the finite horizon HTOC problem $\mathbb{P}_N(x),\ x \in \mathbb{X}$ in the context of the simple homothetic tube MPC. Namely, we consider the simple homothetic tube model predictive controller $\kappa_N^0(\cdot) : \mathcal{X}_N \to \mathbb{U}$ given by:

$$\kappa_N^0(x) = \pi_0^0(x, X_0^0(x), U_0^0(x)) = v_0^0(x) + \nu(x - z_0^0(x)). \tag{V.1}$$

Under Assumptions 1–4, Proposition 3 establishes that the optimizer $\mathbf{d}_N^0(\cdot)$ is single–valued and continuous implying, in view of (V.1) and Assumption 3, that the control law $\kappa_N^0(\cdot) : \mathcal{X}_N \to \mathbb{U}$ is also a single–valued and continuous function. The simple homothetic tube model predictive controller $\kappa_N^0(\cdot) : \mathcal{X}_N \to \mathbb{U}$ induces the controlled, uncertain, dynamics given, for all $x \in \mathcal{X}_N$, by:

$$x^+ \in F(x) := \{Ax + B\kappa_N^0(x) + w : w \in \mathbb{W}\}, \tag{V.2}$$

and it ensures, by construction, that for all $x \in \mathcal{X}_N$:

$$F(x) \subseteq X_1^0(x) = z_1^0(x) \oplus \alpha_1^0(x)S \subseteq \mathcal{X}_N, \tag{V.3}$$

as evident from (II.6) and Proposition 3.

Any state sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by (V.1) with $x_0 \in \mathcal{X}_N$ and the corresponding control actions sequence $\{u_k\}_{k\in\mathbb{N}}$ with $u_k = \kappa_N^0(x_k)$ for each k, for any admissible disturbance sequence $\{w_k\}_{k\in\mathbb{N}}$ with $w_k \in \mathbb{W}$ for each k, however, lie in the simple HTMPC state and control tubes $\{X_0^0(x_k)\}_{k\in\mathbb{N}}$ and $\{U_0^0(x_k)\}_{k\in\mathbb{N}}$, respectively. Therefore, for all $k \in \mathbb{N}$ it holds that:

$$x_k \in X_0^0(x_k) = z_0^0(x_k) \oplus \alpha_0^0(x_k)S \subseteq \mathcal{X}_N \subseteq \mathbb{X} \text{ and} \tag{V.4a}$$

$$\kappa_N^0(x_k) \in U_0^0(x_k) = v_0^0(x_k) \oplus \alpha_0^0(x_k)R \subseteq \mathbb{U}. \tag{V.4b}$$

The limiting behavior of the simple HTMPC state and control tubes $\{X_0^0(x_k)\}_{k\in\mathbb{N}}$ and $\{U_0^0(x_k)\}_{k\in\mathbb{N}}$, respectively, is, therefore, completely induced from the limiting behavior of the sequences $\{z_0^0(x_k)\}_{k\in\mathbb{N}}$, $\{\alpha_0^0(x_k)\}_{k\in\mathbb{N}}$ and $\{v_0^0(x_k)\}_{k\in\mathbb{N}}$. We examine the latter by utilizing the value function $V_N^0(\cdot) : \mathcal{X}_N \to \mathbb{R}_+$ as a Lyapunov function to establish the exponential convergence of the sequences $\{z_0^0(x_k)\}_{k\in\mathbb{N}}$, $\{\alpha_0^0(x_k)\}_{k\in\mathbb{N}}$ and $\{v_0^0(x_k)\}_{k\in\mathbb{N}}$ to the triplet $(0, \bar{\alpha}, 0)$ in a stable manner, i.e. that $z_0^0(x_k) \to 0$, $\alpha_0^0(x_k) \to 1$ and $v_0^0(x_k) \to 0$, as $k \to \infty$, exponentially fast and in a stable fashion for any realized state sequence $\{x_k\}_{k\in\mathbb{N}}$ arising due to an admissible disturbance sequence $\{w_k\}_{k\in\mathbb{N}}$. Proposition 2 establishes that the value function $V_N^0(\cdot) : \mathcal{X}_N \to \mathbb{R}_+$ is a convex and continuous function, which is, in addition, such that for all $x \in \bar{S}$ it holds that $V_N^0(x) = 0$ or equivalently, that for all $x \in \bar{S}$ it holds that $(z_0^0(x), \alpha_0^0(x), v_0^0(x)) = (0, \bar{\alpha}, 0)$. Before proceeding, let:

$$\mathbf{y}_0^0(x) := (z_0^0(x), \alpha_0^0(x), v_0^0(x)) \text{ and } \bar{\mathbf{y}} := (0, \bar{\alpha}, 0). \tag{V.5}$$

