These are notes from a talk at the Oxford $p$-adic Hodge Theory study group in February 2022. They aim to cover the content of [1, I.5] and [4, III.1].

1. $(F,G)$-Regular Rings

**Notation:** $K$ is a $p$-adic field, and $G_K := \text{Gal}(\overline{K}/K)$. Write $C_K$ for the completion of $\overline{K}$, and $\mathbb{F}$ for the residue field of $K$.

Let $F$ be a field and $G$ a group. Suppose that $B$ is a commutative $F$-algebra such that

- $B$ has an action of $G$ by $F$-algebra automorphisms,
- $E := B^G$ is a field,
- $B$ is a domain.

Aim: construct a functor from finite dimensional $F$-representations of $G$ to finite dimensional $E$-vector spaces. Depending on $B$, these $E$-vector spaces will typically naturally have extra structure.

Let $C$ be the fraction field of $B$. There is a unique extension of the action of $G$ on $B$ to an action of $G$ on $C$ by $F$-algebra automorphisms.

**Definition 1.1.** $B$ as above is called $(F,G)$-regular, if

- $B^G = C^G$,
- If $b \in B \setminus 0$ is such that $F \cdot b$ is $G$-stable, then $b \in B^\times$.

From this we see the necessity of the requirement that $B^G$ is a field.

**Example 1.2.** If $B$ is a field, then $B$ is trivially $(F,G)$-regular.

We are mostly interested in $(\mathbb{Q}_p, G_K)$-regular rings.

**Example 1.3.** Any field extension of $\mathbb{Q}_p$ with an action of $G_K$ is $(\mathbb{Q}_p, G_K)$-regular. In particular, both $\overline{K}$ and $C_K$ are $(\mathbb{Q}_p, G_K)$-regular.

Write $\chi : G_K \to \mathbb{Z}_p^\times \subset \mathbb{Q}_p^*$ for the $p$-adic cyclotomic character.

**Definition 1.4.** If $M$ is a $\mathbb{Z}_p[G_K]$-module, then for $n \in \mathbb{Z}$, define $M(n)$ to be the $\mathbb{Z}_p[G_K]$-module, which is $M$ as a $\mathbb{Z}_p$-module, but with “twisted” action, $g \cdot m := \chi(g)^n g(m)$.

More generally, if $\eta : G_K \to \mathbb{Z}_p^\times$ is a continuous character, set $M(\eta)$ to be $M$ with action $g \cdot m = \eta(g)^n g(m)$.

Therefore, we can talk about the action of $G_K$ on $C_K(n)$ for $n \in \mathbb{Z}$.

**Theorem 1.5** (Sen - Tate Theorem). Let $n \in \mathbb{Z}$. Then the $G_K$-invariants,

$$C_K(n)^{G_K} = \begin{cases} K & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}.$$

Put another way, this says that

- There are no transcendental invariants of $C_K$: $C_K^{G_K} = \overline{K}^{G_K} = K$,
- If $n \neq 0$, and $x \in C_K$ with $x = \chi(g)^n g(x)$ for all $g \in G_K$, then $x = 0$.

Actually we will need a more general result, which sometimes goes by the same name.
Theorem 1.6. Suppose that \( \eta : G_K \to \mathbb{Z}_p^\times \) is a continuous character. Then the \( G_K \)-invariants,

\[
C_K(\eta)^{G_K} = \begin{cases} 
K & \text{if } \eta(I_K) \text{ is finite}, \\
0 & \text{if } \eta(I_K) \text{ is infinite}.
\end{cases}
\]

Example 1.7. \( B_{\text{HT}} \) can be non-canonically identified with \( C_K[T, 1/T] \). The action under this identification of \( G_K \) on \( C_K[T, 1/T] \) is given by

\[
g \cdot \left( \sum_{i \in \mathbb{Z}} a_i T^i \right) = \sum_{i \in \mathbb{Z}} g(a_i)\chi(g)^i T^i,
\]

where \( \chi \) is the cyclotomic character. Then we claim this is \((\mathbb{Q}_p, G_K)\)-regular. We first need to compute the invariant subring of \( C_K[T, 1/T] \) and of its fraction field \( C_K(T) \). To do so, we consider both inside the formal Laurent power series ring \( C_K((T)) \), the fraction field of \( C_K[[T]] \). The inclusion \( C_K(T) \hookrightarrow C_K((T)) \) is \( G_K \)-equivariant when \( C_K((T)) \) is given the action \( g \cdot \left( \sum_{i \geq i_0} a_i T^i \right) = \sum_{i \geq i_0} \chi(g)^i g(a_i) T^i \).

