RAPOPORT ZINK SPACES: THE LUBIN-TATE CASE

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Notation. Let K be a finite extension of $\mathbb{Q}_p$, with integer ring $\mathcal{O}$, a uniformiser $\pi \in \mathcal{O}$ and residue field $\mathcal{O} \twoheadrightarrow k$. For $h \geq 1$, let K_h be the unique unramified extension of K of degree h, and $\mathcal{O}_h$ its ring of integers. Let $\breve{K}$ be the maximal unramified extension of K, and $\breve{\mathcal{O}}$ its ring of integers. For A a commutative ring, write $\mathbf{Alg}_A$ for the category of commutative unital associative A-algebras.

Let $\mathbf{Nilp}_{\mathcal{O}}$ be the category of $\mathcal{O}$ algebras for which π is nilpotent, $\mathcal{C}_{\mathcal{O}}$ the full subcategory of artinian local $\mathcal{O}$-algebras with residue field k.

1. Background On Formal Group Laws

Definition 1.1. Let R be a commutative ring. A formal group law[1] over R is a power series $F(X,Y) \in R[[X,Y]]$ such that,

(1) The linear term of $F(X,Y)$ is $F(X,Y) \equiv_2 X + Y$,
(2) $F(F(X,Y),Z) = F(X, F(Y,Z))$,
(3) $F(X,Y) = F(Y,Z)$.

Definition 1.2. Let R be a commutative ring, and $F(X,Y), G(X,Y)$ formal group laws over R. Then an homomorphism of $h : F(X,Y) \to G(X,Y)$ is a power series $h(X) \in XR[[X]]$ such that,

$$h(F(X,Y)) = F(h(X), h(Y)).$$

Morphisms can be composed by composing the associated power series. Using the formal group law $G(X,Y)$, the set $\mathrm{Hom}_R(F,G)$ of homomorphisms of formal group laws over R from $F(X,Y)$ to $G(X,Y)$ becomes an abelian group:

$$h_1(X) + h_2(X) := G(h_1(X), h_2(X)),$$

and $\mathrm{End}_R(F(X,Y))$ becomes a ring.

Definition 1.3. Let A be a commutative ring, and $R \in \mathbf{Alg}_A$. A formal A-module law over R is a formal group law $F(X,Y)$ over R, together with a ring homomorphism $[-] : A \to \mathrm{End}_R(F)$, such that for any $a \in A$, the linear term is,

$$[a](X) \equiv_2 aX.$$

For two formal A-modules laws $F(X,Y), G(X,Y)$ over R, we write $\mathrm{Hom}_{R/A}(F,G)$ for the set of homomorphisms of formal A-module laws: homomorphisms of formal group laws over R which respect the action of A. As before, using $A \to \mathrm{End}_R(G(X,Y))$ this set is canonically an A-module.

Example 1.4. If $R \in \mathbf{Alg}_S$, where S is one of the rings $\mathbf{Z}, \mathbf{Z}_{(p)}, \mathbf{Z}_p$, then formal S-module laws over R are the same as formal group laws over R: for any formal group law $F(X,Y)$ over R there exists a unique formal S-module law structure on $F(X,Y)$.

[1] we only consider commutative 1-dimensional formal group laws

1.1. **Universal Formal $\mathcal{O}$-Modules.** Consider the functor,

$$\mathbf{FGL}_{\mathcal{O}} : \mathbf{Alg}_{\mathcal{O}} \to \mathbf{Set},$$

sending $R \in \mathbf{Alg}_{\mathcal{O}}$ to the set of isomorphism classes of formal $\mathcal{O}$-module laws over R. It is not hard to show this is representable. What is more interesting, is the following presentation of the universal algebra as a polynomial ring $\Lambda_{\mathcal{O}} \in \mathbf{Alg}_{\mathcal{O}}$, and of the universal formal $\mathcal{O}$-module law over $\Lambda_{\mathcal{O}}$.

