Real Representations of C_2-Graded Groups

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Overview

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All work is joint with Dmitriy Rumynin.

Representations Over $\mathbb{R}$

Representations Over $\mathbb{R}$

How to construct all irreducible representations of G over $\mathbb{R}$?

Let V be an irreducible complex representation of G.

Let $V_{\mathbb{R}}$ be the restriction of scalers V to $\mathbb{R}G$.

Let W be a irreducible subrepresentation of $V_{\mathbb{R}}$.

$W_{\mathbb{C}} = V \otimes_{\mathbb{R}G} \mathbb{C}G$, extension of scalers.

There are three possibilities.

Representations Over $\mathbb{R}$

$\mathrm{End}_{\mathbb{R}G}(W)$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathrm{End}_{\mathbb{C}G}(W_{\mathbb{C}})$	$\mathbb{C}$	$\mathbb{C} \times \mathbb{C}$	$M_2(\mathbb{C})$
$W_{\mathbb{C}}$	V	$V \oplus \overline{V}$	$V \oplus V$
$V_{\mathbb{R}}$	$W \oplus W$	W	W
$\dim_{\mathbb{R}} W$	n	$2n$	$4n$
$\dim_{\mathbb{C}} V$	n	n	$2n$
$V \cong \overline{V}$?	Yes	No	Yes
V Realisable?	Yes	No	No
$\exists$ G-invariant bil. form?	Yes (sym.)	No	Yes (alt.)
$\mathcal{F}_{\mathbb{C}}(V)$	1	0	-1

Representations Over $\mathbb{R}$

$$\mathcal{F}_{\mathbb{C}}(V) = \frac{1}{|G|} \sum_{\mathbf{g} \in G} \chi(\mathbf{g}^2).$$

Representations Over $\mathbb{R}$

$\mathrm{End}_{\mathbb{R}G}(W)$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathrm{End}_{\mathbb{C}G}(W_{\mathbb{C}})$	$\mathbb{C}$	$\mathbb{C} \times \mathbb{C}$	$M_2(\mathbb{C})$
$W_{\mathbb{C}}$	V	$V \oplus \overline{V}$	$V \oplus V$
$V_{\mathbb{R}}$	$W \oplus W$	W	W
$\dim_{\mathbb{R}} W$	n	$2n$	$4n$
$\dim_{\mathbb{C}} V$	n	n	$2n$
$V \cong \overline{V}$?	Yes	No	Yes
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Real Groups

Real Groups

A *Real* or C_2-*graded* group is a pair $G \leq \widehat{G}$ where G is a subgroup of $\widehat{G}$ of index 2.

This is also called a Real structure on G.

Write $\pi : \widehat{G} \to C_2 = \{\pm 1\}$ for the structure map

$$1 \to G \to \widehat{G} \to C_2 \to 1.$$

Example (Real Groups)

- The standard Real structure is $G \leq G \times C_2$.
- The cyclic group C_n has other Real structures: $C_n \leq C_{2n}$, D_{2n}.
- $A_n \leq S_n$

Real Groups

For a Real group $G \leq \widehat{G}$ there is an associated *Real conjugation* action of $\widehat{G}$ on G:

$$\mathbf{z} \cdot \mathbf{g} = \mathbf{z}\mathbf{g}^{\pi(\mathbf{z})}\mathbf{z}^{-1}.$$

Example (Real Conjugation)

- The Real conjugacy classes of the standard Real structure $G \leq G \times C_2$ are $(\mathbf{g})_G \cup (\mathbf{g}^{-1})_G$.
- $C_n \leq D_{2n}$ has $(\!(\mathbf{g})\!) = \{\mathbf{g}\}$.
- $C_n \leq C_{2n}$ has $(\!(\mathbf{g})\!) = \{\mathbf{g}\} \cup \{\mathbf{g}^{-1}\}$

Real Groups

Example (Real Conjugation)

$A_n \leq S_n$ has: If $\mathbf{g} \in A_n$, then

$$(\!(\mathbf{g})\!) = \begin{cases} (\mathbf{g})_{A_n} & \text{if } (\mathbf{g})_{A_n} \text{ is not self-inverse,} \\ (\mathbf{g})_{S_n} & \text{if } (\mathbf{g})_{A_n} \text{ is self-inverse.} \end{cases}$$

Thus we always have $(\mathbf{g})_{A_n} \subset (\!(\mathbf{g})\!) \subset (\mathbf{g})_{S_n}$.

