THE BASS-QUILLEN CONJECTURE

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Abstract. These are notes from a talk giving an overview of the Bass-Quillen Conjecture and Quillen's original proof of Serre's problem on projective modules.

Throughout we follow [1], where all uncited results we state can be found. For a ring R, we write $\mathbf{P}(R)$ for the set of isomorphism classes of finitely generated projective R-modules.

1. The Conjecture

Definition 1.1. Suppose that R is a ring, $n \geq 1$, and M is an $R[t_1, ..., t_n]$-module. Then M is called *extended* (from R) if

$$M \cong R[t_1, ..., t_n] \otimes_R N$$

for some R-module N.

Remark 1.2. Such an N as above is necessarily unique, recovered as

$$M/(t_1, ..., t_n)M \cong \frac{R[t_1, ..., t_n]}{(t_1, ..., t_n)} \otimes_R M \cong \frac{R[t_1, ..., t_n]}{(t_1, ..., t_n)} \otimes_R R[t_1, ..., t_n] \otimes_R N \cong N.$$

The Bass-Quillen Conjecture is the conjecture that for all $n \geq 1$ and all commutative regular noetherian rings A, all vector bundles on $\mathbb{A}^n_A$ arise via pullback from a vector bundle on $\mathrm{Spec}(A)$. In other words:

Conjecture 1.3 (Bass-Quillen Conjecture)**.** *Suppose that A is a commutative regular noetherian ring, and $n \geq 1$. Then*

$$\mathbf{P}(A) \to \mathbf{P}(A[t_1, ..., t_n]), \qquad M \mapsto A[t_1, ..., A_n] \otimes_A M$$

is a bijection. Equivalent, every $M \in \mathbf{P}(A[t_1, ..., t_n])$ is extended from A.

When $A = k$ is a field, the Bass-Quillen Conjecture is known as the following.

Theorem 1.4 (Serre's Problem on Projective Modules)**.** *Suppose that $n \geq 1$ and k is a field. Then any finitely generated projective modules over $k[t_1, ..., t_n]$ is free.*

In this talk we will describe the relationship between this problem and K-theory, and discuss two ways to prove this. This first involves unimodular rows, and it more elementary. The second is Quillen's original solution, which involves his patching lemma, and Horrocks' Theorem.

2. Stably Free Modules and K_0

Let us first outline how the Bass-Quillen conjecture relates to K-theory, and try to motivate some of the assumptions that appear in the statement.

Recall that for a ring R, $K_0(R)$ is the group completion of $\mathbf{P}(R)$ with respect to $\oplus$. In particular, $P, Q \in \mathbf{P}(R)$ are equal in $K_0(R)$ if and only if there is some $P' \in \mathbf{P}(R)$ with

$$P \oplus P' \cong Q \oplus P'.$$

Furthermore, because free modules are cofinal in $\mathbf{P}(R)$, this is equivalent to the statement that P and Q are *stably isomorphic*:

$$P \oplus R^m \cong Q \oplus R^m$$

for some $m \geq 1$.

Date: November 20, 2024.

2.1. **Regularity.** If the Bass-Quillen conjecture is true for a ring R, the it is certainly true that the natural map

$$K_0(R) \to K_0(R[t_1, ..., t_n]) \tag{1}$$

is an isomorphism.

Theorem 2.1 (Grothendieck's Theorem)**.** *If R is left regular, then* (1) *is an isomorphism.*

Remark 2.2. Note that for any ring R, the map (1) is always injective, as K_0 is a functor and $R \hookrightarrow R[t_1, ..., t_n]$ is split by $t_i \mapsto 0$.

Here R is called *left regular* if R is left noetherian and any finitely generated left R-module admits a finite projective resolution. When R is commutative and noetherian, this agrees with the usual definition of regular in commutative algebra (that $R_{\mathfrak{m}}$ is a regular local ring for any maximal ideal $\mathfrak{m}$ of R).

It is not known exactly for which rings Grothendieck's Theorem holds. However, one can say something regarding the corresponding question for $\mathrm{Pic}(R)$.

