FORMALISM OF ADMISSIBLE REPRESENTATIONS

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These are notes from a talk at the Oxford p-adic Hodge Theory study group in February 2022. They aim to cover the content of [1, I.5] and [4, III.1].

1. (F, G)-Regular Rings

Notation: K is a p-adic field, and $G_K := \mathrm{Gal}(\overline{K}/K)$. Write C_K for the completion of $\overline{K}$, and $\mathbb{F}$ for the residue field of K.

Let F be a field and G a group. Suppose that B is a commutative F-algebra such that

- B has an action of G by F-algebra automorphisms,
- $E := B^G$ is a field,
- B is a domain.

Aim: construct a functor from finite dimensional F-representations of G to finite dimensional E-vector spaces. Depending on B, these E-vector spaces will typically naturally have extra structure.

Let C be the fraction field of B. There is a unique extension of the action of G on B to an action of G on C by F-algebra automorphisms.

Definition 1.1. B as above is called (F, G)-regular, if

- $B^G = C^G$,
- If $b \in B \setminus 0$ is such that $F \cdot b$ is G-stable, then $b \in B^\times$.

From this we see the necessity of the requirement that B^G is a field.

Example 1.2. If B is a field, then B is trivially (F, G)-regular.

We are mostly interested in $(\mathbb{Q}_p, G_K)$-regular rings.

Example 1.3. Any field extension of $\mathbb{Q}_p$ with an action of G_K is $(\mathbb{Q}_p, G_K)$-regular. In particular, both $\overline{K}$ and C_K are $(\mathbb{Q}_p, G_K)$-regular.

Write $\chi : G_K \to \mathbb{Z}_p^\times \subset \mathbb{Q}_p^\times$ for the p-adic cyclotomic character.

Definition 1.4. If M is a $\mathbb{Z}_p[G_K]$-module, then for $n \in \mathbb{Z}$, define $M(n)$ to be the $\mathbb{Z}_p[G_K]$-module, which is M as a $\mathbb{Z}_p$-module, but with "twisted" action,

$$g \cdot m := \chi(g)^n g(m).$$

More generally, if $\eta : G_K \to \mathbb{Z}_p^\times$ is a continuous character, set $M(\eta)$ to be M with action $g \cdot m = \eta(g)^n g(m)$.

Therefore, we can talk about the action of G_K on $C_K(n)$ for $n \in \mathbb{Z}$.

Theorem 1.5 (Sen - Tate Theorem)**.** *Let $n \in \mathbb{Z}$. Then the G_K-invariants,*

$$C_K(n)^{G_K} = \begin{cases} K & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}.$$

Put another way, this says that

- There are no transcendental invariants of C_K: $C_K^{G_K} = \overline{K}^{G_K} = K$,
- If $n \neq 0$, and $x \in C_K$ with $x = \chi(g)^n g(x)$ for all $g \in G_K$, then $x = 0$.

Actually we will need a more general result, which sometimes goes by the same name.

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Theorem 1.6. *Suppose that $\eta : G_K \to \mathbb{Z}_p^\times$ is a continuous character. Then the G_K-invariants,*

$$C_K(\eta)^{G_K} = \begin{cases} K & \text{if } \eta(I_K) \text{ is finite,} \\ 0 & \text{if } \eta(I_K) \text{ is infinite.} \end{cases}$$

Example 1.7. B_{HT} can be non-canonically identified with $C_K[T, 1/T]$. The action under this identification of G_K on $C_K[T, 1/T]$ is given by

$$g \cdot \left(\sum_{i \in \mathbb{Z}} a_i T^i \right) = \sum_{i \in \mathbb{Z}} g(a_i) \chi(g)^i T^i,$$

where χ is the cyclotomic character. Then we claim this is $(\mathbb{Q}_p, G_K)$-regular. We first need to compute the invariant subring of $C_K[T, 1/T]$ and of its fraction field $C_K(T)$. To do so, we consider both inside the formal Laurent power series ring $C_K((T))$, the fraction field of $C_K[[T]]$. The inclusion $C_K(T) \hookrightarrow C_K((T))$ is G_K-equivariant when $C_K((T))$ is given the action

$$g \cdot \left(\sum_{i \geq i_0} a_i T^i \right) = \sum_{i \geq i_0} \chi(g)^i g(a_i) T^i.$$

A formal Laurent series as above is G_K-invariant if and only if each $a_i \in C_K(i)^{G_K}$, and so by the Theorem above of Sen-Tate, $C_K((T))^{G_K} = K$.

