CHEVALLEY'S RESTRICTION THEOREM

JAMES TAYLOR

1. Introduction

Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field k of characteristic 0. Here we give an overview of the proof of Chevalley's theorem, which is the technical heart of the proof that the Harish-Chandra homomorphism is an isomorphism. The presentation here is a synthesis of that of [1, Chap. 8, Sect. 8] and [2, Sect. 23].

1.1. **Notational Conventions.** All Lie algebras are over an algebraically closed field k of characteristic 0. $\mathfrak{a}$ will denote a Lie algebra, $\mathfrak{g}$ a semisimple lie algebra, with fixed cartan subalgebra $\mathfrak{h}$. W is the weyl group of $\mathfrak{g}$ and Φ the root system, relative to $\mathfrak{h}$.

2. Polynomial functions

Let V be a k-vector space. We first make clear the relationship between four different perspectives of polynomial functions on V, and how $\mathfrak{a}$ acts on each when V is a representation of $\mathfrak{a}$.

Let $\mathcal{O}(V) := S(V^*)$, the symmetric algebra of V^*. Choose a basis $e_1, ..., e_n$ of V, with corresponding dual basis $e_1^*, ..., e_n^* \in V^*$. Then there is a isomorphism of k-algebras $k[e_1^*, ..., e_n^*] \cong S(V^*)$, where

$$(e_1^*)^{m_1} \cdots (e_n^*)^{m_n} \leftrightarrow (e_1^*)^{\otimes m_1} \otimes \cdots \otimes (e_n^*)^{\otimes m_n}.$$

There is a natural map $S(V^*) \to \mathrm{Func}(V, k)$, given by

$$f_1 \otimes \cdots \otimes f_m \mapsto [v \mapsto f_1(v) \cdots f_m(v)].$$

Because k is an infinite field, this is injective[1], and we denote the image by $k[V] \subset \mathrm{Func}(V, k)$, the algebra of polynomial functions on V. If we write an element of V as $(a_1, ..., a_n)$ with respect to the basis $e_1, .., e_n$ above, then for example, under the above identifications the polynomial $e_1^* e_2^* - (e_3^*)^2$ corresponds to the function

$$(a_1, ..., a_n) \mapsto a_1 a_2 - a_3^2,$$

as one might expect. In fact, a function in $\mathrm{Func}(V, k)$ can be written as a polynomial (as above) with respect to one basis, if and only if it can be written as a polynomial with respect to any basis of V. This is because $k[e_1^*, ..., e_n^*] \cong S(V^*) \cong k[V]$, independently of the basis chosen.

Another common way to view this algebra $S(V^*)$, is to first consider the dual of the symmetric algebra on V, $S(V)^*$. If we write $S(V) = \oplus_{m \geq 0} S^m(V)$ as a direct sum of homogeneous parts, then $S(V)^* = \prod_{m \geq 0} S^m(V)^*$. $S(V)^*$ is a commutative k-algebra, the multiplication induced from the coalgebra structure on $S(V)$. By the universal property of $S(V^*)$, we can extend the linear map $V^* \to V^* = S^1(V)^* \subset S(V)^*$ to a k-algebra homomorphism $\phi : S(V^*) \to S(V)^*$. Explicitly, ϕ sends,

$$\phi(f_1 \otimes \cdots \otimes f_m) = \left[(v_1 \otimes \cdots \otimes v_m) \mapsto \sum_{\sigma \in S_n} f_1(x_{\sigma(1)}) \cdots f_m(x_{\sigma(m)}) \right],$$

a linear form on $S^m(V)$. In other words, this tells us the multiplication in $S(V)^*$: the product of $f_1, ..., f_m$ inside $S(V)^*$ is the linear functional on $S^m(V)$ shown to the right. The image is

[1]Over a finite field, this is not an isomorphism. In fact, these rings are not isomorphic - the ring of functions is finite whilst the other is not.

