Håvard Damm-Johnsen - Reductive Groups

George Robinson

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Outline:

1. Definitions and examples
2. Lie algebras
3. Reductive groups
4. Root systems and dual groups

References:

- Murnaghan's notes
- Humphrey's book on LAGs
- Malle-Testerman
- Springer, Borel, Milne,...

1 Motivation

We want to study representations of $G(\mathbb{A}_F)$ where F is a number field and G is something like GL_n. We want to know about the structure of such groups, so for example their structure as algebraic varieties, the structure of their subgroup, and their representations.

For these reasons we need a good theory of algebraic group groups.

2 Definitions and Examples

Definition 2.1. An **algebraic group** over a field k is a group object in the category of varieties over k, which is denoted $\underline{\mathrm{Var}}_k$. More concretely, it is a variety G with maps $\mu : G \times G \to G$, $i : G \to G$ and $e : * \to G$ which correspond to multiplication, inversion and identity. These should satisfy the usual bunch of axioms given by commutative diagrams, for example

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mu \times id} & G \times G \\ \Big\downarrow{\scriptstyle id \times \mu} & & \Big\downarrow{\scriptstyle \mu} \\ G \times G & \xrightarrow{\quad \mu \quad} & G \end{array}$$

which encodes associativity.

For example, we could consider the variety $\mathbb{A}^1_k$ which is an algebraic group with addition, which we denote $\mathbb{G}_a$. We also have the multiplicative group $\mathbb{G}_m(k)$.

Fact: (Yoneda Lemma) any group variety is determined by its functor of points, which is the functor

$$\begin{aligned} \underline{\text{Ring}} &\longrightarrow \underline{\text{Group}} \\ R &\longmapsto G(R) \end{aligned}$$

For example the multiplicative group is defined by

$$\mathbb{G}_m(S) = S^\times = \mathrm{Hom}_k\left(\frac{k[x,y]}{xy-1}, S\right)$$

and so $\mathbb{G}_m$ is the variety defined by $\frac{k[x,y]}{xy-1}$. Similarly, we can define GL_n by the functor

$$S \mapsto \mathrm{GL}_n(S) = \mathrm{Hom}_k(R, S)$$

where $R = \frac{k[x_{11},\dots,x_{nn},t]}{\det(x_{ij})t-1}$. Elliptic curves and abelian varieties are also examples of algebraic groups.

Definition 2.2. A **morphism of algebraic groups** is a map of varieties which preserves the group structure (i.e. some diagrams are commutative).

Definition 2.3. An **isogeny** is a map of algebriac groups which has finite kernel. (Do these need to be surjective also?)

Fact: the category of algebraic groups has kernels, cokernels, products etc.

Definition 2.4. G is said to be a **linear** algebraic group if G is an algebraic group which embeds as a subgroup of GL_n for some $n \in \mathbb{N}$.

For example, the multiplicative group is clearly linear. The additive group is linear via

$$\begin{aligned} \mathbb{G}_a &\longrightarrow \mathrm{GL}_2 \\ a &\longmapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \end{aligned}$$

as are the orthogonal groups and the symplectic groups. In fact, all affine algebriac groups are linear.

Non-examples: elliptic curves and abelian varieties are not linear.

Fact: Consider $H \leq G$ a closed subgroup, then G/H exists as a quasi-projective variety, and is a linear algebraic group if and only if H is normal.

Definition 2.5. A group G is soluble if it is soluble as an abstract group. This means that the derived series $G^{(n)} = 0$ for all $n \gg 0$, where $G^{(0)} = G$ and $G^{(n)} = \left[G^{(n-1)}, G^{(n-1)}\right]$.

Definition 2.6. An element $g \in G$, a linear algebriac group, is **unipotent** if $\rho(g)-1$ is nilpotent for any (equivalently for all) faithful linear representations $\rho : G \to \mathrm{GL}_n$.