Recall that for a commutative ring R, $\mathrm{Pic}(R)$ is the set of isomorphism classes of finitely generated projective modules over R, which is a group with respect to the tensor product. This is related to $K_0(R)$ through the natural group homomorphism

$$\mathrm{Pic}(R) \to K_0(R)^*. \tag{2}$$

Theorem 2.3. *Suppose that R is an integral domain with field of fractions K. Then the following are equivalent.*

(1) $\mathrm{Pic}(R) \to \mathrm{Pic}(R[t_1, ..., t_n])$ *is an isomorphism for* some $n \geq 1$.
(2) $\mathrm{Pic}(R) \to \mathrm{Pic}(R[t_1, ..., t_n])$ *is an isomorphism for* all $n \geq 1$.
(3) If $a \in K$ and $a^m \in R$ for sufficiently large m, then $a \in R$.
(4) R is semi-normal*: If $a \in K$ and $a^2, a^3 \in R$, then $a \in R$.*

Proof. [1, Thm. 5.11]. □

Remark 2.4. Similarly to above, this map is always injective, and the content is about the surjectivity.

Remark 2.5. Semi-normal is a weakening of normal, and a regular commutative noetherian ring is normal (as any (commutative noetherian) regular local ring is a UFD).

This motivates the regularity condition in Conjecture 1.3. Whilst we are here, let us mention what is known about the relationship between regular schemes and homotopy invariance for K_n. More generally than Grothendieck's Theorem, algebraic K-theory is homotopy invariant for regular schemes, meaning that the natural map

$$K_n(X) \to K_n(X \times \mathbb{A}^1)$$

is an isomorphism for all $n \in \mathbb{Z}$. This is not an isomorphism in general for non-regular X.

Definition 2.6. For a scheme X and $n \geq 1$, X is said to be K_n-regular if the above map is an isomorphism.

Vorst's Conjecture, proven by Cortiñas, Hasemeyer and Weibel, states that if R is a commutative ring, essentially finite type over a characteristic 0 field[1], then the property that R is $K_{\dim(R)+1}$-regular implies that R is regular.

[1]meaning a localisation of a finite type algebra over the field

2.2. **Commutativity.** Let us also note that the Bass-Quillen Conjecture isn't true if A is not commutative. For example, let D be a division algebra that is not a field, and fix a pair (a, b) of non-commuting elements. Then the module

$$P \coloneqq \{(r, s) \in D[x, y]^2 \mid r(x + a) = s(y + b)\}$$

satisfies

$$P \oplus D[x, y] \cong D[x, y]^2$$

and is finitely generated projective, but one can show that P is not free [1, Cor. 3.6]. In fact, for some division rings (such as the quarternions), you can show that $\mathrm{Pic}(D[x, y])$ is infinite!

3. Hermite Rings

Suppose now that R is a ring such that (1) is an isomorphism. For example, R could be a commutative regular noetherian ring, as in the assumptions of Bass-Quillen.

Then we see that any $P \in \mathbf{P}(R)$ satisfies

$$P \oplus R[t_1, ..., t_n]^m \cong R[t_1, ..., t_n] \otimes_R M \oplus R[t_1, ..., t_n]^m \cong R[t_1, ..., t_n] \otimes_R (M \oplus R^m)$$

for some $m \geq 1$ and $M \in \mathbf{P}(R)$, and therefore P is *stably extended*. Therefore, *to show the Bass-Quillen Conjecture, using Grothendieck's Theorem it is enough to show that stably extended modules are actually extended.*

In particular, Serre's Problem is equivalent to the statement that stably free is equivalent to free over $k[t_1, ..., t_n]$, or that $k[t_1, ..., t_n]$ is Hermite, where:

Definition 3.1. A ring R is called *Hermite* if any stably free module is free.

It is not relevant, but let us note that stably free modules are a bit strange, in the sense that one can show that the non-finitely generated ones are simpler by using a trick due to Gabel.

Lemma 3.2 ([1, Prop. I..4.2])**.** *Suppose that P is an R-module that is not finitely generated. Then if $P \oplus R^m$ is free for some $m \geq 1$, then P is free.*

4. Hermite Rings and Unimodular Rows

In this section, we want to give a matrix characterisation of Hermitian rings. Throughout, for simplicity, we assume that R is commutative.