$\phi(S(V^*)) = \oplus_{m\geq 0} S^m(V)^*$. Furthermore, this is injective, and the inverse on $\phi(S(V^*))$ is, for $f \in S^m(V)^*$,

$$\phi^{-1}(f)(v) = \frac{1}{n!} f(v, ..., v),$$

considered as an element of $k[V]$. Indeed, one can verify that this element of $\mathrm{Func}(V, k)$ actually lies in $k[V]$, by choosing a basis of V and writing

$$\phi^{-1}(f)(v) = \frac{1}{n!} \sum_{i_1,...,i_m=1}^{n} f(e_{i_1}, ..., e_{i_m}) e^*_{i_1}(v) \cdots e^*_{i_m}(v).$$

By construction, ϕ restricts to a linear isomorphism $S^m(V^*) \to S^m(V)^*$. Note that by the universal property of the tensor product of vector spaces over k, we will consider $S^m(V)^*$ also as m-multilinear maps on V^m which are symmetric: invariant under permutations of their entries.

In summary, we have four different ways to view the polynomial algebra $\mathcal{O}(V)$. Namely $S(V^*)$, $k[V]$, $k[e^*_1, ..., e^*_n]$, and $\phi(S(V^*)) \subset S(V)^*$.

2.1. **Representations.** Now suppose that V is additionally a representation of Lie algebra $\mathfrak{a}$. The natural action of $\mathfrak{a}$ on $S(V)$ is

$$x \cdot (v_1 \otimes \cdots \otimes v_m) = \sum_{i=1}^{m} v_1 \otimes \cdots \otimes x \cdot v_i \otimes \cdots \otimes v_m,$$

for all $x \in \mathfrak{a}$ and $v_1, ..., v_m \in V$. Therefore, the natural action of $\mathfrak{a}$ on $S(V)^*$ is by

$$(x \cdot f)(v_1, ..., v_m) = -\sum_{i=1}^{m} f(v_1, ..., x \cdot v_i, ..., v_m),$$

for all $x \in \mathfrak{a}$, $f \in S^m(\mathfrak{a})^*$, and $v_1, ..., v_m \in V$.

The natural action of $\mathfrak{a}$ on V^* is by $(x \cdot f)(v) = -f(x \cdot v)$, and the natural action of of $\mathfrak{a}$ on $S(V^*)$ is by

$$x \cdot (f_1 \otimes \cdots \otimes f_m) = \sum_{i=1}^{m} f_1 \otimes \cdots \otimes x \cdot f_i \otimes \cdots \otimes f_m.$$

Suppose that V is also a representation of a group G. Then G acts on V^* via $(g \cdot f)(v) = f(g^{-1}v)$, and so too is $S(V^*)$ via

$$g \cdot (f_1 \otimes \cdots \otimes f_m) = g \cdot f_1 \otimes \cdots \otimes g \cdot f_m.$$

Additionally, G acts on V, so on $S(V)$, and thus on $S(V)^*$ too in the usual way:

$$(g \cdot f)(v_1, ..., v_m) = f(g^{-1}v_1, ..., g^{-1}v_m),$$

for all $g \in G$, $f \in S^m(\mathfrak{a})^*$, and $v_1, ..., v_m \in V$.

Lemma 2.1. *$\phi : S(V^*) \to S(V)^*$ is both $\mathfrak{g}$-equivariant and G-equivariant with respect to the actions described above.*

Via the identification of $S(V^*)$ with $k[V]$, $g \in G$ acts on a polynomial function $f \in k[V]$ by

$$(g \cdot f)(v) = f(g^{-1}v).$$

2.2. **Nilpotents and exponentials.** Let (V, ρ) be a representation of the Lie-algebra $\mathfrak{a}$.

First we need a quite general lemma, which can be proven by a direct computation and the binomial expansion.

Lemma 2.2. *Let R be a $\mathbb{Q}$-algebra, and $r \in R$ nilpotent, say $r^{n+1} = 0$. Then*

$$\exp(r) := 1 + r + \cdots + \frac{r^n}{n!}$$

is invertible, with inverse $\exp(-r)$.