This also holds when $\widehat{G}$ has all conjugacy classes self inverse.

Antilinear Representations

Antilinear Representations

Definition

An antilinear representation of $G \leq \widehat{G}$ is a $\mathbb{C}$-vector space V with C_2-graded homomorphism

$$\rho : \widehat{G} \to \mathsf{GL}^*(V).$$

Homomorphisms of such representations are $\mathbb{C}$-linear maps that commute with the action of $\widehat{G}$.

$\mathrm{Hom}_A(V, W)$ forms a real (not complex) vector space.

Antilinear Representations

Phrased differently, antilinear representations are modules over $\mathbb{C} * \widehat{G}$: complex skew group algebra with

- Basis: $\widehat{G}$
- Multiplication: $a\mathbf{g} \cdot b\mathbf{h} = a \cdot {}^{\pi(\mathbf{g})}b\mathbf{gh}$.

By the Artin-Wedderburn Theorem, this algebra is isomorphic to a finite product of matrix rings over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$.

Antilinear Representations

How does Real theory generalise the real representation theory of G?

The group G admits the standard Real structure $G \leq G \times C_2$.

Proposition

The following categories are equivalent:

$$\mathbb{R}\text{-representations of } G \quad \overset{\cong}{\longleftrightarrow} \quad A\text{-representations of } G \leq G \times C_2 \,.$$

Antilinear Representations

Example

We can construct n pairwise non-isomorphic 1-dimensional representations of $C_n \leq D_{2n}$.

Let $C_n = \langle x \rangle$, and $D_{2n} = \langle x, b \rangle$.

So $x \cdot v = \zeta v$ and $b \cdot v = \overline{v}$.

These are all the irreducible representations of $C_n \leq D_{2n}$.

Note: the complex irreducible representations of D_{2n} are mostly two dimensional.

Antilinear Representations

Example

Consider $C_n \leq C_{2n} = \langle y \rangle$.

A representation boils down to a choice of A for the action y satisfying

$$(A\overline{A})^n = 1.$$

For example when n is even it turns out:

$$\mathbb{C} * C_{2n} \cong M_2(\mathbb{R}) \times \prod_{i=1}^{\frac{n-2}{2}} M_2(\mathbb{C}) \times \mathbb{H}.$$

Questions:

- When does a complex representation admit an extension to an antilinear representation?
- If it does, is this extension unique?
- Can we obtain all irreducible antilinear representations from a knowledge of complex representations?
- The endomorphism ring still falls into three cases - to what extent we can generalise the classical table?

Representations Over $\mathbb{R}$

$\text{End}_{\mathbb{R}G}(W)$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\text{End}_{\mathbb{C}G}(W_{\mathbb{C}})$	$\mathbb{C}$	$\mathbb{C} \times \mathbb{C}$	$M_2(\mathbb{C})$
$W_{\mathbb{C}}$	V	$V \oplus \overline{V}$	$V \oplus V$
$V_{\mathbb{R}}$	$W \oplus W$	W	W
$\dim_{\mathbb{R}} W$	n	$2n$	$4n$
$\dim_{\mathbb{C}} V$	n	n	$2n$
$V \cong \overline{V}$?	Yes	No	Yes
V Realisable?	Yes	No	No
$\exists$ G-invariant bil. form?	Yes (sym.)	No	Yes (alt.)
$\mathcal{F}_{\mathbb{C}}(V)$	1	0	-1

More specifically, what are the correct generalisations of:

- Complexification and Realification
- Realisability
- G-invariant forms
- The Frobenius-Schur indicator

Antilinear Representations

The main theory comes down to how the following rings interact.

$$\begin{array}{ccc} \mathbb{R}G & \hookrightarrow & \mathbb{C}G \\ \downarrow & & \downarrow \\ \mathbb{R}\widehat{G} & \hookrightarrow & \mathbb{C}*\widehat{G} \end{array}$$

$W \downarrow_{\mathbb{C}G}^{\mathbb{C}*\widehat{G}}$ is called the *Complexification of* W.