Suppose that $P \in \mathbf{P}(R)$ is stably free. To this, one can choose an isomorphism $R^m \oplus P \cong R^n$ $(m \geq n)$, and let M be the matrix $m \times n$ matrix of the associated projection $f \colon R^n \twoheadrightarrow R^m$. The matrix M is right invertible: there is an $n \times m$ matrix N such that $MN = I_m$ (taking N as the matrix of the inclusion $R^m \hookrightarrow R^m \oplus P$. One recovers P (up to isomorphism) as the kernel of the matrix M.

Lemma 4.1. *Let P, $f \colon R^n \twoheadrightarrow R^m$ be as above. Then P is free if and only if exists an isomorphism*

$$\hat{f} \colon R^n \xrightarrow{\sim} R^m \oplus R^r$$

for some $r \geq 0$ such that $\pi \circ \hat{f} = f$, where $\pi \colon R^m \oplus R^r \to R^m$ is the projection.

Corollary 4.2. *Let P, M be as above. Then P is free if and only if M can be extended to an invertible matrix (by adding $n - m$ rows).*

Proof. If A is the matrix of $\hat{f}$, and X is the matrix of π, then X is the $m \times n$ made of I_m with zeros to the right, hence $XA = N$ exactly corresponds to the first m rows of A being equal to N. □

Of special interest is the case when $m = 1$.

Definition 4.3. A *unimodular row* is a vector $(b_1, ..., b_n) \in R^n$ such that $b_1 R + \cdots + b_n R = R$. We write $\mathrm{Um}_n(R)$ for the set of unimodular rows of length $n \geq 1$ in R.

Remark 4.4. For a unimodular row of length n, the corresponding projective module has rank $n - 1$.

Because it is sufficient to show that stably free implies free when $m = 1$, we have the following.

Corollary 4.5. *A commutative ring R is Hermite if and only if every unimodular row can be extended (by adding $(n-1)$ rows) to an invertible matrix.*

For any $n \geq 1$, there is a natural action of $\mathrm{GL}_n(R)$ on $\mathrm{Um}_n(R)$ by multiplication on the right.

Lemma 4.6. *The correspondence between right invertible matrices and stably free $P \in \mathbf{P}(R)$ induces a bijection between $\mathrm{Um}_n(R)$ are $P \in \mathbf{P}(R)$ with $P \oplus R \cong R^n$. The free module R^{n-1} corresponds to $(1, 0, ..., 0)$.*

Example 4.7. If R is commutative, P has rank 1, and

$$P \oplus R \cong R^2,$$

then $P \cong R$. Indeed, to show this, we just need to show that any $(a, b) \in \mathrm{Um}_2(R)$ is completable to an invertible matrix. Indeed, as (a, b) is unimodular, there are $(e, f) \in R^2$ with $ae + bf = 1$, and so taking $c = -f$, $d = e$ we can extend (a, b) to an invertible matrix.

Equivalently, the only orbit of $\mathrm{GL}_2(R)$ on $\mathrm{Um}_2(R)$ is $(1, 0)$. To see this, note that, for the same c, d found as above, we can take

$$(a, b) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Remark 4.8. In fact, one can show that any rank 1 stably free $P \in \mathbf{P}(R)$ is actually free, or in other words that the map

$$\mathrm{Pic}(R) \to K_0(R)$$

is injective, because P can be recovered from $P \oplus R^{n-1}$ as the determinant line bundle.

For any $d \geq 0$, the following are equivalent:

(1) Any stably free $P \in \mathbf{P}(R)$ of rank $> d$ is free,
(2) Any unimodular row of length $n \geq d + 2$ can be completed to an invertible matrix.
(3) $\mathrm{GL}_n(R)$ acts transitively on $\mathrm{Um}_n(R)$ for $n \geq d + 2$.

Definition 4.9. We call a (commutative) ring which satisfies the above d-Hermite.

Remark 4.10. We see that Hermite = 0-Hermite = 1-Hermite.

Theorem 4.11 (Bass)**.** *If R is noetherian and has Krull dimension d, then R is d-Hermite.*

Note that this proves Serre's conjecture for $k[t]$ (which we already knew as $k[t]$ is a PID).

Using unimodular rows, after Quillen and Suslin's (independent) proofs of Serre's problem, Suslin and Vaserstein both discovered elementary solutions using unimodular rows by showing that there is only one $\mathrm{GL}_n(R)$ orbit.