A formal Laurent series as above is \( G_K \)-invariant if and only if each \( a_i \in C_K(i)^{G_K} \), and so by the Theorem above of Sen-Tate, \( C_K((T))^{G_K} = K \).

Now to prove the second property, let \( b \in C_K[T, 1/T] \setminus 0 \), and suppose that \( \mathbb{Q}_p \cdot b \) is \( G_K \)-stable. Then the action of \( G_K \) on \( b \) defines a group homomorphism \( \eta : G_K \to \mathbb{Q}_p^\times \). For any \( i \in \mathbb{Z} \) and \( g \in G_K \), we have

\[
\eta(g) a_i = \chi(g)^i g(a_i).
\]

Some \( a_{i_0} \neq 0 \), and \( \eta(g) = \chi(g)^{i_0}(g(a_{i_0})/a_{i_0}) \). Then because \( \chi^{i_0} \) and the action of \( g \) on \( C_K \) are continuous, \( \eta \) is continuous. Because \( \eta \) is continuous, \( \eta(G_K) \subset \mathbb{Z}_p^\times \) - this is because the composition \( \| \cdot \| \circ \eta : G_K \to \mathbb{R} \) is continuous, and so \( \| \eta(G_K) \| \subset \mathbb{R} \) is compact.

If \( a_i \neq 0 \), then \( a_i = (\chi^i \eta^{-1})(g)g(a_i) \) for all \( g \in G_K \), so \( a_i \in C_K(\chi^i \eta^{-1})^{G_K} \). Therefore, by the stronger version of the Sen-Tate Theorem above, \( (\chi^i \eta)(I_K) \) is finite. Therefore, if \( a_i \neq 0 \neq a_j \), then \( (\chi^i \eta^{-1} \chi^{-j} \eta)(I_K) = \chi^{i-j}(I_K) \) is finite, so \( i = j \). Thus \( b = a_i T^i \) and is a unit. So \( C_K[T, 1/T] \) is \((\mathbb{Q}_p, G_K)\)-regular.

Example 1.8. The ring \( B_{\text{cris}}^+ \) is a complete DVR with uniformiser \( t \in B_{\text{cris}}^+ \), and \( G_K \) acts on \( t \) via the \( p \)-adic cyclotomic character. \( B_{\text{cris}}^+ \) is not \((\mathbb{Q}_p, G_K)\)-regular, because \( \mathbb{Q}_p \cdot t \) is a \( G_K \)-stable subspace, but \( t \) is not a unit of \( B_{\text{cris}}^+ \). However, the fraction field is \( B_{\text{cris}} = B_{\text{cris}}^+[1/t] \) and being a field is \((\mathbb{Q}_p, G_K)\)-regular. There is a natural filtration on \( B_{\text{cris}} \), with associated graded \( B_{\text{HT}} \), and using this one can see that \( B_{\text{HT}}^{G_K} = K \) (as a consequence, \( B_{\text{HT}}^{G_K} = K \) too).

Example 1.9. We also will construct subrings \( B_{\text{crist}} \subset B_{\text{st}} \) of \( B_{\text{cris}} \). These are both not fields, but will turn out to be \((\mathbb{Q}_p, G_K)\)-regular, with invariant ring \( K_0 = W(\mathbb{F})[1/p] \).

Example 1.10. How does this relate to the rings in Talk 3? Let \( E \) be a field of characteristic \( p > 0 \), and let \( E^s \) be a separable closure of \( E \). Let \( G_E = \text{Gal}(E^s/E) \). Then \( B = E^s \) is \((\mathbb{F}_p, G_E)\)-admissible. The ring \( B = \widehat{O}_{E^s} \) does not naturally fit into this framework, because even though it has an action of \( G_E \), \( B^{G_E} = \mathcal{O}_E \) is not a field. However, for \( E \) the fraction field of \( \mathcal{O}_E \), \( B = \widehat{O}_{E^s} \) is \((\mathbb{Q}_p, G_E)\)-regular, with \( B^G = E \).