$V \uparrow_{\mathbb{C}G}^{\mathbb{C}*\widehat{G}}$ is called the *Realification of* V.

Antilinear Representations

Definition

A $\mathbb{C}G$-module V is called *realisable* if it is the restriction of some antilinear representation.

A necessary condition is that $V \cong \mathbf{w} \cdot \overline{V}$.

Antilinear Representations

We call a bilinear form B on V $\mathbf{w}$-*invariant* if

$$B(\mathbf{g}u, \mathbf{wgw}^{-1}v) = B(u, v) \text{ for all } \mathbf{g} \in G,\ u, v \in V\,.$$

We call a $\mathbf{w}$-invariant bilinear form B on V $\mathbf{w}$-*symmetric* if

$$B(u, \mathbf{w}^2 v) = B(u, v) \text{ for all } u, v \in V\,.$$

and $\mathbf{w}$-*alternating* if

$$B(u, \mathbf{w}^2 v) = -B(u, v) \text{ for all } u, v \in V\,.$$

Antilinear Representations

Theorem

Let W be an irreducible A-representation,

V an irreducible subrepresentation of $W\downarrow = W\downarrow^{\mathbb{C}*\widehat{G}}_{\mathbb{C}G}$.

Let $V\uparrow = V\uparrow^{\mathbb{C}*\widehat{G}}_{\mathbb{C}G}$ and $\mathbf{w}$ a fixed odd element.

Then W and V are as in one of the following table.

Antilinear Representations

$\mathrm{End}_A(W)$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathrm{End}_{\mathbb{C}G}(W)$	$\mathbb{C}$	$\mathbb{C} \times \mathbb{C}$	$M_2(\mathbb{C})$
$W\downarrow$	V	$V \oplus \mathbf{w} \cdot \overline{V}$	$V \oplus V$
$V\uparrow$	$W \oplus W$	W	W
$\dim_{\mathbb{C}} W$	n	$2n$	$2n$
$\dim_{\mathbb{C}} V$	n	n	n
$V \cong \mathbf{w} \cdot \overline{V}$?	Yes	No	Yes
V Realisable?	Yes	No	No
$\exists$ **w**-inv. bil. form?	Yes (**w**-sym.)	No	Yes (**w**-alt.)
$\mathcal{F}(V)$	1	0	-1

Antilinear Representations

Only bit to explain: bottom line.

$\mathcal{F}(V)$ is the *Real Frobenius-Schur indicator*.

$$\mathcal{F}(V) = \frac{1}{|G|} \sum_{\mathbf{z} \in \widehat{G} \setminus G} \chi(\mathbf{z}^2).$$

For the standard real structure this is the usual FS indicator.

How to relate this to the three types?

Antilinear Representations

The trick before doesn't work.

There is no analogous decomposition into symmetric and alternating squares.

How to get around this?

$$\begin{aligned}\mathcal{F}(V) &= \frac{2}{\left|\widehat{G}\right|}\sum_{\mathbf{g}\in\widehat{G}}\chi(\mathbf{g}^2) - \frac{1}{|G|}\sum_{\mathbf{g}\in G}\chi(\mathbf{g}^2) \\ &= \frac{1}{2}\widehat{\mathcal{F}}_{\mathbb{R}}(V\downarrow^{\mathbb{C}G}_{\mathbb{R}G}\uparrow^{\mathbb{R}\widehat{G}}_{\mathbb{R}G}) - \mathcal{F}_{\mathbb{C}}(V).\end{aligned}$$

Antilinear Representations

$$
\begin{array}{ccc}
\mathbb{R}G & \hookrightarrow & \mathbb{C}G \\
\big\downarrow & & \big\downarrow \\
\mathbb{R}\widehat{G} & \hookrightarrow & \mathbb{C}{*}\widehat{G}
\end{array}
$$

Antilinear Representations

The conjugation by $\mathbf{w}$ defines an automorphism ξ of all four algebras.

Let $e \in \mathbb{R}G$ be a central primitive idempotent.