We can apply the above to the ring $R = \mathrm{End}(V)$, and consider the group

$$H := \langle \exp(\rho(a)) \mid \rho(a) \text{ is nilpotent} \rangle \triangleleft \mathrm{Aut}(V) = \mathrm{End}(V)^*,$$

which comes with defining representation V.

Because V is a representation of H, then $S(V^*)$ is too in the natural way, and so for each $a \in A$ with $\rho(a)$ nilpotent, we have a linear automorphism of $S(V^*)$,

$$\exp(\rho(a)) \cdot (f_1 \otimes \cdots \otimes f_m) = \exp(\rho(a))f_1 \otimes \cdots \otimes \exp(\rho(a))f_m,$$

with $(\exp(\rho(a))f_i)(v) = f_i(\exp(\rho(a)^{-1})(v)) = f_i(\exp(-\rho(a))(v))$. We write this automorphism of $S(V^*)$ as $S^*(\exp(\rho(a)))$. Alternatively, because V is a representation of $\mathfrak{a}$, then $S(V^*)$ is also a representation of $\mathfrak{a}$ as described above; call it $(S(V^*), \sigma)$. The following can be shown by explicitly by writing out both maps, and slogging through.

Proposition 2.3. *If $\rho(a)$ is nilpotent, then so is $\sigma(a)$. Furthermore,* $\exp(\sigma(a)) = S^*(\exp(\rho(a)))$.

For a finite dimensional representation of a semisimple Lie-algebra $\mathfrak{g}$, the invariants under both $\mathfrak{g}$ and H are the same.

Lemma 2.4. *Let (ρ, V) be a finite dimensional representation of $\mathfrak{g}$, a semisimple Lie-algebra. Then $v \in V$ is $\mathfrak{g}$-invariant if and only if v is H-invariant:* $\exp(\rho(x))v = v$ *for all $x \in \mathfrak{g}$ with $\rho(x)$ nilpotent.*

Proof. If v is invariant under $\mathfrak{g}$, then for all $x \in \mathfrak{g}$ with $\rho(x)$ nilpotent, $\rho(x)^n \cdot v = 0$ for all $n \geq 1$, thus $\exp(\rho(x))v = v$. Conversely, suppose that $x \in \mathfrak{g}$ with $\rho(x)$ nilpotent, so that there is some $m \geq 0$ with $\rho(x)^m = 0$. Then for all $t \in k$, by assumption,

$$0 = \exp(t\rho(x))v - v = t\rho(x)v + \frac{1}{2!}t^2\rho(x)^2v + \cdots + \frac{1}{(m-1)!}t^{(m-1)}\rho(x)^{(m-1)}v,$$

and thus

$$\rho(x)v = t(\rho(x)^2 + \cdots + \frac{1}{(m-1)!}t^{(m-1)-2}\rho(x)^{(m-1)}v),$$

which holds for all $t \in k^*$, hence $\rho(x)v = 0$. Therefore, v is invariant for all $x \in \mathfrak{g}$ where $\rho(x)$ is nilpotent. We can write $\mathfrak{h} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^-$ and $\mathfrak{n}^+$ are nilpotent Lie-algebras. All elements of a nilpotent Lie algebra act by nilpotent matrices in any finite dimensional representation, so v is fixed by all $x \in \mathfrak{n}^- \oplus \mathfrak{n}^+$, and as this subspace generates $\mathfrak{g}$, then v is fixed by all $x \in \mathfrak{g}$. □

3. Chevalley's Theorem

Let $\mathfrak{g}$ be a semisimple lie algebra, with a fixed cartan subalgebra $\mathfrak{h}$. We can form the algebras $\mathcal{O}(\mathfrak{g}) := S(\mathfrak{g}^*)$, $\mathcal{O}(\mathfrak{h}) := S(\mathfrak{h}^*)$ as above. The dual of the inclusion, $\mathfrak{g}^* \to \mathfrak{h}^*$, induces a homomorphism of algebras, $\hat{\theta} : S(\mathfrak{g}^*) \to S(\mathfrak{h}^*)$. Viewed as a map from $k[\mathfrak{g}] \to k[\mathfrak{h}]$ via our previous identifications, this is just the restriction of functions. Our aim is to prove the following.