2. \textit{B-Admissible Representations}

From now on, we will assume that \( B \) is \((F, G)\)-regular.

Definition 2.1. We define a functor from the category of finite dimensional \( F \)-representations of \( G \) to \( E \)-vector spaces,

\[
D_B : \text{Rep}_F(G) \to \text{Vect}_E,
\]

by

\[
D_B(V) := (V \otimes_F B)^G.
\]
Here $G$ acts on $V \otimes_F B$ by $g(b \otimes v) = gb \otimes gv$, and $(V \otimes_F B)^G$ is a $E = B^G$-vector space induced from the $B$-module structure of $V \otimes_B B$ where $b'(v \otimes b) = v \otimes b'b$. We also have a natural map,

$$\alpha_V : D_B(V) \otimes_E B \to (V \otimes_F B) \otimes_E B = V \otimes_F (B \otimes_E B) \to V \otimes_F B.$$  
With $G$ acting on $D_B(V) \otimes_E B$ through the second factor $B$, then this is a $G$-equivariant $B$-linear map.

**Example 2.2.** Suppose that $V = F$ is the trivial representation. Then $D_B(F) = B^G$, and $\alpha_V$ is the identity map.

**Example 2.3.** Suppose that $B = F$. Then $D_B(V) = (V \otimes_F F)^G = V^G$ is the functor from $\text{Rep}_F(G)$ to $\text{Vect}_F$ taking $V$ to its $G$-invariants.

At this point, it is not really obvious why $D_B(V)$ is finite dimensional.

**Lemma 2.4.** If $V \in \text{Rep}_F(G)$, then $\alpha_V : D_B(V) \otimes_E B \to V \otimes_F B$ is (B-linear, G-equivariant, and) injective.

**Proof.** If $C$ is the fraction field of $B$, we have a commutative square:

$$
\begin{array}{c}
D_B(V) \otimes_E B \\
\downarrow
\end{array}
\xrightarrow{\alpha_V}

\begin{array}{c}
V \otimes_F B \\
\downarrow
\end{array}
\quad
\begin{array}{c}
D_C(V) \otimes_E C \\
\downarrow
\end{array}
\xrightarrow{\beta_V}

\begin{array}{c}
V \otimes_F C
\end{array}
$$

Therefore it is sufficient to prove that $\beta_V$ is injective. $\beta_V$ is injective if and only if $\beta_V$ maps an $E$-basis to a $C$-linearly independent set of $V \otimes_F C$. Therefore, it is sufficient to show that if for some $\{x_1, ..., x_n\} \subset (V \otimes_F C)^G$ are $E$-linearly independent, then they are $C$-linearly independent.

Suppose not, and take a $C$-linear dependence of minimal length. So for some $r \geq 1$, $x_r = \sum_{i<r} c_i x_i$, and $r$ is the minimal length of such a relation. Then for all $g \in G$,

$$x_r = g(x_r) = \sum_{i<r} g(c_i) x_i.$$  

Equivalently,

$$0 = \sum_{i<r} (g(c_i) - c_i) x_i,$$  

and so by minimality, $g(c_i) = c_i$ for all $i$. But then for all $i$, $c_i \in C^G = E$, a contradiction. \qed

**Theorem 2.5.** For $V \in \text{Rep}_F(G)$, we have the bound,

$$\dim_E D_B(V) \leq \dim_F(V).$$

In particular, $D_B(V)$ is a finite dimensional $E$-vector space. Furthermore, for $d = \dim_F(V)$, the following are equivalent:

1. $\dim_E D_B(V) = \dim_F(V)$,
2. $\alpha_V$ is an isomorphism,
3. There is an isomorphism $V \otimes_F B \cong B^d$ as $B$-modules which is $G$-equivariant,
4. There is a $B$-basis of $V \otimes_F B$ consisting of $d$ elements of $(V \otimes_F B)^G$.