Theorem 1.5. *Let,*

$$\Lambda_{\mathcal{O}} := \mathcal{O}[t_2, t_3, ...] \in \boldsymbol{Alg}_{\mathcal{O}},$$

and let $\sigma : \Lambda_{\mathcal{O}} \to \Lambda_{\mathcal{O}}$ be the $\mathcal{O}$-algebra endomorphism defined by $\sigma(t_i) = t_i^q$ for any $i \geq 2$. For each $i \geq 1$, write $v_i := t_{q^i}$. Let $f^u(X) \in \Lambda_{\mathcal{O}}[1/\pi][[X]]$ defined by the functional equation,

$$f^u(X) = X + \sum_{i=1}^{\infty} t_i X^i - \sum_{i=1}^{\infty} v_i X^{q^i} + \sum_{i=1} \frac{v_i}{\pi^i} \sigma_* f(X^{q^i}),$$

and let $F^u(X,Y) := f^{-1}(f(X) + f(Y)) \in \Lambda_{\mathcal{O}}[[X,Y]]$ be the unique formal $\mathcal{O}$-module law with logarithm $f^u(X)$. Then the pair $(\Lambda_{\mathcal{O}}, F^u(X,Y))$ is a universal $\mathcal{O}$-module law.

1.2. **Height.** Let L be a field extension of k.

Proposition 1.6. *Let $f : F_1(X,Y) \to F_2(X,Y)$ be a morphism of formal $\mathcal{O}$-module laws over L. Then if $f(X) \neq 0$, there is a unique $m \in \mathbb{N}_{\geq 1}$ and $g(X) \in L[[X]]$ such that,*

$$f(X) = g(X^q),$$

and $g'(0) \neq 0$. We set $\mathrm{ht}(f) := m$ if $f(X) \neq 0$, and $\mathrm{ht}(0) := \infty$.

Definition 1.7. Let $F(X,Y)$ be a formal $\mathcal{O}$-module over L. Then we define *the height of* $F(X,Y)$ to be,

$$\mathrm{ht}(F(X,Y)) := \mathrm{ht}([\pi]) \in \mathbb{N}_{\geq 1} \cup \{\infty\}.$$

Theorem 1.8. *Suppose that L algebraically closed. Then taking the height defines an bijection from the set of formal $\mathcal{O}$-modules over L to the set $\mathbb{N}_{\geq 1} \cup \{\infty\}$.*

Remark. The representatives of each isomorphism class are defined over k. The additive formal group $\mathbb{G}_a(X,Y)$ has height ∞. For each $h \geq 1$, the formal $\mathcal{O}$-module of height h, $F_h(X,Y)$ has endomorphisms,

$$\mathrm{End}_{k/\mathcal{O}}(F_h(X,Y)) = \mathcal{O}[\Pi] \hookrightarrow \mathcal{O}_D = \mathrm{End}_{k_h/\mathcal{O}}(F_{h,k_h}(X,Y)) = \mathrm{End}_{\overline{k}/\mathcal{O}}(F_{h,\overline{k}}(X,Y)),$$

2. The Moduli Functor

Let $h \geq 1$. We fix a formal $\mathcal{O}$-module X_h of height h over k.
We are interested the functor,

$$\breve{\mathcal{M}}_{\mathrm{RZ},h} : \mathbf{Nilp}_{\breve{\mathcal{O}}} \to \mathbf{Set},$$

is defined by sending $R \in \mathbf{Nilp}_{\breve{\mathcal{O}}}$ to the set of isomorphism classes of pairs (X, ρ), where,

- X is a π-divisible formal $\mathcal{O}$-module over R,
- $\rho : F_{R/\pi R} \dashrightarrow \mathbf{X}_{R/\pi R}$, is an $\mathcal{O}$-linear quasi-isogeny.

In fact, this makes sense over $\mathcal{O}$, and by general Rapoport-Zink theory the height will break this up a disjoint union formal schemes indexed by the height, so we are really interested in showing representability of the functor,

$$\mathcal{M}_{\mathrm{RZ},h,0} : \mathbf{Nilp}_{\mathcal{O}} \to \mathbf{Set},$$

is defined by sending $R \in \mathbf{Nilp}_{\mathcal{O}}$ to the set of isomorphism classes of pairs (X, ρ), where,

- X is a π-divisible formal $\mathcal{O}$-module over R,
- $\rho : F_{R/\pi R} \dashrightarrow \mathbf{X}_{R/\pi R}$, is an $\mathcal{O}$-linear quasi-isogeny of height 0.

Now classically, in the papers by Lubin-Tate and Gross-Hopkins, they consider the following functor on $\mathcal{C}_{\mathcal{O}}$. Let,

$$\mathrm{LT}_h : \mathcal{C}_{\mathcal{O}} \to \mathbf{Set},$$

send an artinian local $\mathcal{O}$-algebra R with residue field k, to the set of formal $\mathcal{O}$-module laws $F(X,Y)$ over R which reduces to $F_h(X,Y)$ modulo $\mathfrak{m}_R$ up to equivalence. Here two lifts $F_1(X,Y), F_2(X,Y)$ are called equivalent if there is an isomorphism $f : F_1(X,Y) \to F_2(X,Y)$ such that $f(X) \equiv X$ modulo $\mathfrak{m}_R$.