Theorem 3.1 (Chevalley)**.** *The restriction homomorphism,*

$$\hat{\theta} : \mathcal{O}(\mathfrak{g}) \to \mathcal{O}(\mathfrak{h}),$$

induces an isomorphism,

$$\theta : \mathcal{O}(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{O}(\mathfrak{h})^W.$$

Here the action of W on $\mathcal{O}(\mathfrak{h}) := S(\mathfrak{h}^*)$ is that induced by the defining action of W on $\mathfrak{h}^*$:

$$w \cdot (f_1 \otimes \cdots \otimes f_n) = w \cdot f_1 \otimes \cdots \otimes w \cdot f_n,$$

for all $w \in W$, $f_i \in \mathfrak{h}^*$. The action of $\mathfrak{g}$ on $\mathcal{O}(\mathfrak{g}) = S(\mathfrak{g}^*)$ is as described in Section 2.1, namely,

$$x \cdot (f_1 \otimes \cdots \otimes f_m) = \sum_{i=1}^{m} f_1 \otimes \cdots \otimes x \cdot f_i \otimes \cdots \otimes f_m,$$

where $(x \cdot f_i)(y) = -f_i([x, y])$.

We prove this in four steps, namely that

(1) $\mathcal{O}(\mathfrak{g})^{\mathfrak{g}} = \mathcal{O}(\mathfrak{g})^G$, for G the group of elementary automorphisms,
(2) θ is well defined: $\hat{\theta}(\mathcal{O}(\mathfrak{g})^G) \subset \mathcal{O}(\mathfrak{h})^W$,
(3) θ is injective, using a density argument from affine algebraic geometry,
(4) θ is surjective, by explicitly exhibiting a spanning set of W-invariant polynomials.

We now proceed with the first, and rephrase the invariants $\mathcal{O}(\mathfrak{g})^{\mathfrak{g}}$ as invariants under all inner automorphisms of $\mathfrak{g}$. The following is a special case of the group H of Section 2.2, with V the adjoint representation of $\mathfrak{g}$.

Definition 3.2. Let

$$G := \langle \exp(\mathrm{ad}_x) \mid x \text{ is ad-nilpotent} \rangle \lhd \mathrm{Aut}(\mathfrak{g}),$$

the normal subgroup of *elementary automorphisms* of $\mathfrak{g}$.

We define our action of G on $\mathcal{O}(\mathfrak{g})$ as in Section 2.1, by

$$(g \cdot f)(v) = f(g^{-1}v).$$

Proposition 3.3. *The invariant subspaces of $\mathcal{O}(\mathfrak{g})$ from the actions of $\mathfrak{g}$ and G are equal:*

$$\mathcal{O}(\mathfrak{g})^{\mathfrak{g}} = \mathcal{O}(\mathfrak{g})^G.$$

Proof. By Lemma 2.4, the invariants $\mathcal{O}(\mathfrak{g})^{\mathfrak{g}}$ are the same as the invariants under the group generated by the $\exp(\sigma(x))$ for $x \in \mathfrak{g}$ ad-nilpotent, in the notation of Section 2.2. But then by Proposition 2.3, these are exactly the invariants under G, with the action on $\mathcal{O}(\mathfrak{g})$ described above. □

Therefore, from now on we work with $\mathcal{O}(\mathfrak{g})^G$ instead of $\mathcal{O}(\mathfrak{g})^{\mathfrak{g}}$.

3.1. **Well-Definedness.** In order to show that each $\mathfrak{g}$-invariant polynomial function on $\mathfrak{g}$ is W-invariant when restricted to $\mathfrak{h}$, we first consider elementary automorphisms τ_α of $\mathfrak{g}$, for $\alpha \in \Phi$, which correspond to the generators s_α of W.