Now to prove the second property, let $b \in C_K[T, 1/T] \setminus 0$, and suppose that $\mathbb{Q}_p \cdot b$ is G_K-stable. Then the action of G_K on b defines a group homomorphism $\eta : G_K \to \mathbb{Q}_p^\times$. For any $i \in \mathbb{Z}$ and $g \in G_K$, we have

$$\eta(g) a_i = \chi(g)^i g(a_i).$$

Some $a_{i_0} \neq 0$, and $\eta(g) = \chi(g)^{i_0} (g(a_{i_0})/a_{i_0})$. Then because χ^{i_0} and the action of g on C_K are continuous, η is continuous. Because η is continuous, $\eta(G_K) \subset \mathbb{Z}_p$ - this is because the composition $|| \cdot || \circ \eta : G_K \to \mathbb{R}$ is continuous, and so $||\eta(G_K)|| \subset \mathbb{R}$ is compact.

If $a_i \neq 0$, then $a_i = (\chi^i \eta^{-1})(g) g(a_i)$ for all $g \in G_K$, so $a_i \in C_K(\chi^i \eta^{-1})^{G_K}$. Therefore, by the stronger version of the Sen-Tate Theorem above, $(\chi^i \eta)(I_K)$ is finite. Therefore, if $a_i \neq 0 \neq a_j$, then $(\chi^i \eta^{-1} \chi^{-j} \eta)(I_K) = \chi^{i-j}(I_K)$ is finite, so $i = j$. Thus $b = a_i T^i$ and is a unit. So $C_K[T, 1/T]$ is $(\mathbb{Q}_p, G_K)$-regular.

Example 1.8. The ring B_{dR}^+ is a complete DVR with uniformiser $t \in B_{\mathrm{dR}}^+$, and G_K acts on t via the p-adic cyclotomic character. B_{dR}^+ is not $(\mathbb{Q}_p, G_K)$-regular, because $\mathbb{Q}_p \cdot t$ is a G_K-stable subspace, but t is not a unit of B_{dR}^+. However, the fraction field is $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[1/t]$ and being a field is $(\mathbb{Q}_p, G_K)$-regular. There is a natural filtration on B_{dR}, with associated graded B_{HT}, and using this one can see that $B_{\mathrm{dR}}^{G_K} = K$ (as a consequence, $(B_{\mathrm{dR}}^+)^{G_K} = K$ too).

Example 1.9. We also will construct subrings $B_{\mathrm{cris}} \subset B_{\mathrm{st}}$ of B_{dR}. These are both not fields, but will turn out to be $(\mathbb{Q}_p, G_K)$-regular, with invariant ring $K_0 = W(\mathbb{F})[1/p]$.

Example 1.10. How does this relate to the rings in Talk 3? Let E be a field of characteristic $p > 0$, and let E^s be a separable closure of E. Let $G_E = \mathrm{Gal}(E^s/E)$. Then $B = E^s$ is $(\mathbb{F}_p, G_E)$-admissible. The ring $B = \widehat{\mathcal{O}_{\mathcal{E}}^{nr}}$ does not naturally fit into this framework, because even though it has an action of G_E, $B^{G_E} = \mathcal{O}_{\mathcal{E}}$ is not a field. However, for $\mathcal{E}$ the fraction field of $\mathcal{O}_{\mathcal{E}}$, $B = \widehat{\mathcal{E}^{nr}}$ is $(\mathbb{Q}_p, G_E)$-regular, with $B^G = \mathcal{E}$.

2. B-Admissible Representations

From now on, **we will assume that B is (F, G)-regular**.

Definition 2.1. We define a functor from the category of finite dimensional F-representations of G to E-vector spaces,

$$D_B : \mathrm{Rep}_F(G) \to \mathrm{Vect}_E,$$

by

$$D_B(V) := (V \otimes_F B)^G.$$

Here G acts on $V \otimes_F B$ by $g(b \otimes v) = gb \otimes gv$, and $(V \otimes_F B)^G$ is a $E = B^G$-vector space induced from the B-module structure of $V \otimes_F B$ where $b'(v \otimes b) = v \otimes b'b$. We also have a natural map,

$$\alpha_V : D_B(V) \otimes_E B \to (V \otimes_F B) \otimes_E B = V \otimes_F (B \otimes_E B) \to V \otimes_F B.$$

With G acting on $D_B(V) \otimes_E B$ through the second factor B, then this is a G-equivariant B-linear map.