**Proof.** Again let $C$ be the fraction field of $B$. If we tensor $\alpha_V : D_B(V) \otimes_F B \to V \otimes_F B$ with $C$ over $B$, we obtain,

$$\alpha_V \otimes_B C : D_B(V) \otimes_E B \to V \otimes_F B.$$  

The $C$-dimension of the term of the left is $\dim_E(D_B(V))$, and the $C$-dimension of the term of the right is $\dim_F(V)$. Because $\alpha_V$ is injective and $C$ is flat over $B$, we see that in general $\dim_E(D_B(V)) \leq \dim_F(V)$. Furthermore, clearly if $\alpha_V$ is an isomorphism (1), we have the equality (2). Now, to prove that (2) $\Rightarrow$ (1), suppose that $\dim_E(D_B(V)) = \dim_F(V)$. We want to show that $\alpha_V$ is an isomorphism.

Let $\{e_i\}$ be an $E$-basis of $D_B(V)$, and $\{v_j\}$ be an $F$ basis of $V$. By assumption, these have the same size $d$, and we can express the $B$-linear map $\alpha_V$ by the $d \times d$ matrix $(b_{ij})$, so
is an open subgroup of \( G \). By assumption, \( \alpha_V \otimes_B C \) is an isomorphism, so \( \det(b_{ij}) \neq 0 \), when considered an element of \( C \).

We know that \( b := \det(b_{ij}) \in B \setminus 0 \), and want to show that \( b \in B^\times \). To do this we use the fact that \( B \) is \((F,G)\)-regular, so we want to show that \( F \cdot b \) is stable under the action of \( G \).

Passing to the \( d \)th exterior power (as \( B\)-modules),
\[
\wedge^d (\alpha_V) (e_1 \wedge \cdots \wedge e_d) = b(v_1 \wedge \cdots v_d)
\]
Because \( \alpha_V \) is \( G \)-equivariant, and \( e_i \) are \( G \)-invariant, \( G \) acts on the left trivially, and so
\[
g(b) g(v_1 \wedge \cdots v_d) = b(v_1 \wedge \cdots v_d).
\]
But \( g(v_1 \wedge \cdots v_d) = \det(g)(v_1 \wedge \cdots v_d) \), hence
\[
g(b) = \det(g)^{-1} b,
\]
and \( Q_p \cdot b \) is stable under the action of \( G \).

The equivalence of (3) and (4) is immediate, so we just need to show that (3) is equivalent to (1) and (2). If \( \alpha_V \) is an isomorphism (1), then because \( \dim_E(D_B(V)) = d \), by choosing an \( E \) basis of \( D_B(V) \) we have \( B^d \cong D_B(V) \otimes_E B \), and this is \( G \)-equivariant because the action of \( G \) on \( D_B(V) \) is trivial. Therefore, composing this with \( \alpha_V \) we have (3). Conversely, if \( B^d \cong D_B(V) \otimes_E B \), then \( \dim_E(D_B(V)) = d \) and we have (1). \( \square \)

**Definition 2.6.** We call \( V \in \text{Rep}_F(G) \) \( B \)-admissible if \( \dim_E D_B(V) = \dim_F(D_B(V)) \). We write \( \text{Rep}^B_F(G) \) for the full-subcategory of finite dimensional \( B \)-admissible \( F \)-representations of \( G \). If \((F,G) = (Q_p,G_K)\), and \( B \) is one of \( B_{\text{HT}}, B_{\text{dR}}, B_{\text{st}} \) or \( B_{\text{cris}} \), then we call \( B \)-admissible \( p \)-adic representations Hodge-Tate, de Rham, Semistable and Crystalline respectively.

For the next example, we need a couple of definitions. Let \( R \) be a commutative topological ring, and \( \Gamma \) a topological group, such that \( \Gamma \) acts on \( R \) continuously. Then a \( R \)-representation \([3, \text{Def. 2.2}]\) is an \( R \)-module of finite type equipped with a continuous semi-linear action of \( \Gamma \). This is called free of rank \( d \) if the underlying \( R \)-module is. This is called trivial if it has a basis consisting of \( \Gamma \)-invariant elements. For a fixed \( d \geq 1 \), there is a one-to-one correspondence between free \( R \)-representations of rank \( d \) and elements of \( H^1_{\text{cts}}(\Gamma, \text{GL}_d(R)) \). Furthermore, a free \( R \)-representation \( X \) is trivial if and only if it corresponds to the trivial element of \( H^1_{\text{cts}}(\Gamma, \text{GL}_d(R)) \) \([3, \text{Prop. 2.6}]\).