A further relevant property of the value function $V_N^0(\cdot)$ is stated next:

Proposition 4: Suppose Assumptions 1–4 hold. Then there exists a scalar pair $(c_7, c_8) \in (0, \infty) \times (0, \infty)$ such that for all $x \in \mathcal{X}_N$ it holds that:

$$c_7|\mathbf{y}_0^0(x) - \bar{\mathbf{y}}| \le V_N^0(x) \le c_8|\mathbf{y}_0^0(x) - \bar{\mathbf{y}}| \text{ and} \tag{V.6a}$$

$$\forall x^+ \in F(x),\ V_N^0(x^+) - V_N^0(x) \le -c_7|\mathbf{y}_0^0(x) - \bar{\mathbf{y}}|, \tag{V.6b}$$

where $F(\cdot)$, $\mathbf{y}_0^0(\cdot)$ and $\bar{\mathbf{y}}$ are given by (V.2) and (V.5), respectively.

A direct, but relevant, consequence of Proposition 4 is the following fact:

Corollary 1: Suppose Assumptions 1–4 hold. Then there exists a scalar pair $(a_N, b_N) \in (0,1) \times (0,\infty)$ such that the inequalities

$$\forall k \in \mathbb{N},\ V_N^0(x_k) \leq a_N^k V_N^0(x_0) \text{ and} \tag{V.7a}$$
$$\forall k \in \mathbb{N},\ |\mathbf{y}_0^0(x_k) - \bar{\mathbf{y}}| \leq a_N^k b_N |\mathbf{y}_0^0(x_0) - \bar{\mathbf{y}}| \tag{V.7b}$$

hold true for any $x_0 \in \mathcal{X}_N$ and any corresponding state sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by (V.2).

We also need the following technical lemma:

Lemma 3: Suppose Assumptions 1–4 hold. Then there exists a scalar pair $(c_9, c_{10}) \in (0,\infty) \times (0,\infty)$ such that for all $x \in \mathcal{X}_N$ it holds that:

$$c_9|\mathbf{y}_0^0(x) - \bar{\mathbf{y}}| \leq H(\mathcal{B}_2^n, X_0^0(x), \bar{S}) \leq c_{10}|\mathbf{y}_0^0(x) - \bar{\mathbf{y}}| \tag{V.8}$$

The sequence of our preliminary results now easily yields our main result:

Theorem 1: Suppose Assumptions 1–4 hold. Then:

(i) The N–step homothetic tubes controllability set $\mathcal{X}_N$ is a robust positively invariant set for the system (V.2) and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$, i.e. for all $x \in \mathcal{X}_N \subseteq \mathbb{X}$ it holds that $\kappa_N^0(x) \in \mathbb{U}$ and $F(x) \subseteq \mathcal{X}_N$ where $\kappa_N^0(\cdot)$ and $F(\cdot)$ are given by (V.1) and (V.2), respectively.

(ii) There exists a scalar pair $(\tilde{a}_N, \tilde{b}_N) \in [0,1) \times \mathbb{R}_+$ such that, for all $k \in \mathbb{N}$,:

$$H(\mathcal{B}_2^n, X_0^0(x_k), \bar{S}) \leq \tilde{a}_N^k \tilde{b}_N H(\mathcal{B}_2^n, X_0^0(x_0), \bar{S}), \text{ and,}$$
$$H(\mathcal{B}_2^m, U_0^0(x_k), \bar{R}) \leq \tilde{a}_N^k \tilde{b}_N,$$

holds true for all state sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by (V.2) with arbitrary $x_0 \in \mathcal{X}_N$.

(iii) The set $\bar{S}$ is robustly exponentially stable for the controlled uncertain system (V.2) with the basin of attraction being the N–step homothetic tubes controllability set $\mathcal{X}_N$.