Example 2.2. Suppose that $V = F$ is the trivial representation. Then $D_B(F) = B^G$, and α_V is the identity map.

Example 2.3. Suppose that $B = F$. Then $D_B(V) = (V \otimes_F F)^G = V^G$ is the functor from $\mathrm{Rep}_F(G)$ to Vect_F taking V to its G-invariants.

At this point, it is not really obvious why $D_B(V)$ is finite dimensional.

Lemma 2.4. *If $V \in \mathrm{Rep}_F(G)$, then $\alpha_V : D_B(V) \otimes_E B \to V \otimes_F B$ is (B-linear, G-equivariant, and) injective.*

Proof. If C is the fraction field of B, we have a commutative square:

$$\begin{array}{ccc} D_B(V) \otimes_E B & \xrightarrow{\alpha_V} & V \otimes_F B \\ \downarrow & & \downarrow \\ D_C(V) \otimes_E C & \xrightarrow{\beta_V} & V \otimes_F C \end{array}$$

Therefore it is sufficient to prove that β_V is injective. β_V is injective if and only if β_V maps an E-basis to a C-linearly independent set of $V \otimes_F C$. Therefore, it is sufficient to show that if $\{x_1, ..., x_m\} \subset (V \otimes_F C)^G$ are E-linearly independent, then they are C-linearly independent.

Suppose not, and take a C-linear dependence of minimal length. So for some $r \geq 1$, $x_r = \sum_{i<r} c_i x_i$, and r is the minimal length of such a relation. Then for all $g \in G$,

$$x_r = g(x_r) = \sum_{i<r} g(c_i) x_i.$$

Equivalently,

$$0 = \sum_{i<r} (g(c_i) - c_i) x_i,$$

and so by minimality, $g(c_i) = c_i$ for all i. But then for all i, $c_i \in C^G = E$, a contradiction. □

Theorem 2.5. *For $V \in \mathrm{Rep}_F(G)$, we have the bound,*

$$\dim_E D_B(V) \leq \dim_F(V).$$

In particular, $D_B(V)$ is a finite dimensional E-vector space. Furthermore, for $d = \dim_F(V)$, the following are equivalent:

1. *$\dim_E D_B(V) = \dim_F(V)$,*
2. *α_V is an isomorphism,*
3. *There is an isomorphism $V \otimes_F B \cong B^d$ as B-modules which is G-equivariant,*
4. *There is a B-basis of $V \otimes_F B$ consisting of d elements of $(V \otimes_F B)^G$.*

Proof. Again let C be the fraction field of B. If we tensor $\alpha_V : D_B(V) \otimes_E B \to V \otimes_F B$ with C over B, we obtain,

$$\alpha_V \otimes_B C : D_B(V) \otimes_E C \to V \otimes_F C.$$

The C-dimension of the term of the left is $\dim_E(D_B(V))$, and the C-dimension of the term of the right is $\dim_F(V)$. Because α_V is injective and C is flat over B, we see that in general $\dim_E(D_B(V)) \leq \dim_F(V)$. Furthermore, clearly if α_V is an isomorphism (1), we have the equality (2). Now, to prove that (2) ⇒ (1), suppose that $\dim_E(D_B(V)) = \dim_F(V)$. We want to show that α_V is an isomorphism.

Let $\{e_i\}$ be an E-basis of $D_B(V)$, and $\{v_j\}$ be an F basis of V. By assumption, these have the same size d, and we can express the B-linear map α_V by the $d \times d$ matrix (b_{ij}), so

$\alpha_V(e_i) = \sum_j b_{ij} \otimes v_j$. By assumption, $\alpha_V \otimes_B C$ is an isomorphism, so $\det(b_{ij}) \neq 0$, when considered an element of C.

We know that $b := \det(b_{ij}) \in B \setminus 0$, and want to show that $b \in B^\times$. To do this we use the fact that B is (F, G)-regular, so we want to show that $F \cdot b$ is stable under the action of G.

Passing to the dth exterior power (as B-modules),

$$\wedge^d(\alpha_V)(e_1 \wedge \cdots \wedge e_d) = b(v_1 \wedge \cdots v_d)$$

Because α_V is G-equivariant, and e_i are G-invariant, G acts on the left trivially, and so

$$g(b)g(v_1 \wedge \cdots \wedge v_d) = b(v_1 \wedge \cdots v_d).$$

But $g(v_1 \wedge \cdots \wedge v_d) = \det(g)(v_1 \wedge \cdots \wedge v_d)$, hence

$$g(b) = \det(g)^{-1}b,$$

and $\mathbb{Q}_p \cdot b$ is stable under the action of G.