**Example 2.7.** \((F,G) = (Q_p,G_K), B = \overline{K}\). Then we claim that \( V \) is \( \overline{K} \)-admissible if and only if the action of \( G_K \) on \( V \) factors through some finite quotient. This property is called being potentially trivial, and is the same as \( V \) being smooth as a \( G_K \)-representation, or that the action of \( G_K \) is discrete: continuous when \( V \) is given the discrete topology.

To see why, suppose that the action of \( G_K \) on \( V \) is potentially trivial, so there is an open subgroup of \( G_K \) which acts trivially on \( V \). Let \( X = V \otimes_{Q_p} \overline{K} \). The action of \( G_K \) on \( X \) by \( g(v \otimes \lambda) = g(v) \otimes g(\lambda) \) also factors through some open subgroup, because this is true for the action of \( G_K \) on \( \overline{K} \). Let \( d = \dim_{Q_p} V = \dim_{\overline{K}} X \), and fix the discrete topology on \( \overline{K} \). Then \( X \) is a free rank \( d \) \( \overline{K} \)-representation of \( G_K \).

By a strong version of Hilbert’s Theorem 90, we have that for all \( d \geq 1 \),
\[
H^1_{\text{cts}}(G_K, \text{GL}_d(\overline{K})) = 0.
\]
Therefore, by the above discussion, \( X = (V \otimes_{Q_p} \overline{K}) \) has a \( \overline{K} \)-basis of \( G_K \)-invariant elements. But this means that \( V \) is \( \overline{K} \)-admissible by Theorem 2.5.

Showing the converse is easier: suppose that \((V,\rho)\) is \( \overline{K} \)-admissible and choose a basis of \( G_K \)-invariant elements \( \{e_1, \ldots, e_d\} \). We want to show that the stabiliser of any element \( x = \sum_i \lambda_i e_i \) is an open subgroup. For any \( g \in G_K \), \( g(x) = \sum_i g(\lambda_i) e_i \), and so
\[
(gG_K)x = \cap_i (G_K)_{\lambda_i} e_i,
\]
is an open subgroup of \( G_K \). But then for a \( Q_p \)-basis \( \{v_1, \ldots, v_d\} \) of \( V \), letting \( x = v_i \otimes 1 \) in turn, we see that \( \ker(\rho) = \cap_i (G_K)_{v_i \otimes 1} \) is open.
Example 2.8. Let $P := \bar{K}^w$. We have $\text{Gal}(\bar{P}/P) = I_K$. Let $(F,G) = (\mathbb{Q}_p, G_K)$, $B = \bar{P}$. Then a $p$-adic representation $V$ of $G_K$ is $\bar{P}$-admissible if and only if the action of $I_K$ is discrete [3, Prop. 3.53]. This property is called being potentially unramified.

We have that $\bar{K} \subset \bar{P} \subset C_K$. In fact the $C_K$-admissible representations are the same as the $\bar{P}$-admissible representations:

Example 2.9. $(F,G) = (\mathbb{Q}_p, G_K)$, $B = C_K$. Then a $p$-adic representation $V$ is $C_K$-admissible if and only if the action of $I_K$ factors through some finite quotient. Then a $p$-adic representation $V$ of $G_K$ is $\bar{P}$-admissible if and only if the action of $I_K$ is discrete [3, Prop. 3.55].

Concretely, if $(\rho, V)$ is a $p$-adic representation of $G_K$, then

- $V$ is $\bar{K}$-admissible iff $\ker(\rho) \leq G_K$ is open,
- $V$ is $\bar{P}$-admissible iff $V$ is $C_K$-admissible iff $\ker(\rho) \cap I_K \leq I_K$ is open (in $I_K$).

Example 2.10. In talk 5 we shall see an equivalent definition of a Hodge-Tate representation, not in terms of the period ring.

Example 2.11. In talk 3, $B = E^s$ is a $(\mathbb{F}_p, G_E)$-regular ring. Then the $E^s$-admissible $\mathbb{F}_p$-representations of $G_E$ are exactly the continuous representations (where $G_E$ has the Krull topology and $V$ the discrete topology). The reasoning is analogous to that of Example 2.5.