For each root $\alpha \in \Phi$, define

$$\tau_\alpha = \exp(\mathrm{ad}(e_\alpha)) \exp(\mathrm{ad}(-e_{-\alpha})) \exp(\mathrm{ad}(e_\alpha)) \in G.$$

Lemma 3.4. $\tau_\alpha(h) = h - \alpha(h)h_\alpha$ *for all* $h \in \mathfrak{h}$.

Proof. We can write $\mathfrak{h} = \ker \alpha \oplus \langle h_\alpha \rangle_k$. If $\alpha(h) = 0$, then $[e_\alpha, h] = 0 = [e_{-\alpha}, h]$, because $e_\alpha \in \mathfrak{g}_\alpha$, $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$. Thus $\tau_\alpha(h) = h = h - \alpha(h)h$. Furthermore, one can use the relations

$$\begin{aligned} [e_\alpha, h_\alpha] &= -2e_\alpha, \\ [e_{-\alpha}, h_\alpha] &= 2e_{-\alpha}, \\ [e_\alpha, e_{-\alpha}] &= h_\alpha, \end{aligned}$$

to explicitly compute that $\tau_\alpha(h_\alpha) = -h_\alpha$, and $-h_\alpha = h_\alpha - 2h_\alpha = h_\alpha - \alpha(h_\alpha)h_\alpha$. □

Recall that for $\alpha \in \Phi$, the corresponding generator of W is $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}*$, $(s_\alpha \cdot f)(h) = (f - f(h_\alpha)\alpha)(h)$.

Corollary 3.5. *For all $\alpha \in \Phi$ and all $f \in \mathfrak{h}^*$,*

$$(s_\alpha \cdot f)(h) = f(\tau_\alpha(h)),$$

for all $h \in \mathfrak{h}$.

Proof. Using the definition of s_α,

$$\begin{aligned} (s_\alpha \cdot f)(h) &= (f - f(h_\alpha)\alpha)(h), \\ &= f(h) - f(h_\alpha)\alpha(h), \\ &= f(h - \alpha(h)h_\alpha), \\ &= f(\tau_\alpha(h)). \end{aligned}$$

□

Remark. One might expect that this gives a way of realising W inside of G, as $\langle \tau_\alpha | \alpha \in \Phi \rangle$. However in general, this subgroup of G can have order strictly larger than W. One example is given by $\mathfrak{sl}_3(k)$. For more details on extending automorphisms of $\mathfrak{h}$ to automorphisms of $\mathfrak{g}$, see [2, Thm. 14.2].

Lemma 3.6. *The restriction of a $\mathfrak{g}$-invariant polynomial function on $\mathfrak{g}$ to a polynomial function on $\mathfrak{h}$ is W-invariant: if $f \in \mathcal{O}(\mathfrak{g})^G$, then $\hat{\theta}(f) \in \mathcal{O}(\mathfrak{h})^W$. Therefore θ is well-defined.*

Proof. Take $f \in \mathcal{O}(\mathfrak{g})^G$. It is enough to show that $\hat{\theta}(f)$ is fixed by each s_α, for $\alpha \in \Phi$, as these generate W. Indeed, a polynomial function is of the form

$$f = \sum_{m \geq 0} f_{m,1} \otimes \cdots \otimes f_{m,m},$$

and $f \in \mathcal{O}(\mathfrak{g})^G$ means that f fixed by τ_α^{-1}. Therefore considering $f \in k[\mathfrak{h}]$, for $y \in \mathfrak{h}$,

$$\begin{aligned}
(s_\alpha \cdot f)(y) &= \sum_{m \geq 0} (s_\alpha \cdot f_{m,1})(y) \cdots (s_\alpha \cdot f_{m,m})(y), \\
&= \sum_{m \geq 0} f_{m,1}(\tau_\alpha y) \cdots f_{m,m}(\tau_\alpha y), \\
&= \sum_{m \geq 0} (\tau_\alpha^{-1} \cdot f_{m,1})(y) \cdots (\tau_\alpha^{-1} \cdot f_{m,m})(y), \\
&= \sum_{m \geq 0} f_{m,1}(y) \cdots f_{m,m}(y), \\
&= f(y).
\end{aligned}$$

□

3.2. **Injectivity.** That θ is injective is shown using a density argument from affine algebraic geometry. We show that if a G-invariant polynomial function on $\mathfrak{g}$ vanishes on $\mathfrak{h}$, then in fact it must vanish on the set of regular elements of $\mathfrak{g}$, which is a dense subset in the zariski topology, so f must actually vanish on all of $\mathfrak{g}$.