We have a filtration of important subgroups of an algebraic group

$$\{1\} \subset G_3 \subset G_2 \subset G_1$$

where

- G_3 is the unique maximal connected unipotent subgroup of G,
- G_2 is the unique maximal connected solvable normal subgroup of G,
- G_1 is the unique maximal connected linear subgroup of G.

3 Lie algebras

Remark: For a Lie group, we can define this analytically.

Definition 3.1 (/Lemma)**.** Let X be a variety of a field k, $\mathcal{O}_X$ is a sheaf of rings, $x \in X$, and $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$. Then $\mathfrak{m}_x/\mathfrak{m}_x^2$ is a finite dimensional k-vector space. This is called the Zariski cotangent space. We define the Zariski tangent space to be $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee =: T_xX$.

Proof. Consider $\alpha + \mathfrak{m}_x \in \mathcal{O}_{X,x}/\mathfrak{r}$, and $a + \mathfrak{m}_x^2 \in \mathfrak{m}_x/\mathfrak{m}_x^2$, then

$$(\alpha + \mathfrak{m}_x)(a + \mathfrak{m}_x^2) = \alpha \cdot a + \mathfrak{m}_x^2.$$

□

How do we turn this into a Lie algebra? First, we consider a linear algebraic group G with ring of functions $A = k[G]$, then G acts on A by $g \cdot f(x) = f(g^{-1}x)$. The key idea is to look at the left-invariant derivations.

Definition 3.2. An element $\delta \in \mathrm{End}(A)$ is a **derivation** if $\delta(fg) = f\delta(g) + g\delta(f)$. It is left-invariant if $g \cdot \delta(f) = \delta(g \cdot f)$.

It is an easy exercise to show that if $\delta_1, \delta_2 \in \mathrm{End}(A)$ are both left invariant differentials, then so is their Lie bracket $\delta_1\delta_2 - \delta_2\delta_1$. This turns the vector space of left-invariant differentials into a Lie algebra.

Proposition 3.3. There is an isomorphism

$$\begin{aligned}\{\text{left-invariant derivations}\} &\longrightarrow T_eG \\ \delta &\longmapsto (f \mapsto \delta(f)(e)).\end{aligned}$$

In particular, we have a Lie algebra structure on T_eG.

Definition 3.4. $\mathfrak{g} = T_eG = \mathfrak{Lie}(G)$.

Proposition 3.5. The functor $G \mapsto \mathfrak{g}$ is fully faithful.

Definition 3.6. Given an element $g \in G$, we get the conjugation map $c_g : G \to G$ with $x \mapsto gxg^{-1}$. By functoriality of the Lie algebra construction, we get a map $d(c_g) : \mathfrak{g} \to \mathfrak{g}$. Therefore we get a morphism of linear algebraic groups

$$\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g}),$$

called the adjoint representation of G. Taking the derivative of this map we now get a map

$$\mathrm{ad}_\mathfrak{g} = d(\mathrm{Ad}_G)_e : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}),$$

called the adjoint representation of $\mathfrak{g}$.

Exercise: look up the proof of the fact that $\mathfrak{gl}_n = \mathrm{Mat}_n$ and try to work out what $\mathfrak{sl}_n$ is.

Proposition 3.7. 1. If $H \leq G$ is a closed subgroup, then $\mathfrak{h} \leq \mathfrak{g}$ is a subalgebra.

2. If the characteristic of k is 0, then the kernel of the adjoint representation is the center $Z(G)$. If the characteristic is p, then this may not hold. For example, an example worked out in Chevalley's theory of Lie groups concerns the group

$$\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a^p & b \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_3(\overline{\mathbb{F}_p})$$

4 Reductive Groups

Definition 4.1. A **reductive group** is a LAG G such that $\{1\}$ is the only connected unipotent normal subgroup.

Definition 4.2. A Borel subgroup is a maximal connected solvable subgroup.

For example, the upper triangular matrices in GL_n form a Borel subgroup.

Proposition 4.3. GL_n and SL_n are reductive.