Since $\mathbf{w}^2 \in G$, ξ^2 is an inner automorphism of $\mathbb{R}G$ and $\xi^2(e) = e$. There are two cases to consider:

- **unsplit case:** $\xi(e) = e$ so that $f := e$ is central in $\mathbb{C}{*}\widehat{G}$,
- **split case:** $\xi(e) \neq e$ so that $f := e + \xi(e)$ is central in $\mathbb{C}{*}\widehat{G}$.

Antilinear Representations

By *an antilinear block* we mean the below square, obtained from the central idempotent f:

$$\begin{array}{ccc} \mathcal{A} := f\mathbb{R}G & \hookrightarrow & \mathcal{C} := f\mathbb{C}G \\ \big\downarrow & & \big\downarrow \\ \mathcal{B} := f\mathbb{R}\widehat{G} & \hookrightarrow & \mathcal{D} := f\mathbb{C}*\widehat{G} \end{array}$$

Antilinear Representations

Theorem (Dyson's Theorem)

There are 10 possible structures of an antilinear-block.

	$\mathbb{F}_a$	$\mathbb{F}_b$	$\mathbb{F}_d$	$\lvert\mathcal{A}^\vee\rvert$	$\lvert\mathcal{B}^\vee\rvert$	$\lvert\mathcal{C}^\vee\rvert$	$\lvert\mathcal{D}^\vee\rvert$	$G \leq \widehat{G}$	S_c	DL
I	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	1	2	1	1	$C_1 \leq C_2$	$\mathbb{C}_{tr}$	RR
II	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	1	1	1	1	$C_2 \leq C_4$	$\mathbb{C}_{sn}$	QR
III	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	2	1	2	1	$K_4 \leq D_8$	$\mathbb{C}_+$	CR
IV	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	1	2	2	1	$C_3 \leq C_6$	$\mathbb{C}_{\mathbf{w}}$	$CC2$
V	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}$	1	1	2	2	$C_3 \leq D_6$	$\mathbb{C}_{\mathbf{w}}$	RC
VI	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$	1	1	2	2	$C_4 \leq Q_8$	$\mathbb{C}_i$	QC
VII	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	2	1	4	2	$C_8 \leq C_8 \rtimes C_2$	$\mathbb{C}_\alpha$	$CC1$
VIII	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{H}$	1	2	1	1	$Q_8 \leq Q_8 \times C_2$	$\mathbb{C}^2$	QQ
IX	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{R}$	1	1	1	1	$Q_8 \leq Q_8 \rtimes C_2$	$\mathbb{C}^2$	RQ
X	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$	2	1	2	1	$Q_8 \times C_2 \leq G_{32}^8$	$\mathbb{C}^2$	CQ

Antilinear Representations

Corollary

$\mathcal{F}(V)$ returns the right values.

Proof.

	I	II	III	IV	V	VI	VII	VIII	IX	X
$\mathbb{F}_d$	$\mathbb{R}$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{R}$	$\mathbb{C}$
$\widehat{\mathcal{F}}_{\mathbb{R}}(V\downarrow\uparrow)$	4	0	2	0	2	-2	0	-4	0	-2
$\mathcal{F}_{\mathbb{C}}(V)$	1	1	1	0	0	0	0	-1	-1	-1
$\mathcal{F}(V)$	1	-1	0	0	1	-1	0	-1	1	0

□

Antilinear Representations

$\mathrm{End}_A(W)$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathrm{End}_{\mathbb{C}G}(W)$	$\mathbb{C}$	$\mathbb{C} \times \mathbb{C}$	$M_2(\mathbb{C})$
$W\downarrow$	V	$V \oplus \mathbf{w} \cdot \overline{V}$	$V \oplus V$
$V\uparrow$	$W \oplus W$	W	W
$\dim_{\mathbb{C}} W$	n	$2n$	$2n$
$\dim_{\mathbb{C}} V$	n	n	n
$V \cong \mathbf{w} \cdot \overline{V}$?	Yes	No	Yes
V Realisable?	Yes	No	No
$\exists$ **w**-inv. bil. form?	Yes (**w**-sym.)	No	Yes (**w**-alt.)
$\mathcal{F}(V)$	1	0	-1

Antilinear Representations

Theorem

If W_1, W_2 are A-representations with A-characters χ_1, χ_2, then

$$\dim_{\mathbb{R}} \operatorname{Hom}_A(W_1, W_2) = \langle \chi_1, \chi_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product of class functions on G.