3.3. **Algebraic geometry.** We will need the following standard results.

Lemma 3.7. *Let $n \geq 1$. Any non-empty zariski-open subset of $\mathbb{A}_k^n$ is dense.*

This follows because $\mathbb{A}_k^n$ is irreducible, and a topological space is irreducible if and only if every non-empty open subset is dense.

Lemma 3.8. *Let $n \geq 1$. If $f \in k[x_1, ..., x_n]$ vanishes on a dense subset of $\mathbb{A}_k^n$, then $f = 0$ in $k[x_1, ..., x_n]$.*

Proof. Suppose that f vanishes on a dense subset D. The complement in $\mathbb{A}_k^n$ of the vanishing set of f, $D_f = \mathbb{A}_k^n \setminus V(f)$, is an open subset, which does not intersect D, thus by the density of D, must be empty. Now because k is an infinite field (being algebraically closed), f must be the zero element of $k[x_1, ..., x_n]$. □

3.4. **Regular elements and Cartan subalgebras.** A subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is said to be toral if all $x \in \mathfrak{a}$ have $x = x_s$ in the abstract jordan decomposition, i.e. $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is semisimple (diagonalisable). A cartan subalgebra is a maximal toral subalgebra. These exist, are abelian, and any semisimple element lies in a maximal toral subalgebra. Further, any two cartan subalgebras are conjugate by an element of G.

An element $x \in \mathfrak{g}$ is said to be *regular* if $\mathfrak{g}_{0,x}$, the generalised zero eigenspace of ad_x, is of minimal dimension over all $x \in \mathfrak{g}$. If x is regular, then $\mathfrak{g}_{0,x}$ is a cartan subalgebra, and all cartan subalgebras are of this form.

3.5. **Proof of injectivity.** Let $e_1, ..., e_n$ be a basis of $\mathfrak{g}$. We identify $S(\mathfrak{g}^*)$ with $k[e_1^*, ..., e_n^*]$, and through this give $S(\mathfrak{g}^*)$ the zariski topology. For an element $x \in \mathfrak{g}$, write the characteristic polynomial of $\text{ad}_x : \mathfrak{g} \to \mathfrak{g}$ as,

$$\rho_x(T) = \sum_{k=0}^{n} c_k(x) T^n$$

where T is an indeterminate. The matrix of ad_x is an $n \times n$ matrix with entries in $\mathfrak{g}$, which are linear, hence polynomial, functions on $\mathfrak{g}$. Therefore, each $c_k : \mathfrak{g} \to k$ is a polynomial function on $\mathfrak{g}$. Let m be the smallest integer with c_m not identically zero. One might ask for which elements this is non-zero.

Proposition 3.9. *$c_m(x) \neq 0$ if and only if $x \in \mathfrak{g}$ is regular. Furthermore, $m = \dim(\mathfrak{h})$.*

Proof. By the definition of c_m, $c_m(x) \neq 0$ if and only x has the smallest factor of T in the minimal polynomial of ad_x, which is exactly the condition that the generalised 0-eigenspace $g_{0,x}$ is of minimal dimension over all $x \in \mathfrak{g}$, i.e. x is regular. Therefore, for a regular element x, $m = \dim(g_{0,x})$, which for the dimension of any cartan subalgebra. □