In order to relate these two functors, we introduce an intermediate functor,

$$\mathcal{M}_h : \mathbf{Nilp}_{\mathcal{O}} \to \mathbf{Set}\,.$$

Definition 2.1. Let $R \in \mathbf{Nilp}_{\mathcal{O}}$, and $h : F(X,Y) \to G(X,Y)$ a morphism of formal $\mathcal{O}$-module laws over R. Then h is said to be a $\star$-isomorphism if there is a nilpotent ideal I of R such that,

$$f(X) \equiv_I X.$$

Remark. Note that if h is a $\star$-isomorphism, then the coefficient of X is in $1 + I \subset R$, hence in $R^\times$ as I is nilpotent. Therefore, the morphism h is invertible.

Definition 2.2. Fix $F_h(X,Y)$, a 1-dimensional formal group law over k of finite height $h \in \mathbb{N}_{\geq 1}$. Let,

$$\mathcal{M}_h : \mathbf{Nilp}_{\mathcal{O}} \to \mathbf{Set},$$

be the functor assigning to $R \in \mathbf{Nilp}_{\mathcal{O}}$, the set of $\star$-isomorphism classes of formal $\mathcal{O}$-modules $F(X,Y)$ over R such that,

$$F_{R/IR}(X,Y) = F_{h,R/IR}(X,Y),$$

over R/IR for some nilpotent ideal $I \subset R$ with $\pi \in I$.

The relationship between $\mathcal{M}_h$ and $\mathcal{M}_{\mathrm{RZ},h,0}$ is the following.

Proposition 2.3. *There is a natural isomorphism of functors* $\mathbf{Nilp}_{\mathcal{O}} \to$ ***Set****,*

$$\mathcal{M}_h \xrightarrow{\sim} \mathcal{M}_{RZ,h,0}.$$

Proof Sketch. Send a formal $\mathcal{O}$-module $F(X,Y)$ in $\mathcal{M}_h(R)$ to the pair $(F(X,Y),\alpha)$, where $\alpha : F_{R/\pi R} \dashrightarrow F_{h,R/\pi R}$ is the unique quasi-isogeny lifting the identity $F_{R/IR} \to F_{h,R/IR}$. The quasi-isogeny α will have height 0 as the identity has height 0. □

On the other hand, the link between $\mathcal{M}_h$ and LT_h is the following.

Proposition 2.4. *The restriction of* $\mathcal{M}_h$ *to the full subcategory* $\mathcal{C}_{\mathcal{O}}$ *is naturally isomorphic to* LT_h.

Now in order to show the representability of our original functor $\breve{\mathcal{M}}_{\mathrm{RZ},h}$, we want to show the representability of $\mathcal{M}_h$.

Definition 2.5. For $S \in \mathbf{Compl}_{\mathcal{O}}$, we let $\mathrm{Spf}(S) : \mathbf{Nilp}_{\mathcal{O}} \to \mathbf{Set}$, be the functor sending $R \in \mathbf{Nilp}_{\mathcal{O}}$ to,

$$\mathrm{Spf}(S)(R) := \mathrm{Hom}_{\mathcal{O},\mathrm{cts}}(S,R),$$

where R is viewed with the discrete topology.

Example 2.6. For example, if $S = \mathcal{O}[[X_1,...,X_n]]$, then for $R \in \mathbf{Nilp}_{\mathcal{O}}$,

$$\mathrm{Spf}(S)(R) \xrightarrow{\sim} \mathrm{Nil}(R)^n,$$

sending $f : S \to R$ to the tuple $(f(x_1),...,f(x_n))$.

Let $\overline{g}_h : \Lambda_{\mathcal{O}} \to k$ be the universal morphism corresponding to $F_h(X,Y)$ over k defining $\mathcal{M}_h$. Note that this has $\overline{g}_h(v_i) = 0$ for all $i = 1,...,h-1$ because $F_h(X,Y)$ has height h. Let $r : \mathcal{O}[[X_1,...,X_{h-1}]] \to k$ be the map reducing modulo the maximal ideal $(\pi, X_1,...,X_{h-1})$.