Proof. The unipotent radical would have to be contained inside any Borel subgroups, but the lower triangular matrices also form a Borel subgroup, and their intersection is the diagonal matrices. The only unipotent diagonal matrix is the identity. □

Definition 4.4. A torus $\mathbb{T}$ is a closed subgroup of G such that $\mathbb{T} \cong \mathbb{G}_m$ over $\overline{k}$. A character of $\mathbb{T}$ is a map $\mathbb{T} \to \mathbb{G}_m$, and a cocharacter (often called a 1-parameter subgroup) is a map $\mathbb{G}_m \to \mathbb{T}$.

For example the diagonal matrices define a torus inside GL_n. If we consider the adjoint representation of G, then we can diagonalise the action of any torus, and we get

$$\mathfrak{g} = \bigoplus_{\alpha \in X(\mathbb{T})} g_\alpha, g_\alpha := \{x \in \mathfrak{g} : \mathrm{Ad}(t) \cdot x = \alpha(t), \forall t \in \mathbb{T}\}.$$

The key point here is that for all but finitely many $\alpha \in \mathbb{T}$ the space on the right is 0. The set of $\alpha \neq 0$ for which the space is not zero are called the roots of G with respect to $\mathbb{T}$, denoted $\Phi(G, \mathbb{T}) \subset X(\mathbb{T})$.

Proposition 4.5. Let G be a connected reductive group and $\mathbb{T}$ a maximal torus with Lie algebra $\mathfrak{t}$.

- $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$
- For all $\alpha \in \Phi$, $\mathbb{T}_\alpha := (\ker \alpha)^0$ is a torus of codimension 1.
- For all $\alpha \in \Phi$, there exists a unique $\mathrm{Ad}_{\mathbb{T}_\alpha}$-stable subgroup $U_\alpha \leq G$ called the root group, and these are permuted by $W(G, \mathbb{T}) = N_G(\mathbb{T})/Z_G(\mathbb{T})$ which is called the Weyl group.
- $G = \langle U_\alpha, \mathbb{T} : \alpha \in \Phi(G, \mathbb{T}) \rangle$.

5 Root Systems

Definition 5.1. An **abstract root system** in a finite dimensional $\mathbb{R}$-vector space is a set Φ along with maps $\{s_\alpha \in \mathrm{End}(V) : \alpha \in \Phi\}$ such that

- $0 \notin \Phi$,

- $s_\alpha(\Phi) \subset \Phi$
- If $\alpha, \beta \in \Phi$, then $s_\alpha(\beta) - \beta \in \alpha\mathbb{Z}$.

Fact: if $\mathbb{T}$ is a maximal torus in G, then $\Phi(G, \mathbb{T})$ gives a root system in $\mathbb{Z}[\Phi] \otimes_{\mathbb{Z}} \mathbb{R}$. Any cocharacter/character of $\mathbb{G}_m$ is of the form $x \mapsto x^n$ for some $n \in \mathbb{Z}$. Therefore there is a pairing

$$\begin{aligned} Y(\mathbb{T}) \times X(\mathbb{T}) &\longrightarrow \mathbb{Z} \\ \langle \chi, \phi \rangle &\longmapsto \chi \circ \phi \in \mathrm{End}(\mathbb{G}_m) \cong \mathbb{Z} \end{aligned}$$

This is a perfect pairing. It allows us to define the dual root

Definition 5.2. If $\alpha \in \Phi$, then there is a unique dual root $\alpha^\vee \in Y(\mathbb{T})$ such that

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha$$

(check this).

Definition 5.3. An **abstract root datum** is a tuple $\Psi = (\Phi, \Phi^\vee, X(\mathbb{T}), Y(\mathbb{T}))$ satisfying a bunch of axioms. Given a root datum Ψ, we can define the dual datum by $\Psi^\vee := (\Phi^\vee, \Phi, Y, X)$.

Fact: if Ψ (equivalently $\Psi^\vee$) is reduced, then there is a unique reductive group with that root datum.

Definition 5.4. This root datum corresponds to a unique connected reductive group, which is called the dual group $\widehat{G}$.