Proposition 3.10. *θ is injective.*

Proof. Suppose that $f \in \mathcal{O}(\mathfrak{g})^G$, with $f|_{\mathfrak{h}}$ identically zero. The set of regular elements of $\mathfrak{g}$ is open by Proposition 3.9, hence dense by Lemma 3.7. Therefore, to show $f = 0$, it suffices to show f vanishes on the set of regular elements, by Lemma 3.8. Let $x \in \mathfrak{g}$ be regular. Then because $x \in \mathfrak{g}_{0,x}$, and the cartan subalgebra $\mathfrak{g}_{0,x}$ is conjugate to $\mathfrak{h}$ (both being cartan subalgebras), there is some $g \in G$ with $h = g \cdot x \in \mathfrak{h}$. Therefore, $f(x) = f(g^{-1}h) = (g \cdot f)(h) = f(h) = 0$. □

3.6. **Surjectivity.**

3.7. **Partial orders and dominant weights.** Recall the integral weight lattice $P = \{\chi \in \mathfrak{h}^* \mid \chi(h_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Phi\} \subset Q = \mathbb{Z}\Phi \subset \mathfrak{h}^*$. If $\Pi \subset \Phi$ is a chosen set of simple roots, then $\Phi^+ = \mathbb{N}\Pi \cap \Phi$ is the set of positive roots. $P^+ = \{\chi \in \mathfrak{h}^* \mid \chi(h_\alpha) \in \mathbb{N} \text{ for all } \alpha \in \Phi^+\} \subset P$ is the set of dominant integral weights. There is a bijection between $\mathfrak{h}^*$ and simple modules in $\mathcal{O}$, associating $\chi \in \mathfrak{h}^*$ with the unique simple quotient $L(\chi)$ of the Verma module $M(\chi)$ of χ. Furthermore, $L(\chi)$ is finite dimensional if and only if $\chi \in P^+$. There is a partial order on $\mathfrak{h}^*$ defined by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q^+ = \mathbb{N}\Phi^+$. For a fixed dominant weight $\lambda \in P^+$, the number of dominant weights $\mu \leq \lambda$ is finite [2, Lemma 13.2 B].

We will make use of some facts about the weights of $L(\lambda)$. The first is that the dimension of the weight space $\dim(L(\lambda))_\mu$ is constant on the orbits of μ under the action of the weyl group [2, Thm. 21.2]. The second is that $\dim(L(\lambda))_\lambda = 1$, and all non-zero weights in $L(\lambda)$ have $\mu \leq \lambda$ [2, Thm. 20.2].

3.8. **Proof of surjectivity.** For each $n \geq 0$, we consider the functions

$$f_{\rho,n} : x \mapsto \text{tr}(\rho(x)^n)$$

on $\mathfrak{g}$, where ρ is a finite dimensional representation of $\mathfrak{g}$.

Lemma 3.11. *$f_{\rho,n}$ is a $\mathfrak{g}$-invariant polynomial function on $\mathfrak{g}$.*

Proof. Consider, $g \in S^n(\mathfrak{g})^*$,

$$g(x_1, ..., x_n) = \sum_{\sigma \in S_n} \text{tr}(\rho(x_{\sigma(1)}) \cdots \rho(x_{\sigma(n)})),$$

and write $g_\sigma(x_1, ..., x_n) = \mathrm{tr}(\rho(x_{\sigma(1)}) \cdots \rho(x_{\sigma(n)}))$. For any $x \in \mathfrak{g}$,

$$\begin{aligned}(x \cdot g_\sigma)(x_1, ..., x_n) &= -\sum_{i=1}^{n} g_\sigma(x_1, ..., [x, x_i], ..., x_n) \\ &= -\sum_{i=1}^{n} \mathrm{tr}(\rho(x_{\sigma(1)})[\rho(x), \rho(x_{\sigma(i)})]\rho(x_{\sigma(n)})) \\ &= \mathrm{tr}(\rho(x_{\sigma(1)}) \cdots \rho(x_{\sigma(n)})\rho(x)) - \mathrm{tr}(\rho(x)\rho(x_{\sigma(1)}) \cdots \rho(x_{\sigma(n)})) \\ &= 0,\end{aligned}$$

as the trace of finite dimensional linear operators is invariant under cyclic permutations. Therefore $g \in (S^n(\mathfrak{g})^*)^{\mathfrak{g}}$, and thus

$$f_{\rho,n}(x) = \frac{1}{n!} g(x, ..., x),$$

is a $\mathfrak{g}$-invariant polynomial function. □

Our aim now is to show that these polynomial functions on $\mathfrak{g}$ span $\mathcal{O}(\mathfrak{h})^W$, when restricted to W.