Antilinear Representations

Corollary

$\dim_{\mathbb{R}} Z(\mathbb{C} * \widehat{G}) = \#(\textit{Conjugacy Classes of } G)$.

Theorem

$\#(\textit{Irreducible A-Representations}) = \#(\textit{Real Conjugacy Classes})$.

Antilinear Representations

Let $\chi_1, \dots, \chi_n$ be all distinct irreducible complex characters of G.

Proposition

Define $r : G \to \mathbb{N}$ *by* $r(\mathbf{h}) = \#\{\mathbf{z} \in \widehat{G} \setminus G \mid \mathbf{z}^2 = \mathbf{h}\}$. *Then*

$$r(\mathbf{h}) = \sum_{j=1}^{n} \mathcal{F}(\chi_j)\chi_j(\mathbf{h})\,.$$

Corollary

If $G \leq \widehat{G}$ *has no A-representations of type* $\mathbb{H}$, *then* $r : G \to \mathbb{N}$ *attains its maximum value at the identity.*

Antilinear Representations

Let $\chi_1, \ldots, \chi_n$ be all distinct irreducible complex characters of G.

Proposition

Define $r' : G \to \mathbb{N}$ *by* $r'(\mathbf{h}) = \#\{\mathbf{z} \in G \mid \mathbf{z}^2 = \mathbf{h}\}$. *Then*

$$r'(\mathbf{h}) = \sum_{j=1}^{n} \mathcal{F}_{\mathbb{C}}(\chi_j)\chi_j(\mathbf{h})\,.$$

Corollary

If G *has no real representations of type* $\mathbb{H}$, *then* $r' : G \to \mathbb{N}$ *attains its maximum value at the identity.*

Antilinear Representations

Earlier, we saw the Real conjugacy classes of $A_n \leq S_n$.

All $\mathbb{R}S_n$-modules are of type $\mathbb{R}$.

Thus the only possible A-block structures are types I, III or V.

Antilinear Representations

	$\mathbb{F}_a$	$\mathbb{F}_b$	$\mathbb{F}_d$	$\lvert\mathcal{A}^\vee\rvert$	$\lvert\mathcal{B}^\vee\rvert$	$\lvert\mathcal{C}^\vee\rvert$	$\lvert\mathcal{D}^\vee\rvert$	$G \leq \widehat{G}$	S_c	DL
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VI	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$	1	1	2	2	$C_4 \leq Q_8$	$\mathbb{C}_i$	QC
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IX	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{R}$	1	1	1	1	$Q_8 \leq Q_8 \rtimes C_2$	$\mathbb{C}^2$	RQ
X	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$	2	1	2	1	$Q_8 \times C_2 \leq G_{32}^8$	$\mathbb{C}^2$	CQ

Antilinear Representations

For $A_n \leq S_n$, this

- Recovers the classical result that $\mathbb{R}A_n$ has no simple modules of quaternionic type.
- Tell us $\mathbb{C}{*}S_n$ has no simple modules of quaternionic type.

Antilinear Representations

If V is a complex representation of A_n, then we can consider V, $\overline{V}$, $\mathbf{w} \cdot V$ and $\mathbf{w} \cdot \overline{V}$.

In type I, $(\mathcal{F}_{\mathbb{C}}(V), \mathcal{F}(V)) = (1,1)$ and all four are isomorphic.

In type III, $(\mathcal{F}_{\mathbb{C}}(V), \mathcal{F}(V)) = (1,0)$ and $V \cong \overline{V} \ncong \mathbf{w} \cdot V \cong \mathbf{w} \cdot \overline{V}$.

In type V, $(\mathcal{F}_{\mathbb{C}}(V), \mathcal{F}(V)) = (0,1)$ and $V \cong \mathbf{w} \cdot \overline{V} \ncong \overline{V} \cong \mathbf{w} \cdot V$.