5. Quillen Patching and Horrocks' Theorem

In this section we briefly discuss Quillen's original solution to Serre's problem, which involves his famous patching Theorem.

Definition 5.1. For a commutative ring R, we set $R\langle t \rangle$ to be the localisation of $R[t]$ at the set of monic polynomials in t.

Theorem 5.2 ((Local) Horrocks' Theorem, Algebraic)**.** *Suppose that R is a commutative local ring, and $P \in \mathbf{P}(R[t])$. Then if $P\langle t \rangle := R\langle t \rangle \otimes_{R[t]} P$ is free, P is free.*

This also has a geometric version:

Theorem 5.3 ((Local) Horrocks' Theorem, Geometric)**.** *Suppose that R is a commutative local ring. Then the only vector bundle on $\mathbb{A}^1_R$ which extends to $\mathbb{P}^1_R$ is the trivial bundle.*

To get a feel for why these are equivalent, consider the case when $R = k$ is a field. In this case, $R\langle t\rangle = k(t)$ is the field of rational functions.

If the algebraic version is true, and a vector bundle on $\mathbb{A}^1_R$ defined by P extends to $\mathbb{P}^1_k$, then $k(t) \otimes_{k[t]} P$ is the stalk at the point at infinity, hence is free, and thus by the algebraic version the vector bundle defined by P is free.

Conversely, if the geometric version is true, then for any such P, if $k(t) \otimes_{k[t]} P$ is free, this means there is a closed and open neighborhood around ∞ where P is free. Therefore, P can be extended to a vector bundle over $\mathbb{P}^1_k$, which by the geometric version implies that P is free.

Quillen patching allows one to extend this beyond the local case. Let us state it now (much more general versions exist).

Theorem 5.4 (Quillen Patching)**.** *Suppose that R is a commutative ring, and let $P \in \mathbf{P}(R[t_1, ..., t_n])$. Then P is extended from R if and only if each $P_\mathfrak{m}$ is free over $R_\mathfrak{m}[t_1, ..., t_n]$ for all maximal ideal $\mathfrak{m}$ of R.*

Quillens result immediately implies the follow "affine" version of Horrock's theorem.

Theorem 5.5 ((Affine) Horrocks' Theorem, Algebraic)**.** *Suppose that R is a commutative ring, and $P \in \mathbf{P}(R[t])$. Then if $P\langle t\rangle \coloneqq R\langle t\rangle \otimes_{R[t]} P$ is extended from R, P is extended from $P_0 \in \mathbf{P}(R)$.*

This has an analogous geometric version too.

Theorem 5.6 ((Affine) Horrocks' Theorem, Geometric)**.** *Suppose that R is a commutative ring. Then the only vector bundles on $\mathbb{A}^1_R$ which extends to $\mathbb{P}^1_R$ are those pulled back from* $\operatorname{Spec}(R)$*.*

Remark 5.7. The same proof as above shows that the geometric implies the algebraic, but the proof that the geometric implies the algebraic doesn't generalise. However, both statements are true, using Quillen patching and local Horrocks for each.

This allows us to now prove Serre's problem.

Proof of Serre's Problem. We prove by induction on $n \geq 0$ that any finitely generated projective module over $k[t_1, ..., t_n]$ is free. When $n = 0$ this is trivial, so let $n \geq 1$, and set

$$A = k[t_2, ..., t_n].$$

Then

$$A[t] \subset k\langle t\rangle[t_2, ..., t_n] \subset A\langle t\rangle.$$

By induction, as $k\langle t\rangle = k(t)$ is a field,

$$k(t)[t_2, ..., t_n] \otimes_{A[t]} P$$

is free, hence $P\langle t\rangle$ is also free. Therefore by affine Horrocks' Theorem, P is extended from $P/tP \in \mathbf{P}(A)$. But then again but induction, a finitely generated projective modules over A are free, thus P is free. □

Remark 5.8. The above proof also shows that the Bass-Quillen conjecture is true when k is a PID, as for such rings $k\langle t\rangle$ is also a PID, which was the only property that was used.

6. Current Status

Lindel (1981) showed that the Bass-Quillen conjecture holds for all commutative regular rings which are essentially finite type over a field of characteristic 0.

References

[1] T. Y. Lam. *Serre's problem on projective modules.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2006.