Remark 3: Theorem 1 establishes the stable and exponential convergence of the simple HTMPC state tubes $\{X_0^0(x_k)\}_{k\in\mathbb{N}}$ to the set $\bar{S}$ as well as the exponential convergence of the simple HTMPC control tubes $\{U_0^0(x_k)\}_{k\in\mathbb{N}}$ to the set $\bar{R}$. The established property is equivalent to the stability of the set $\bar{S}$ and the exponential convergence of the possible state trajectories $\{x_k\}_{k\in\mathbb{N}}$ to the set $\bar{S}$ in the sense that the sequence $\{h(\mathcal{B}_2^n, \{x_k\}, \bar{S})\}_{k\in\mathbb{N}}$ converges exponentially fast to 0 in a stable manner. In addition, the corresponding control actions sequence $\{u_k\}_{k\in\mathbb{N}}$ with $u_k = \kappa_N^0(x_k)$ converges exponentially fast to the set $\bar{R}$ in the sense that the sequence $\{h(\mathcal{B}_2^m, \{u_k\}, \bar{R})\}_{k\in\mathbb{N}}$ converges exponentially fast to 0.

Remark 4: As far as the implementation of the simple HTMPC is concerned, the main computational burden is the off–line determination of the sets S, R, S^+ and G_f. However, a number of methods exists in the literature that can be utilized to execute the necessary computations [3], [24], [25], [29]–[37]. Once the off–line computations are performed, the on–line implementation of the simple HTMPC can be performed by utilizing the standard convex optimization software (since the finite horizon (HTOC) problem $\mathbb{P}_N(x)$ given in (II.9) is, for any fixed $x \in \mathcal{X}_N$ a strictly convex quadratic programming problem). Finally, since $\forall x \in \bar{S},\ V_N^0(x) = 0$, $\mathbf{d}_N^0(x) = \bar{\mathbf{d}}_N$ and, consequently, $\forall x \in \bar{S},\ \kappa_N^0(x) = \nu(x)$ we note that on–line optimization can be terminated once the state x_k enters the set $\bar{S}$.

VI. YET SIMPLER HTMPC

We outline a further plausible simplification of the simple HTMPC. Namely, the set of constraints in (II.6e) or its equivalent reformulation (III.4) (under Assumption 3) can be enforced as described in Section III. Indeed, the set of constraints in (III.4) can be enforced by setting for all $k \in \mathbb{N}_{N-1}$:

$$z_{k+1} = Az_k + Bv_k \text{ and } \alpha_{k+1} = \lambda\alpha_k + \mu, \tag{VI.1}$$

where λ and μ are given as in (III.10). In this case, we introduce the set:

$$G_S = \{(z, \alpha, v)\ :\ z \oplus \alpha S \subseteq \mathbb{X},\ v \oplus \alpha R \subseteq \mathbb{U} \text{ and } \alpha \geq 0\}. \tag{VI.2}$$

We note that, under Assumptions 2 and 3, the set G_S is a C–polytopic set in $\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^m$.

In order to simplify further the simple HTMPC we utilize (VI.1) and (VI.2) and invoke the following set of constraints on the decision variable $\mathbf{d}_N$:

$$x \in z_0 \oplus \alpha_0 S, \tag{VI.3a}$$
$$\forall k \in \mathbb{N}_{N-1},\ z_{k+1} = Az_k + Bv_k, \tag{VI.3b}$$
$$\forall k \in \mathbb{N}_{N-1},\ \alpha_{k+1} = \lambda\alpha_k + \mu, \tag{VI.3c}$$
$$\forall k \in \mathbb{N}_{N-1},\ (z_k, \alpha_k, v_k) \in G_S, \text{ and,} \tag{VI.3d}$$
$$(z_N, \alpha_N) \in G_f. \tag{VI.3e}$$

Let, similarly as in Section IV, the set D_{SN} be given by:

$$D_{SN} := \{(x, \mathbf{d}_N)\ :\ \text{(VI.3) holds}\} \tag{VI.4}$$

Under Assumptions 1–4, similarly as in Lemma 2, the set D_{SN} is a non–trivial polytopic set in $\mathbb{R}^{n+N(n+m+1)+n+1}$. The simplified HTOC problem $\mathbb{P}_{SN}(x)$ is given, for all $x \in \mathbb{X}$, by:

$$V_{SN}^0(x) := \inf_{\mathbf{d}_N}\{V_N(\mathbf{d}_N)\ :\ (x, \mathbf{d}_N) \in D_{SN}\}, \text{ and,}$$
$$\mathbf{d}_{SN}^0(x) := \arg\inf_{\mathbf{d}_N}\{V_N(\mathbf{d}_N)\ :\ (x, \mathbf{d}_N) \in D_{SN}\}. \tag{VI.5}$$

Under Assumptions 1–4, the effective domain of the value function $V_{SN}^0(\cdot)$ and its optimizer $\mathbf{d}_{SN}^0(\cdot)$ is the set:

$$\mathcal{X}_{SN} := \text{Projection}_{\mathbb{R}^n} D_{SN}, \tag{VI.6}$$

which is a non–trivial C–polytopic set in $\mathbb{R}^n$ such that $\bar{S} \subseteq \mathcal{X}_0 \subseteq \mathcal{X}_{SN} \subseteq \mathbb{X}$.