The equivalence of (3) and (4) is immediate, so we just need to show that (3) is equivalent to (1) and (2). If α_V is an isomorphism (1), then because $\dim_E(D_B(V)) = d$, by choosing an E basis of $D_B(V)$ we have $B^d \cong D_B(V) \otimes_E B$, and this is G-equivariant because the action of G on $D_B(V)$ is trivial. Therefore, composing this with α_V we have (3). Conversely, if $B^d \cong D_B(V) \otimes_E B$, then $\dim_E(D_B(V)) = d$ and we have (1). □

Definition 2.6. We call $V \in \mathrm{Rep}_F(G)$ *B-admissible* if $\dim_E D_B(V) = \dim_F(D_B(V))$. We write $\mathrm{Rep}_F^B(G)$ for the full-subcategory of finite dimensional B-admissible F-representations of G. If $(F, G) = (\mathbb{Q}_p, G_K)$, and B is one of B_{HT}, B_{dR}, B_{st} or B_{cris}, then we call B-admissible p-adic representations Hodge-Tate, de Rham, Semistable and Crystalline respectively.

For the next example, we need a couple of definitions. Let R be a commutative topological ring, and Γ a topological group, such that Γ acts on R continuously. Then a R-representation [3, Def. 2.2] is an R-module of finite type equipped with a continuous semi-linear action of Γ. This is called free of rank d if the underlying R-module is. This is called *trivial* if it has a basis consisting of Γ-invariant elements. For a fixed $d \geq 1$, there is a one-to-one correspondence between free R-representations of rank d and elements of $H^1_{\mathrm{cts}}(\Gamma, \mathrm{GL}_d(R))$. Furthermore, a free R-representation X is trivial if and only if it corresponds to the trivial element of $H^1_{\mathrm{cts}}(\Gamma, \mathrm{GL}_d(R))$ [3, Prop. 2.6].

Example 2.7. $(F, G) = (\mathbb{Q}_p, G_K)$, $B = \overline{K}$. Then we claim that V is $\overline{K}$-admissible if and only if the action of G_K on V factors through some finite quotient. This property is called being *potentially trivial*, and is the same as V being smooth as a G_K-representation, or that the action of G_K is discrete: continuous when V is given the discrete topology.

To see why, suppose that the action of G_K on V is potentially trivial, so there is an open subgroup of G_K which acts trivially on V. Let $X = V \otimes_{\mathbb{Q}_p} \overline{K}$. The action of G_K on X by $g(v \otimes \lambda) = g(v) \otimes g(\lambda)$ also factors through some open subgroup, because this is true for the action of G_K on $\overline{K}$. Let $d = \dim_{\mathbb{Q}_p} V = \dim_{\overline{K}} X$, and fix the discrete topology on $\overline{K}$. Then X is a free rank d $\overline{K}$-representation of G_K.

By a strong version of Hilbert's Theorem 90, we have that for all $d \geq 1$,

$$H^1_{\mathrm{cts}}(G_K, \mathrm{GL}_d(\overline{K})) = 0.$$

Therefore, by the above discussion, $X = (V \otimes_{\mathbb{Q}_p} \overline{K})$ has a $\overline{K}$-basis of G_K-invariant elements. But this means that V is $\overline{K}$-admissible by Theorem 2.5.

Showing the converse is easier: suppose that (V, ρ) is $\overline{K}$-admissible and choose a basis of G_K-invariant elements $\{e_1, ..., e_d\}$. We want to show that the stabiliser of any element $x = \sum_i \lambda_i e_i$ is an open subgroup. For any $g \in G_K$, $g(x) = \sum_i g(\lambda_i)e_i$, and so

$$(G_K)_x = \cap_i (G_K)_{\lambda_i},$$

is an open subgroup of G_K. But then for a $\mathbb{Q}_p$-basis $\{v_1, ..., v_d\}$ of V, letting $x = v_i \otimes 1$ in turn, we see that $\ker(\rho) = \cap_i (G_K)_{v_i \otimes 1}$ is open.

Example 2.8. Let $P := \widehat{K^{\mathrm{nr}}}$. We have $\mathrm{Gal}(\overline{P}/P) = I_K$. Let $(F,G) = (\mathbb{Q}_p, G_K)$, $B = \overline{P}$. Then a p-adic representation V of G_K is $\overline{P}$-admissible if and only if the action of I_K is discrete [3, Prop. 3.53]. This property is called being *potentially unramified*.