Antilinear Representations

It is well-known that $\mathbb{R}A_n$ does not have a simple module of type $\mathbb{C}$ if and only if $n \in \{2, 5, 6, 10, 14\}$.

We can understand this for $\mathbb{C}*S_n$ now.

Proposition

$A_n \leq S_n$ has no irreducible A-representation of complex type if and only if $n \in \{2, 3, 4, 7, 8, 12\}$.

Linear Representations

Linear Representations

${}^{-1}V = V^*$, ${}^{1}V = V$.

Definition

A *linear representation* of a C_2-graded group $\widehat{G}$ (or a Real group G) is a finitely dimensional $\mathbb{C}$-vector space V with invertible linear maps $\rho(\mathbf{z}) : {}^{\pi(\mathbf{z})}V \to V$ for all $\mathbf{z} \in \hat{G}$, such that $\rho(\mathbf{e}) = \mathbb{1}_V$, and

$$\rho(\mathbf{z}_2\mathbf{z}_1) = \rho(\mathbf{z}_2) \circ {}^{\pi(\mathbf{z}_2)}\rho(\mathbf{z}_1)^{\pi(\mathbf{z}_2)} \circ \mathrm{ev}^{\delta_{\pi(\mathbf{z}_1),\pi(\mathbf{z}_2),-1}}\,.$$

Linear Representations

Each odd element $\mathbf{w}$ defines a non-degenerate bilinear form

$$B_{\mathbf{w}} : V \times V \to \mathbb{K},\ B_{\mathbf{w}}(u, v) := \rho(\mathbf{w})^{-1}(v)(u).$$

In fact, if V is a $\mathbb{C}G$-module, suppose that for each $\mathbf{w} \in \widehat{G} \setminus G$ we have a non-degenerate bilinear form $B_{\mathbf{w}}$.

Then this defines a linear representation if and only if

- Each $B_{\mathbf{w}}$ is $\mathbf{w}$-invariant and $\mathbf{w}$-symmetric.
- $B_{\mathbf{w}_1}(u, v) = B_{\mathbf{w}_2}(u, \mathbf{w}_2\mathbf{w}_1^{-1} v)$ for all $\mathbf{w}_1, \mathbf{w}_2 \in \widehat{G} \setminus G$.

Linear Representations

There are two notions of morphism:

Weak: $\mathbb{C}G$-linear, with for one (hence all) $\mathbf{w} \in \widehat{G} \setminus G$,

$$B_{\mathbf{w}}(u, v) = B_{\mathbf{w}}(f(u), f(v)).$$

Strong: $\mathbb{C}G$-linear, with the the below diagram commuting.

$$\begin{array}{ccc} W^* & \xrightarrow{f^*} & V^* \\ {\scriptstyle \mu(\mathbf{z})}\downarrow & & \downarrow{\scriptstyle \rho(\mathbf{z})} \\ W & \xleftarrow[f]{} & V \end{array}$$

A weak morphism is strong if and only if it is bijective.

Linear Representations

Subrepresentation: $\mathbb{C}G$-submodule, with restrictions of form(s) non-degenerate.

Mashke's theorem holds, with the complement the orthogonal complement of the form.

Krull-Remak-Schmidt Theorem also holds.

Linear Representations

Equivalent categories?

Example

Consider $G = 1$, $V = \mathbb{C}$ the trivial A-representation.

Then $\mathrm{End}_A(V) = \{r\mathbb{1}_V \mid r \in \mathbb{R}^\times\}$.

The only odd element **w** has $B_{\mathbf{w}}$ the standard bilinear form: $\langle v, w\rangle = vw$.

Thus $f = c\mathbb{1}_V : V \to V$ preserves the form ($c \in \mathbb{C}$) if and only if

$$\langle 1, 1\rangle = \langle c \cdot 1, c \cdot 1\rangle = c^2\langle 1, 1\rangle.$$

So $\mathrm{End}_L(V) = \{\pm\mathbb{1}_V\}$.

Linear Representations

What do irreducible representations look like?

Proposition

One of the following mutually exclusive statements holds for an irreducible L-representation V.