$D_{E^s}$ as it stands is not an equivalence of categories - this functor is essentially surjective and faithful, but not full. In talk 3, this is modified to an equivalence of categories, by mapping to finite-dimensional $K$-vector spaces equipped with an injective (Frobenius) semi-linear $\phi$. We have the Frobenius $E^s \rightarrow E^s$, and for any $V \in \text{Rep}_{\mathbb{F}_p}(V)$ we define $\phi$ on $D_{E^s}(V) = (E^s \otimes_{\mathbb{F}_p} V)^{G_E}$ by restricting $\phi : E^s \otimes_{\mathbb{F}_p} V \rightarrow E^s \otimes_{\mathbb{F}_p} V$,

$$\phi(x \otimes v) = x^p \otimes v,$$

to $D_{E^s}(V) = (E^s \otimes_{\mathbb{F}_p} V)^{G_E}$.

3. Properties of $D_B$

Now we summarise the main properties of the functor $D_B$.

Theorem 3.1. The restriction of the functor $D_B$ to the full subcategory $\text{Rep}_F^B(G)$,

$$D_B : \text{Rep}_F^B(G) \rightarrow \text{Vect}_E,$$

is exact and faithful. $\text{Rep}_F^B(G)$ is closed under sub-representations and quotients. Furthermore,

- If $V_1, V_2 \in \text{Rep}_F^B(G)$, then there is a natural isomorphism,
  $$D_B(V_1) \otimes_E D_B(V_2) \rightarrow D_B(V_1 \otimes_F V_2),$$
  and so $V_1 \otimes V_2 \in \text{Rep}_F^B(G)$.
- If $V \in \text{Rep}_F^B(G)$, then $V^* \in \text{Rep}_F^B(G)$, and the natural map,
  $$D_B(V) \otimes_E D_B(V^*) \xrightarrow{\sim} D_B(V \otimes_F V^*) \rightarrow D_B(F) \cong E,$$
  is a perfect pairing.
- $\text{Rep}_F^B(G)$ also is closed under symmetric and exterior powers, and $D_B$ commutes with these constructions.

Proof. We prove that $D_B$ is exact, fully faithful and closed under subquotients. The rest can be found in [1, Part I, Section 5].

Both faithfulness and exactness come down to the fact that $B$ is an algebra over both $F$ and $E = B^G$, and therefore is faithfully flat over $F$ and $E$. Recall that for a ring $R$ and module $M$, $M$ is a faithfully flat $R$-module if and only if the following equivalent conditions hold:

- $M$ is flat and for any $R$-linear $f : N_1 \rightarrow N_2$, then $f$ is non-zero if and only if $f \otimes 1 : M_1 \otimes_R M \rightarrow M_2 \otimes_R M$ is non-zero.
- For any sequence $N_1 \rightarrow N_2 \rightarrow N_3$, this is exact at $N_2$ if and only if $N_1 \otimes_R M \rightarrow N_2 \otimes_R M \rightarrow N_3 \otimes_R M$ is exact at $N_2 \otimes_R M$. 
**Faithful:** Suppose that \( f : V \to W \) is a morphism \( F[G] \)-modules. Then because \( B \) is faithfully flat over \( E \), \( \text{DB}(f) = f \otimes 1 : \text{DB}(V) \to \text{DB}(W) \) is zero if and only if

\[
(f \otimes 1) \otimes 1 : \text{DB}(V) \otimes_E B \to \text{DB}(W) \otimes_E B,
\]
is zero. But because \( V \) and \( W \) are \( B \)-admissible, \( \alpha_V, \alpha_W \) are isomorphisms in the commutative diagram,

\[
\begin{array}{ccc}
\text{DB}(V) \otimes_E B & \xrightarrow{(f \otimes 1) \otimes 1} & \text{DB}(W) \otimes_E B \\
\downarrow \alpha_V & & \downarrow \alpha_W \\
V \otimes_F B & \xrightarrow{f \otimes 1} & W \otimes_F B
\end{array}
\]

and because \( B \) is faithfully flat over \( E \), \( f \otimes 1 \) is zero if and only if \( f : V \to W \) is zero.