Remark 5: Under Assumptions 1–4, all the results established in Section IV apply directly to the simplified HTOC problem $\mathbb{P}_{SN}(x)$ with minor notational changes. In particular, Propositions 2 and 3 hold true with obvious notational changes.

Remark 6: The simplified homothetic tube model predictive controller $\kappa^0_{SN}(\cdot) : \mathcal{X}_{SN} \to \mathbb{U}$ given as in (V.1) (i.e. for all $x \in \mathcal{X}_{SN}$, $\kappa^0_{SN}(x) = v^0_0(x) + \nu(x - z^0_0(x))$) ensures that the controlled uncertain dynamics given as in (V.2) (i.e. $x^+ \in F(x) := \{Ax + B\kappa^0_{SN}(x) + w \ : \ w \in \mathbb{W}\}$,) are well behaved. Indeed, as in Remark 5, under Assumptions 1–4, Propositions 4, Corollary 1, Lemma 3 and Theorem 1 hold true with obvious notational changes. However, since the set of constraints in (VI.1) is more restrictive than the set of constraints (III.4), the computationally beneficial simplifications outlined here come at the expense of the potential degradation of optimality, the speed of convergence and the reduction of the size of the domain of the attraction compared to the HTMPC method discussed in Section V.

VII. Concluding Remarks

We considered the robust model predictive control synthesis problem for constrained linear discrete time systems and proposed two simple homothetic tube model predictive control synthesis methods. Under rather mild assumptions it is demonstrated that the proposed methods are computationally efficient and induce strong system theoretic properties.

References

[1] R. E. Bellman. *Dynamic Programming*. Princeton University Press, USA, 1957.

[2] D. P. Bertsekas. *Dynamic Programming and Optimal Control*. Athena Scientific, 2007. Volumes I and II – 3rd Edition.

[3] F. Blanchini and S. Miani. *Set–Theoretic Methods in Control*. Systems & Control: Foundations & Applications. Birkhäuser, Boston, Basel, Berlin, 2008.

[4] W. Langson, I. Chryssochoos, S. V. Raković, and D. Q. Mayne. Robust model predictive control using tubes. *Automatica*, 40:125–133, 2004.

[5] S. V. Raković and D. Q. Mayne. A simple tube controller for efficient robust model predictive control of constrained linear discrete time systems subject to bounded disturbances. In *Proceedings of the 16th IFAC World Congress IFAC 2005*, Praha, Czech Republic, July 2005. Invited Session.

[6] S. V. Raković and D. Q Mayne. Robust model predictive control of constrained piecewise affine discrete time systems. In *Proceedings of the 6th IFAC Symposium – NOLCOS2004*, Stuttgart, Germany, September 2004.

[7] D. Q. Mayne, M. Seron, and S. V. Raković. Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41:219–224, 2005.

[8] S. V. Raković and D. Q. Mayne. Set robust control invariance for linear discrete time systems. In *Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference CDCECC 2005*, Sevilla, Spain, December 2005.

[9] D. Q. Mayne, S. V. Raković, R. Findeisen, and F. Allgöwer. Robust output feedback model predictive control of constrained linear systems. *Automatica*, 42:1217–1222, 2006.

[10] S. V. Raković, A. R. Teel, D. Q Mayne, and A. Astolfi. Simple Robust Control Invariant Tubes for Some Classes of Nonlinear Discrete Time Systems. In *Proceedings of 45th IEEE Conference on Decision and Control, CDC 2006*, San Diego, CA, USA., December 2006.

[11] S. V. Raković and D. Q. Mayne. Robust model predictive control for obstacle avoidance: Discrete time case. *Lecture Notes in Control and Information Sciences – Assessment and Future Directions of Nonlinear Model Predictive Control*, 358:617–627, 2007.

[12] D. Q. Mayne and E. C. Kerrigan. Tube–Based Robust Nonlinear Model Predictive Control. In *Proceedings of the 7th IFAC Symposium – NOLCOS2007*, Pretoria, South Africa., August 2007.

[13] S. V. Raković. Set theoretic methods in model predictive control. *Lecture Notes in Control and Information Sciences – Nonlinear Model Predictive Control : Towards New Challenging Applications*, 384:41–54, 2009.