Theorem 2.7. *Choose an $\mathcal{O}$-algebra homomorphism $g_h : \Lambda_{\mathcal{O}} \to \mathcal{O}[[X_1, ..., X_{h-1}]]$, such that,*

$$\begin{array}{ccc} \Lambda_{\mathcal{O}} & \xrightarrow{\quad g_h \quad} & \mathcal{O}[[X_1, ..., X_{h-1}]] \\ & {\scriptstyle \bar{g}_h} \searrow \quad \swarrow & \\ & k & \end{array}$$

commutes, with the extra condition that $g_h(v_i) = X_i$ for $i = 1, ..., h-1$, which is possible as $\overline{g}_h(v_i) = 0$ for all $i = 1, ..., h-1$. Let $F^{univ}(X, Y) := g_{h,*}F^u(X, Y)$. Then the pair $(\mathcal{O}[[X_1, ..., X_{h-1}]], F^{univ}(X, Y))$ defines a representation of $\mathcal{M}_h$: the map defined by push-forward of the universal formal group law $F^{univ}(X, Y)$,

$$\mathrm{Spf}(\mathcal{O}[[X_1, ..., X_{h-1}]]) \xrightarrow{\sim} \mathcal{M}_h.$$

is a natural isomorphism.

Remark. In particular, we note that this isomorphism is not canonical.

Remark. In order to even state the theorem, we have to use that $\Lambda_{\mathcal{O}}$ is a polynomial ring.

The proof of this theorem goes via the following steps.

Definition 2.8. A functor

$$G : \mathbf{Nilp}_{\mathcal{O}} \to \mathbf{Set},$$

is said to *satisfy the formal Mayer-Vietoris property* if for any morphisms $R_1 \to S$, $R_2 \to S$ in $\mathbf{Nilp}_{\mathcal{O}}$, such that $R_1 \to S$ surjective with nilpotent kernel, the natural morphism,

$$G(R_1 \times_S R_2) \to G(R_1) \times_{G(S)} G(R_2),$$

is a bijection. The functor G is called *formally smooth* if for any surjection $R \to S$ in $\mathbf{Nilp}_{\mathcal{O}}$ with nilpotent kernel, the map,

$$G(R) \to G(R/I),$$

is surjective.

It is easy to see that $\mathrm{Spf}(\mathcal{O}[[X_1, ..., X_{h-1}]])$ is formally smooth and satisfies the formal Mayer-Vietoris property.

Lemma 2.9. *$\mathcal{M}_h$ is formally smooth and satisfies the formal Mayer-Vietoris property.*

Remark. This would not be true, if in the definition of $\mathcal{M}_h$ we only considered things modulo π and didn't allow for general nilpotent ideals I.

Using this, we can reduce the proof of Theorem 2.7, to proving that the restriction to LT_h is an isomorphism.

Lemma 2.10. *If $\mathrm{Spf}(\mathcal{O}[[X_1, ..., X_{h-1}]])(R) \xrightarrow{\sim} \mathcal{M}_h(R)$ from the statement of Theorem 2.7 is an isomorphism for any $R \in \mathcal{C}_{\mathcal{O}}$, then it is for any $R \in \mathbf{Nilp}_{\mathcal{O}}$.*

Remark. This is generally true for a morphism of functors which are both satisfy the formal Mayer-Vietoris property and are formally smooth.

Then verifying this statement reduces to showing the map is a bijection on certain modules of the form "$k[M]$", for M a k-module, for which one actually uses properties of formal $\mathcal{O}$-modules.

3. The Period Morphism

Here we give an overview of the Gross Hopkins period morphism, and why it is surjective.

In order for the equations to be as explicit as possible, we choose as universal formal $\mathcal{O}$-module $F^{\mathrm{univ}}(X, Y)$ over $\mathcal{O}[[X_1, ..., X_{h-1}]]$ the "canonical lift", where in Theorem 2.7 we take $\mathcal{O}$-linear homomorphism,

$$\Lambda_{\mathcal{O}} \to \mathcal{O}[[X_1, ..., X_{h-1}]],$$

sending $v_i \mapsto X_i$ for $i = 1, ..., h-1$, $v_h \mapsto 1$, and all other variables sent to zero.