We have that $\overline{K} \subset \overline{P} \subset C_K$. In fact the C_K-admissible representations are the same as the $\overline{P}$-admissible representations:

Example 2.9. $(F,G) = (\mathbb{Q}_p, G_K)$, $B = C_K$. Then a p-adic representation V is C_K-admissible if and only if the action of I_K factors through some finite quotient. Then a p-adic representation V of G_K is $\overline{P}$-admissible if and only if the action of I_K is discrete [3, Prop. 3.55].

Concretely, if (ρ, V) is a p-adic representation of G_K, then

- V is $\overline{K}$-admissible iff $\ker(\rho) \leq G_K$ is open,
- V is $\overline{P}$-admissible iff V is C_K-admissible iff $\ker(\rho) \cap I_K \leq I_K$ is open (in I_K).

Example 2.10. In talk 5 we shall see an equivalent definition of a Hodge-Tate representation, not in terms of the period ring.

Example 2.11. In talk 3, $B = E^s$ is a $(\mathbb{F}_p, G_E)$-regular ring. Then the E^s-admissible $\mathbb{F}_p$-representations of G_E are exactly the continuous representations (where G_E has the Krull topology and V the discrete topology). The reasoning is analogous to that of Example 2.7.

D_{E^s} as it stands is not an equivalence of categories - this functor is essentially surjective and faithful, but not full. In talk 3, this is modified to an equivalence of categories, by mapping to finite-dimensional K-vector spaces equipped with an injective (Frobenius) semi-linear ϕ. We have the frobenius $E^s \to E^s$, and for any $V \in \mathrm{Rep}_{\mathbb{F}_p}(V)$ we define ϕ on $D_{E^s}(V) = (E^s \otimes_{\mathbb{F}_p} V)^{G_E}$ by restricting $\phi : E^s \otimes_{\mathbb{F}_p} V \to E^s \otimes_{\mathbb{F}_p} V$,

$$\phi(x \otimes v) = x^p \otimes v,$$

to $D_{E^s}(V) = (E^s \otimes_{\mathbb{F}_p} V)^{G_E}$.

3. Properties of D_B

Now we summarise the main properties of the functor D_B.

Theorem 3.1. *The restriction of the functor D_B to the full subcategory $\mathrm{Rep}_F^B(G)$,*

$$D_B : \mathrm{Rep}_F^B(G) \to \mathrm{Vect}_E,$$

is exact and faithful. $\mathrm{Rep}_F^B(G)$ is closed under sub-representations and quotients. Furthermore,

- *If $V_1, V_2 \in \mathrm{Rep}_F^B(G)$, then there is a natural isomorphism,*
$$D_B(V_1) \otimes_E D_B(V_2) \to D_B(V_1 \otimes_F V_2),$$
and so $V_1 \otimes V_2 \in \mathrm{Rep}_F^B(G)$.
- *If $V \in \mathrm{Rep}_F^B(G)$, then $V^* \in \mathrm{Rep}_F^B(G)$, and the natural map,*
$$D_B(V) \otimes_E D_B(V^*) \xrightarrow{\sim} D_B(V \otimes_F V^*) \to D_B(F) \cong E,$$
is a perfect pairing.
- *$\mathrm{Rep}_F^B(G)$ also is closed under symmetric and exterior powers, and D_B commutates with these constructions.*

Proof. We prove that D_B is exact, fully faithful and closed under subquotients. The rest can be found in [1, Part I, Section 5].

Both faithfulness and exactness come down to the fact that B is an algebra over both F and $E = B^G$, and therefore is faithfully flat over F and E. Recall that for a ring R and module M, M is a faithfully flat R-module iff either of the following equivalent conditions hold:

- M is flat and for any R-linear $f : N_1 \to N_2$, then f is non-zero if and only if $f \otimes 1 : M_1 \otimes_R M \to M_2 \otimes_R M$ is non-zero.
- For any sequence $N_1 \to N_2 \to N_3$, this is exact at N_2 if and only if $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact at $N_2 \otimes_R M$.