**Exact:** Let

\[
0 \to U \to V \to W \to 0,
\]
be a short exact sequence in \( \text{Rep}_F^B(G) \). Then

\[
0 \to U \otimes_F B \to V \otimes_F B \to W \otimes_F B \to 0,
\]
is an exact sequence of \( B \)-modules, and so because \( U, V \) and \( W \) are \( B \)-admissible,

\[
0 \to \text{DB}(U) \otimes_E B \to \text{DB}(V) \otimes_E B \to \text{DB}(W) \otimes_E B \to 0,
\]
is exact. But then \( B \) is faithfully flat over \( E \), hence

\[
0 \to \text{DB}(U) \to \text{DB}(V) \to \text{DB}(W) \to 0,
\]
is exact.

**Closed under subquotients:** Consider a short exact sequence of \( F[G] \)-modules,

\[
0 \to U \to V \to W \to 0,
\]
where \( V \) is \( B \)-admissible. By definition \( \text{DB} \) is left exact, so we also have the exact sequence of \( E \)-modules,

\[
0 \to \text{DB}(U) \to \text{DB}(V) \to \text{DB}(W)
\]

Therefore,

\[
\dim_E \text{DB}(V) \leq \dim_E \text{DB}(U) + \dim_E \text{DB}(W) \leq \dim_F(U) + \dim_F(W) = \dim_F(V).
\]

But then because \( V \) is \( B \)-admissible, these are all equalities, hence \( U \) and \( W \) are \( B \)-admissible.

**Remark.** \( \text{Rep}_F^B(G) \) need not be closed under extensions. For example, if \( B = B_{\text{HT}} \), then [4, Example 1.1.12] exhibits a 2-dimensional representation \( V \) which is not Hodge-Tate but fits into an exact sequence,

\[
0 \to \mathbb{Q}_p \to V \to \mathbb{Q}_p \to 0.
\]

Explicitly, we have \( \log_p : \mathbb{Z}_p^\times \to \mathbb{Q}_p \), defined by the usual power series on \( 1 + p\mathbb{Z}_p \), and on \( \mathbb{Z}_p^\times = \mu_{p-1} \times (1 + p\mathbb{Z}_p) \) by \( \log_p((1 + x)) = \log_p(1 + x) \). Then the action of \( g \in G_K \) on \( V \) is by,

\[
\begin{pmatrix}
1 & \log_p(\chi(g)) \\
0 & 1
\end{pmatrix}
\]

However, one can show that for any \( p \)-adic representation \( W \) which fits into an exact sequence,

\[
0 \to \mathbb{Q}_p(m) \to W \to \mathbb{Q}_p(n) \to 0,
\]
is Hodge-Tate whenever \( m \neq n \). In the above, \( \mathbb{Q}_p = \mathbb{Q}_p(0) \).
4. Image Categories

Recall that we have $B_{\text{cris}} \subset B_{\text{st}} \subset B_{\text{dR}}$, and $B_{\text{HT}}$. For each of these period rings $B$, $E = B^{G_K}$ is $K_0$, $K_0$, $K$ and $K$ respectively.

$D_{B_{\text{dR}}}$ can be modified using the filtration on $B_{\text{dR}}$ to give an exact faithful functor to $\text{Fil}_K$, the category of filtered $K$-vector spaces. However, this functor is not fully faithful.

$D_{B_{\text{cris}}}$, naturally takes values in $\text{MF}_K^\phi$, the category of filtered $\phi$-modules over $K$ (see Definition 7.3.4 [1]).

$D_{\text{st}}$ takes values in $\text{MF}_K^{\phi,N}$, the category of filtered $(\phi,N)$-modules. There is a notion of a weakly admissible object of $\text{MF}_K^{\phi,N}$, the full subcategory these define is denoted $\text{MF}_K^{\phi,N,w.a}$. One can show that any semistable representation is weakly admissible. It is a deep and recent result of Fontaine and Colmez [2] that

$$D_{\text{st}} : \text{Rep}_{\text{st}}^p(G_K) \to \text{MF}_K^{\phi,N,w.a},$$

is an equivalence of categories. One can then pass to objects with vanishing monodromy ($N = 0$), to restrict this to an equivalence of categories,

$$D_{\text{cris}} : \text{Rep}_{\text{cris}}^p(G_K) \to \text{MF}_K^{\phi,w.a}.$$ 

References