Let $\mathcal{M}_{\mathrm{RZ},h} : \mathbf{Nilp}_{\mathcal{O}} \to \mathbf{Set}$ be the Rapoport-Zink functor, which splits up,

$$\mathcal{M}_{\mathrm{RZ},h} = \bigsqcup_{m \in \mathbf{Z}} \mathcal{M}_{\mathrm{RZ},h,m}.$$

We want to construct a morphism,

$$\pi_{\mathrm{GH}} : \mathcal{M}^{\mathrm{ad}}_{\mathrm{RZ},h,\eta} \to \mathbb{P}^{h-1,\mathrm{ad}}_{K}.$$

Once we base change to K_h, the group $D^\times$ acts on $\mathcal{M}^{\mathrm{ad}}_{\mathrm{RZ},h}$, and also on $\mathbb{P}^{h-1,\mathrm{ad}}_{K_h}$ via fractional linear transformations.

Explicitly, the action of $D^\times$ is through the natural action of $\mathrm{GL}_{h-1}(K_h)$, and an embedding $D \hookrightarrow M_{h-1}(K_h)$, such that Π maps to an invertible matrix which induces the automorphism which looks like,

$$\Pi \cdot [c_0 : c_1 : \cdots : c_{h-1}] = [\pi^{-1} c_1 : c_2 : \cdots : c_0].$$

However, even if we stay over K, and don't base change to K_h, we still have this equivariant action of the group $\Pi^{\mathbb{Z}}$. Consider the open subset,

$$\mathbb{D} := \mathrm{Spa}\left(K \left\langle \frac{X_1^h}{\pi^{h-1}}, \cdots, \frac{X_{h-1}^h}{\pi^{h-(h-1)}} \right\rangle \right) \subset \mathrm{Spa}(\mathcal{O}[[X_1, ..., X_{h-1}]])^{\mathrm{ad}}_{\eta}.$$

One shows that π_{GH} induces a (highly non-trivial) isomorphism from this closed polydisk onto the closed polydisk,

$$\mathbb{D}' := \mathrm{Spa}\left(K \left\langle \frac{w_1^h}{\pi^{h-1}}, \cdots, \frac{w_{h-1}^h}{\pi^{h-(h-1)}} \right\rangle \right) \subset \mathbb{P}^{h-1,\mathrm{ad}}_{K},$$

where $w_i := c_i/c_0$ for $i = 1, ..., h-1$ are coordinates of $\mathbb{A}^{h-1,\mathrm{ad}}_{K} \subset \mathbb{P}^{h-1,\mathrm{ad}}_{K}$.

One then uses the $D^\times$-equivariance to show that π_{GH} is surjective, using that the open subsets $\Pi^i \cdot \mathbb{D}'_{K_h}$ for $i \in \mathbf{Z}$ cover $\mathbb{P}^{h-1,\mathrm{ad}}_{K_h}$. With a little extra argument, one can show that the restriction,

$$\pi_{\mathrm{GH}}\,|_0 : \mathcal{M}^{\mathrm{ad}}_{\mathrm{RZ},h,0,\eta} \to \mathbb{P}^{h-1,\mathrm{ad}}_{K},$$

is also surjective, by showing that if R is the ring of integers in a sufficiently large enough finite extension of K, that given $(G(X,Y), \rho) \in \mathcal{M}_{\mathrm{RZ},h,0}(R)$, you can always find some $(G'(X,Y), \rho') \in \mathcal{M}_{\mathrm{RZ},h,0}(R)$ such that,

$$\pi_{\mathrm{GH}}(G(X,Y), \rho) = \pi_{\mathrm{GH}}(G'(X,Y), \rho').$$

To say a little about how the morphism is constructed, let (B, B^+) be a complete sheafy Huber pair over $(K, \mathcal{O})$. Then,

$$\mathcal{M}^{\mathrm{ad}}_{\mathrm{RZ},h,\eta}(B, B^+) = \varinjlim_{B_0 \subset B^+} \mathcal{M}_{\mathrm{RZ},h}(B_0) = \varinjlim_{B_0 \subset B^+} \varprojlim_{n \geq 1} \mathcal{M}_{\mathrm{RZ},h}(B_0/\pi^n B_0).$$

On the other hand, $\mathbb{P}^{h-1,\mathrm{ad}}_{K}(B, B^+)$ is the set of isomorphism classes of invertible B-modules $\mathcal{L}$ with a surjection $B^h \to \mathcal{L}$.

Given any π-adically complete π-torsion free $\mathcal{O}$-algebra R like B_0, the morphism π_{GH} is constructed by taking a point $(G, \rho) \in \mathcal{M}_{\mathrm{RZ},h}(R)$ to $(\mathcal{L}, B^h \to \mathcal{L})$, where $\mathcal{L} := \mathrm{Lie}(G)[1/\pi]$. Now actually $\mathrm{Lie}(G)$ will be free, and it comes down to choosing some nice naturally varying choice of $c_0, c_1, ..., c_{h-1}$ to define the map $B^h \to \mathcal{L}$.