Faithful: Suppose that $f : V \to W$ is a morphism $F[G]$-modules. Then because B is faithfully flat over E, $D_B(f) = f \otimes 1 : D_B(V) \to D_B(W)$ is zero if and only if

$$(f \otimes 1) \otimes 1 : D_B(V) \otimes_E B \to D_B(W) \otimes_E B,$$

is zero. But because V and W are B-admissible, α_V, α_W are isomorphisms in the commutative diagram,

$$\begin{array}{ccc} D_B(V) \otimes_E B & \xrightarrow{(f\otimes 1)\otimes 1} & D_B(W) \otimes_E B \\ \alpha_V \downarrow & & \downarrow \alpha_W \\ V \otimes_F B & \xrightarrow[f\otimes 1]{} & W \otimes_F B \end{array}$$

and because B is faithfully flat over F, $f \otimes 1$ is zero if and only if $f : V \to W$ is zero.

Exact: Let

$$0 \to U \to V \to W \to 0,$$

be a short exact sequence in $\mathrm{Rep}_F^B(G)$. Then

$$0 \to U \otimes_F B \to V \otimes_F B \to W \otimes_F B \to 0,$$

is an exact sequence of B-modules, and so because U, V and W are B-admissible,

$$0 \to D_B(U) \otimes_E B \to D_B(V) \otimes_E B \to D_B(W) \otimes_E B \to 0,$$

is exact. But then B is faithfully flat over E, hence

$$0 \to D_B(U) \to D_B(V) \to D_B(W) \to 0,$$

is exact.

Closed under subquotients: Consider a short exact sequence of $F[G]$-modules,

$$0 \to U \to V \to W \to 0,$$

where V is B-admissible. By definition D_B is left exact, so we also have the exact sequence of E-modules,

$$0 \to D_B(U) \to D_B(V) \to D_B(W)$$

Therefore,

$$\dim_E D_B(V) \leq \dim_E D_B(U) + \dim_E D_B(W) \leq \dim_F(U) + \dim_F(W) = \dim_F(V).$$

But then because V is B-admissible, these are all equalities, hence U and W are B-admissible. □

Remark. $\mathrm{Rep}_F^B(G)$ need not be closed under extensions. For example, if $B = B_{\mathrm{HT}}$, then [4, Example 1.1.12] exhibits a 2-dimensional representation V which is not Hodge-Tate but fits into an exact sequence,

$$0 \to \mathbb{Q}_p \to V \to \mathbb{Q}_p \to 0.$$

Explicitly, we have $\log_p : \mathbb{Z}_p^\times \to \mathbb{Q}_p$, defined by the usual power series on $1 + p\mathbb{Z}_p$, and on $\mathbb{Z}_p^\times = \mu_{p-1} \times (1 + p\mathbb{Z}_p)$ by $\log_p(\zeta(1 + x)) = \log_p(1 + x)$. Then the action of $g \in G_K$ on V is by,

$$\begin{pmatrix} 1 & \log_p(\chi(g)) \\ 0 & 1 \end{pmatrix}$$

However, one can show that for any p-adic representation W which fits into an exact sequence,

$$0 \to \mathbb{Q}_p(m) \to W \to \mathbb{Q}_p(n) \to 0,$$

is Hodge-Tate whenever $m \neq n$. In the above, $\mathbb{Q}_p = \mathbb{Q}_p(0)$.

4. Image Categories

Recall that we have $B_{\text{cris}} \subset B_{\text{st}} \subset B_{\text{dR}}$, and B_{HT}. For each of these period rings B, $E = B^{G_K}$ is K_0, K_0, K and K respectively.

$D_{B_{\text{dR}}}$ can be modified using the filtration on B_{dR} to give an exact faithful functor to Fil_K, the category of filtered K-vector spaces. However, this functor is not fully faithful.

$D_{B_{\text{cris}}}$, naturally takes values in MF_K^{ϕ}, the category of filtered ϕ-modules over K (see Definition 7.3.4 [1]).

D_{st} takes values in $\text{MF}_K^{\phi,N}$, the category of filtered (ϕ, N)-modules. There is a notion of a *weakly admissible* object of $\text{MF}_K^{\phi,N}$, the full subcategory these define is denoted $\text{MF}_K^{\phi,N,w.a.}$. One can show that any semistable representation is weakly admissible. It is a deep and recent result of Fontaine and Colmez [2] that

$$D_{\text{st}} : \text{Rep}_{\mathbb{Q}_p}^{\text{st}}(G_K) \to \text{MF}_K^{\phi,N,w.a.},$$

is an equivalence of categories. One can then pass to objects with *vanishing monodromy* ($N = 0$), to restrict this to an equivalence of categories,

$$D_{\text{cris}} : \text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(G_K) \to \text{MF}_K^{\phi,w.a.}.$$

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