The ultimatum game

Two players must divide a cake. Player 1 proposes a division (c_1, c_2), where c_i is the fraction of the cake going to i (with $0 \leq c_i \leq 1$ and $c_1 + c_2 = 1$). The cake is continuously divisible, so any such division can be proposed. After observing the proposal, player 2 chooses whether to accept or reject it. If she accepts, the proposed division is implemented. If she rejects, the cake is destroyed and both players receive no cake. Each player's payoff is the share of the cake they receive. Restrict attention to pure strategies throughout.

(a) Characterise all pure strategy Nash equilibria of this game and their outcomes. Are all equilibria equally reasonable?

(b) How many subgame-perfect Nash equilibria are there? Characterise them and explain the difference between your answers to parts (a) and (b).

(c) Now suppose that player 2 is averse to receiving less cake than player 1. If she accepts a proposed division of the cake, her payoff is

$$u_2(c_1, c_2) = c_2 - a \max\{c_1 - c_2, 0\},$$

where $a \geq 0$ is a commonly known parameter measuring her degree of inequality aversion. Player 1's payoff from an accepted proposal remains c_1, and rejection still gives both players a payoff of zero. Characterise the subgame-perfect Nash equilibria of this revised game. Explain how the equilibrium division depends on a and give a brief economic interpretation.

(d) Now return to the original model without inequality aversion (i.e., $a = 0$). Suppose that player 2 bakes the cake. She chooses its size $S \geq 0$ at a cost of S^2. Thus, if player 2 accepts the proposed division payoffs are $c_1 S$ to player 1 and $(1-c_1)S - S^2$ to player 2. If she rejects, the cake is destroyed and payoffs are 0 and $-S^2$, respectively.

Compare two timings.

$$2 \text{ chooses } S \to 1 \text{ proposes } (c_1, c_2) \to \ 2 \text{ accepts or rejects.} \tag{1}$$

$$1 \text{ proposes } (c_1, c_2) \to 2 \text{ chooses } S \to \ 2 \text{ accepts or rejects.} \tag{2}$$

Determine the subgame-perfect equilibrium outcomes under the two timings. Would player 1 prefer to choose (c_1, c_2) before or after the cake is baked? Explain the economic intuition.