One constructs for any π-torsion free $\mathcal{O}$-algebra S, a covariant functor,

$$M(-) : \mathbf{FG}_{S/\mathcal{O},\pi-div}(S/\pi) \to \{\text{fin. locally free } S\text{-modules}\},$$

which is defined in terms of quasi-logarithms, and somehow uses this to get these sections.

4. Level Structures

In this section, we consider the base change,

$$\overline{\mathrm{LT}}_h : \mathcal{C}_{\breve{\mathcal{O}}} \to \mathbf{Set},$$

sending R to the set of $\mathrm{LT}_h(R)$, which in this section we consider as the set of equivalence classes of pairs (X, ϕ), where $\phi : X_{\overline{k}} \to X_h$ is an isomorphism of formal $\mathcal{O}$-module laws.

Definition 4.1. Let $m \geq 0$. For $X \in \overline{\mathrm{LT}}_h(R)$, a *Drinfeld Level m Structure on X* is an $\mathcal{O}$-module homomorphism,

$$\eta : (\pi^{-m}\mathcal{O}/\mathcal{O})^h \to (\mathfrak{m}_R, +_X),$$

such that,

$$\prod_{a \in (\pi^{-m}\mathcal{O}/\mathcal{O})^h} (X - \eta(a)) \mid [\pi^m](X),$$

in $R[[X]]$.

Remark. There are quite a few ways one can rephrase this condition: see the notes by Fargues.

Definition 4.2. Let $m \geq 0$. Let $\overline{\mathrm{LT}}_{h,m} : \mathcal{C}_{\breve{\mathcal{O}}} \to \mathbf{Set}$, be the functor sending R to the set of isomorphism classes of triples (X, ϕ, η), where $(X, \phi) \in \overline{\mathrm{LT}}_{h,0}(X)$ and η is a Drinfeld Level m Structure on X.

Remark. Let's try and motive Drinfeld's condition. Suppose instead we had defined a level m-structure as just any homomorphism,

$$\eta : (\pi^{-m}\mathcal{O}/\mathcal{O})^h \to (\mathfrak{m}_R, +_X),$$

and we don't impose Drinfeld's condition. Then this functor is represented by,

$$S_{h,m} := R_{h,0}[[T_1, ..., T_h]]/([\pi^m](T_1), ..., [\pi^m](T_h)),$$

with universal level structure $\zeta_m^{\mathrm{univ}}(e_i) = T_i$, for $e_1, ..., e_h$ the standard basis of $(\pi^{-m}\mathcal{O}/\mathcal{O})^h$.

In the case that $h = 1$, we want the rings $R_{1,m}$ to be integral models of the Lubin-Tate extensions $\breve{K}_m/\breve{K}$. If we have $K = \mathbb{Q}_p$, we can take $X^{\mathrm{univ}} = \mathbb{G}_m(X, Y)$, with $[p](X) = (1 + X)^p - 1$. Then,

$$S_{1,0} = \breve{\mathcal{O}}, \qquad S_{1,1} = \breve{\mathcal{O}}[T]/((1 + T)^p - 1).$$

However, this is slightly too big, as we want the ring of integers in $\mathbb{Q}_p(\zeta_p)$, which has degree $p - 1$ over $\mathbb{Q}_p$, so we want instead,

$$\breve{\mathcal{O}}[\zeta_p] = \breve{\mathcal{O}}[\zeta_p - 1] = \breve{\mathcal{O}}[T]/(\frac{(1 + T)^p - 1}{T}) = \breve{\mathcal{O}}[T]/([p](T)/T).$$

The higher extensions $\breve{\mathcal{O}}_m$ for $m \geq 2$ should be given by,

$$\breve{\mathcal{O}}[\zeta_{p^m}] = \breve{\mathcal{O}}[\zeta_{p^{m-1}}][T]/([p](T) - (\zeta_{p^{m-1}} - 1)).$$

Adding these quotients corresponds exactly to imposing Drinfeld's condition.