[14] D. Q. Mayne, S. V. Raković, R. Findeisen, and F. Allgöwer. Robust output feedback model predictive control of constrained linear systems : Time varying case. *Automatica*, 45:2082–2087, 2009.

[15] F. Blanchini. Control synthesis of discrete time systems with control and state bounds in the presence of disturbance. *Journal of Optimization Theory and Applications*, 65(1):29–40, 1990.

[16] J. R. Gossner, B. Kouvaritakis, and J. A. Rossiter. Stable generalised predictive control in the presence of constraints and bounded disturbances. *Automatica*, 33(4):551–568, 1997.

[17] L. Chisci, J. A. Rossiter, and G. Zappa. Systems with persistent disturbances: predictive control with restricted constraints. *Automatica*, 37:1019–1028, 2001.

[18] Y. I. Lee and B. Kouvaritakis. Constrained receding horizon predictive control for systems with disturbances. *International Journal of Control*, 72(11):1027–1032, 1999.

[19] Y. I. Lee and B. Kouvaritakis. A linear programming approach to constrained robust predictive control. *IEEE Transactions on Automatic Control*, 45(9):1765–1770.

[20] Y. I. Lee and B. Kouvaritakis. Robust receding horizon predictive control for systems with uncertain dynamics and input saturation. *Automatica*, 36(10):1497–1504, 2000.

[21] Y. I. Lee, B. Kouvaritakis, and M. Cannon. Constrained receding horizon predictive control for nonlinear systems. *Automatica*, 38(12):2093–2102, 2002.

[22] R. S. Smith. Robust model predictive control of constrained linear systems. In *Proceedings of the 2004 American Control Conference*, volume 1, pages 245 – 250, Boston, MA, USA, July 2004.

[23] S. V. Raković, B. Kouvaritakis, and R. Findeisen. Min–max homothetic tube model predictive control for constrained linear systems. Technical Report OUEL 2312/09, University of Oxford, Oxford, UK, November 2009.

[24] E. G. Gilbert and K. T. Tan. Linear systems with state and control constraints: the theory and application of maximal output admissible sets. *IEEE Transactions on Automatic Control*, 36(9):1008–1020, 1991.

[25] S. V. Raković and M. Fiacchini. Invariant Approximations of the Maximal Invariant Set of "Encircling the Square". In *Proceedings of the 17th IFAC World Congress IFAC 2008*, Seoul, Korea, 2008.

[26] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, USA, 1970.

[27] R. Schneider. *Convex bodies: The Brunn-Minkowski theory*, volume 44. Cambridge University Press, Cambridge, England, 1993. Encyclopedia of Mathematics and its Applications.

[28] B. Bank, J. Guddat, D. Klatte, B. Kummer, and K. Tammer. *Non–linear Parametric Optimization*. Birkhäuser, Basel, Boston, Stuttgart, 1983.

[29] D. P. Bertsekas. Infinite–time reachability of state-space regions by using feedback control. *IEEE Transaction on Automatic Control*, 17(5):604–613, 1972.

[30] F. Blanchini. Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions. *IEEE Transactions on Automatic Control*, 39(2):428–433, 1994.

[31] I. V. Kolmanovsky and E. G. Gilbert. Theory and computation of disturbance invariant sets for discrete time linear systems. *Mathematical Problems in Engineering: Theory, Methods and Applications*, 4:317–367, 1998.

[32] S. V. Raković, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne. Invariant approximations of the minimal robustly positively invariant set. *IEEE Transactions on Automatic Control*, 50(3):406–410, 2005.

[33] S. V. Raković. *Robust Control of Constrained Discrete Time Systems: Characterization and Implementation*. PhD thesis, Imperial College London, London, United Kingdom, 2005.

[34] S. V. Raković, E. C. Kerrigan, D. Q. Mayne, and K. I. Kouramas. Optimized Robust Control Invariance for Linear Discrete-Time Systems: Theoretical Foundations. *Automatica*, 43:831–841, 2007.

[35] Z. Artstein and S. V. Raković. Feedback and Invariance under Uncertainty via Set Iterates. *Automatica*, 44(2):520–525, 2008.

[36] S. V. Raković and M. Barić. Local Control Lyapunov Functions for Constrained Linear Discrete-Time Systems: The Minkowski Algebra Approach. *IEEE Transactions on Automatic Control*, 54(11):2686–2692, 2009.

[37] S. V. Raković and M. Barić. Parameterized Robust Control Invariant Sets for Linear Systems: Theoretical Advances and Computational Remarks. *IEEE Transactions on Automatic Control*, 2010. To Appear.