First we make some reductions. Because P contains Φ, P spans $\mathfrak{h}^*$, and so any element of $\mathcal{O}(\mathfrak{h})$ is a polynomial in the $\lambda \in P$. Furthermore, by the process of polarisation (see [2, Sect. 23, Ex. 5]), actually λ^k for $\lambda \in P$, $k \geq 0$ span $\mathcal{O}(\mathfrak{h})$. Now, if we write

$$\gamma(f) = \sum_{w \in W} w \cdot f,$$

for $f \in \mathcal{O}(\mathfrak{h})$, this is W-invariant. Furthermore:

Lemma 3.12. *The collection $\gamma(\lambda^k)$ for $\lambda \in P^+$, $k \geq 0$, spans $\mathcal{O}(\mathfrak{h})^W$.*

Proof. Suppose that

$$f = \sum_{\lambda \in P, k \geq 0} a_{\lambda,k} \lambda^k,$$

is W-invariant. Then applying each element $w \in W$ and taking the sum,

$$\begin{aligned}|W| \sum_{\lambda \in P, k \geq 0} a_{\lambda,k} \lambda^k &= \sum_{\lambda \in P, k \geq 0} \sum_{w \in W} a_{\lambda,k} (w \cdot \lambda)^k, \\ &= \sum_{\lambda \in P, k \geq 0} a_{\lambda,k} \sum_{w \in W} (w \cdot \lambda)^k.\end{aligned}$$

Therefore, f is in the span of $\gamma(\lambda^k)$ for $\lambda \in P$, $k \geq 0$, and furthermore, every $\lambda \in P$ is W-conjugate to some $\lambda' \in P^+$ [2, Lemma 13.2 A], so $\gamma(\lambda^k) = \gamma((\lambda')^k)$ and $\gamma(\lambda^k)$ for $\lambda \in P^+$, $k \geq 0$, spans $\mathcal{O}(\mathfrak{h})^W$. □

Proposition 3.13. *θ is surjective.*

For $\lambda \in P^+$, write ρ_λ for the finite dimensional simple representation $L(\lambda)$.

Proof. By the above reductions, it is sufficent to show that $\gamma(\lambda^k)$ for $\lambda \in P^+$, $k \geq 0$ is in the image of θ. We achieve this (for each fixed k) by upward induction on the partial ordering $\geq$ of P^+, which we can do because as mentioned above, the number of $\lambda \in P^+$ with $\lambda \leq \mu$, for a fixed μ, are finite in number. For the base step, take $\lambda \in P^+$ minimal. Because λ is minimal, there is only one W-orbit of weights in $L(\lambda)$, all of the same dimension, and because $\dim L(\lambda)_\lambda = 1$, there are all one-dimensional. Therefore,

$$\theta(f_{\rho_\lambda,k}) = \gamma(\lambda^k).$$

Now for the induction step, choose some λ with $\lambda \in P^+$, $k \geq 0$. Then

$$\theta(f_{\rho_\lambda,k}) = \gamma(\lambda^k) + \sum_{\mu \in P^+ \colon \mu < \lambda} c_\mu \gamma(\mu^k),$$

for the only other weights μ that appear in $L(\lambda)$ have $\mu < \lambda$. By induction, each $\gamma(\mu^k)$ is in the image of θ, and so $\gamma(\lambda^k)$ is too. $\square$

References

[1] Nicolas Bourbaki. *Lie groups and Lie algebras: Chapters 7-9*, volume 3. Springer Science & Business Media, 2008.

[2] James E Humphreys. *Introduction to Lie algebras and representation theory*, volume 9. Springer Science & Business Media, 2012.