By restricting the level structure, we have natural transformations,

$$\overline{\mathrm{LT}} = \overline{\mathrm{LT}}_{h,0} \leftarrow \overline{\mathrm{LT}}_{h,1} \leftarrow \overline{\mathrm{LT}}_{h,2} \leftarrow \cdots.$$

The functor $\overline{\mathrm{LT}}_{h,m}$ has an right action of $\mathrm{GL}_h(\mathcal{O}/\pi^m\mathcal{O}) \times \mathcal{O}_D^\times$, defined by,

$$(X, \phi, \eta) \cdot (g, x) := (X, \phi \circ x, \eta \circ g).$$

Let X^{univ} be a universal element for $\overline{\mathrm{LT}}_{h,0}$.

Theorem 4.3. *For each $m \geq 0$, the functor $\overline{\mathrm{LT}}_{h,m}$ is represented by a regular local ring $R_{h,m}$. There is a universal Drinfeld level m structure,*

$$\phi_m^{\mathrm{univ}} : (\pi^{-m}\mathcal{O}/\mathcal{O})^h \to (\mathfrak{m}_{R_{h,m}}, +_{X^{\mathrm{univ}}}),$$

so that $(X^{\mathrm{univ}}, \phi_m^{\mathrm{univ}})$ represents $\overline{\mathrm{LT}}_{h,m}$. Furthermore,

- *For $n > m$, the extension $R_{h,m} \to R_{h,n}$ is finite and flat,*

- *The extension* $R_{h,0}[1/\pi] \hookrightarrow R_{h,m}[1/\pi]$ *is a finite étale Galois extension, with Galois group* $\mathrm{GL}_h(\mathcal{O}/\pi^m\mathcal{O})$.

One can define $R_{h,m}$ and ϕ_m^{univ} in terms of $R_{h,0}$ and the power series $[\pi](X)$. If we already know these for $m = 0, 1$, then for $m \geq 2$, $R_{h,m}$ and universal level structure ϕ_m^{univ} are inductively defined as follows. Fix some some compatible basis $e_{m,1}, ..., e_{m,1}$ of $(\pi^{-m}\mathcal{O}/\mathcal{O})^h$ as an $\mathcal{O}/\pi^m\mathcal{O}$-module for each $m \geq 0$. Suppose we are given $(R_{h,m}, \phi_m^{\mathrm{univ}})$ for $m \geq 1$, and let $b_i := \phi_m^{\mathrm{univ}}(e_{m,i}) \in \mathfrak{m}_{R_{h,m}}$ for $i = 1, ..., h$. Then,

$$R_{h,m+1} = R_{h,m}[[y_1, ..., y_h]]/([\pi](y_1) - b_1, ..., [\pi](y_h) - b_h),$$

and $\phi_{m+1}^{\mathrm{univ}}(e_{m+1,i}) := y_i$.

Therefore, it only remains to see how to construct $R_{h,1}$ and ϕ_1^{univ}. This itself is done inductively as follows.

Definition 4.4. For $0 \leq r \leq h$, we define the functor $\Phi_r : \mathcal{C}_{\mathcal{O}} \to \mathbf{Set}$, which sends R to the set of isomorphism classes of pairs (X, ϕ, η), where $(X, \phi) \in \overline{\mathrm{LT}}_{h,0}(R)$ and η is an $\mathcal{O}$-module homomorphism,

$$\eta : (\pi^{-1}\mathcal{O}/\mathcal{O})^r \to (\mathfrak{m}_R, +_X),$$

such that,

$$\prod_{a \in (\pi^{-m}\mathcal{O}/\mathcal{O})^r} (X - \eta(a)) \mid [\pi](X),$$

in $R[[X]]$.

This is representable, by L_r and $(X^{\mathrm{univ}}, \phi^{\mathrm{univ}}, \eta_r^{\mathrm{univ}})$, which we define inductively now.

$L_0 = R_{h,0}$ with trivial η_0^{univ}. Then supposing we have defined this for $r - 1$, set,

$$g_{r-1}(T) := \frac{[\pi](T)}{\prod_{a \in (\pi^{-m}\mathcal{O}/\mathcal{O})^{r-1}} (T - \eta_{r-1}^{\mathrm{univ}}(a))}.$$

Then let,

$$L_r := L_{r-1}[[z_r]]/(g_{r-1}(z_r)),$$

and η_r^{univ} extending $\eta_{r-1}^{\mathrm{univ}}$, defined by $\eta_r^{\mathrm{univ}}(e_r) := z_r$. Finally, we can set $R_{h,1} := L_h$, and $\phi_m^{\mathrm{univ}} := \eta_h^{\mathrm{univ}}$, and $\overline{\mathrm{LT}}_{h,1} = \Phi_h$.

Remark. Note that $g_{r-1}(T) = [\pi](T)/T$, and so this agrees with what we would expect from the $h = 1$, $K = \mathbb{Q}_p$ example above.

Corollary 4.5. *For each* $m \geq 0$, *the elements* $b_i := \phi_m^{\mathrm{univ}}(e_{m,i})$ *for* $i = 1, ..., h$ *form a regular system of parameters for the regular local ring* $R_{h,m}$.

4.1. **Connected Components.** Now over $\breve{\mathcal{O}}$, the rings $R_{h,m}$ are all integral domains, hence connected. What can we say about their geometrically connected components?

In what follows, we let $\breve{K}_m$ denote the mth Lubin-Tate extension of $\breve{K}$. Recall that,

$$\breve{K} = \breve{K}_0 \subset \breve{K}_1 \subset \breve{K}_m \subset \cdots,$$

and the extension,

$$\breve{K}_\infty := \cup_{m \geq 0} \breve{K}_m,$$

has a canonical isomorphism,

$$\chi : \mathrm{Gal}(\breve{K}_\infty/\breve{K}) \xrightarrow{\sim} \mathcal{O}^\times.$$

Theorem 4.6. *The ring* $R_{h,m}[1/p]$ *contains the field extension* $\breve{K}_m$ *of* $\breve{K}$. *For any* $m \geq 1$, *and for any finite separable extension* E *of* $\breve{K}_m$,

$$\mathrm{Sp}(R_{h,m}[1/p]) \times_{\mathrm{Sp}(\breve{K})} \mathrm{Sp}(E),$$

has $(q-1)q^{m-1}$ *geometrically connected components, which are the fibres of the morphism,*

$$\mathrm{Sp}(R_{h,m}[1/p]) \times_{\mathrm{Sp}(\breve{K})} \mathrm{Sp}(E) \to \mathrm{Sp}(\breve{K}_m) \times_{\mathrm{Sp}(\breve{K})} \mathrm{Sp}(E) = \mathrm{Sp}(\breve{K}_m \otimes_{\breve{K}} E).$$

The natural decomposition,

$$\breve{K}_m \otimes_{\breve{K}} E = \prod_{\tau \in \mathrm{Gal}(\breve{K}_\infty/\breve{K})} \mathrm{Sp}(E),$$

together with the isomorphism,

$$\chi_m : \mathrm{Gal}(\breve{K}_m/\breve{K}) \xrightarrow{\sim} (\mathcal{O}/\pi^m\mathcal{O})^\times,$$

induces a bijection,

$$\pi_0(\mathrm{Sp}(R_{h,m}[1/p]) \times_{\mathrm{Sp}(\breve{K})} \mathrm{Sp}(E)) \equiv \pi_0(\mathrm{Sp}(\breve{K}_m \otimes_{\breve{K}} E)) \xrightarrow{\sim} (\mathcal{O}/\pi^m\mathcal{O})^\times,$$

which is $\mathrm{GL}_h(\mathcal{O}/\pi^m\mathcal{O}) \times \mathcal{O}_D^\times$*-equivariant if we define the action,*

$$(g, h) \cdot x := \det(g)\,\mathrm{Nrd}(h)^{-1}x,$$

of $\mathrm{GL}_h(\mathcal{O}/\pi^m\mathcal{O}) \times \mathcal{O}_D^\times$ *on* $(\mathcal{O}/\pi^m\mathcal{O})^\times$*.*

Here $\mathrm{Nrd} : \mathcal{O}_D^\times \to \mathcal{O}$ is the reduced norm.

References

[1] Johannes Anschütz. *Lecture Notes:* The Lubin-Tate Tower. Available at: `http://www.math.uni-bonn.de/people/ja/lubintate/lecture_notes_lubin_tate.pdf`.

[2] V. G. Drinfeld. Elliptic modules. *Mat. Sb. (N.S.)*, 94(136):594–627, 656, 1974.

[3] M. J. Hopkins and B. H. Gross. Equivariant vector bundles on the Lubin-Tate moduli space. In *Topology and representation theory (Evanston, IL, 1992)*, volume 158 of *Contemp. Math.*, pages 23–88. Amer. Math. Soc., Providence, RI, 1994.

[4] Jonathan Lubin and John Tate. Formal moduli for one-parameter formal Lie groups. *Bull. Soc. Math. France*, 94:49–59, 1966.

[5] Matthias Strauch. Geometrically connected components of Lubin-Tate deformation spaces with level structures. *Pure Appl. Math. Q.*, 4(4):1215–1232, 2008.