(1) $V\downarrow_{\mathbb{C}G} = W$ is a simple $\mathbb{C}G$-module; $W \cong \mathbf{w}\cdot\overline{W}$ as $\mathbb{C}G$-modules; W is of antilinear type $\mathbb{R}$; $\mathrm{Aut}_L(V) = \{\pm\mathbb{1}\}$.

(2) $V\downarrow_{\mathbb{C}G} = W \oplus W'$ is the sum of two simple $\mathbb{C}G$-modules, both of antilinear type $\mathbb{C}$; $W \ncong W'$ and $W \ncong \mathbf{w}\cdot\overline{W}$ as $\mathbb{C}G$-modules; $\mathrm{Aut}_L(V) \cong \mathbb{C}\setminus 0$.

(3) $V\downarrow_{\mathbb{C}G} = W \oplus W'$ is the sum of two simple $\mathbb{C}G$-modules, both of antilinear type $\mathbb{H}$; $W \cong W'$ and $W \cong \mathbf{w}\cdot\overline{W}$ as $\mathbb{C}G$-modules; $\mathrm{Aut}_L(V) \cong \mathrm{SL}_2(\mathbb{C})$.

Linear Representations

So there is no hope for an equivalence.

However, note that as topological spaces, $\mathbb{R}^\times \simeq \{\pm 1\}$, and $\mathsf{SL}_2(\mathbb{C}) \simeq \mathbb{H}^\times$.

Motivates:

Theorem

The following pairs of ∞-categories are equivalent:

- $[\![\mathsf{Iso}(\mathcal{A}(G))]\!]$ *and* $[\![\mathsf{Iso}(\mathcal{L}(G))]\!]$,
- $[\![\mathsf{Mono}(\mathcal{A}(G))]\!]$ *and* $[\![\mathcal{L}(G)]\!]$.

Here $\mathcal{A}(G)$, $\mathcal{L}(G)$ are the antilinear and linear categories of representations respectively.

Example

Consider $G = 1$.

There is only the trivial antilinear representation.

A $\mathbf{w}$-invariant $\mathbf{w}$-symmetric bilinear form is just a symmetric bilinear form.

So the correspondence just recovers the familiar fact that any symmetric bilinear form over $\mathbb{C}$ is congruent to the identity.

Hermitian Representations

Hermitian Representations

${}^{-1}V = \overline{V}^*$, ${}^{1}V = V$.

Definition

A *hermitian representation* of a C_2-graded group $\widehat{G}$ (or a Real group G) is a finite dimensional $\mathbb{C}$-vector space V with invertible linear maps $\rho(\mathbf{z}) : {}^{\pi(\mathbf{z})}V \to V$ for all $\mathbf{z} \in \hat{G}$, such that $\rho(\mathbf{e}) = \mathbb{1}_V$, and

$$\rho(\mathbf{z}_2\mathbf{z}_1) = \rho(\mathbf{z}_2) \circ {}^{\pi(\mathbf{z}_2)}\rho(\mathbf{z}_1)^{\pi(\mathbf{z}_2)} \circ \mathrm{ev}^{\delta_{\pi(\mathbf{z}_1),\pi(\mathbf{z}_2),-1}}.$$

Hermitian Representations

Each odd element **w** defines a non-degenerate **sesquilinear** form

$$B_{\mathbf{w}} : V \times V \to \mathbb{K},\ B_{\mathbf{w}}(u, v) := \rho(\mathbf{w})^{-1}(v)(u)\,.$$

As before we have strong and weak morphisms, and Mashke's Theorem etc. holds.

Hermitian Representations

Proposition

Let V be an irreducible H-representation. One of the following mutually exclusive statements hold.

(1) $W := V\downarrow_{\mathbb{C}G}$ *is a simple* $\mathbb{C}G$*-module;* $W \cong \mathbf{w} \cdot W$ *as* $\mathbb{C}G$*-modules;* $\mathrm{Aut}_H(V) = \{\lambda \mathbb{1} \mid |\lambda| = 1\}$.

(2) $V\downarrow_{\mathbb{C}G} = W \oplus W'$ *decomposes as the sum of two simple* $\mathbb{C}G$*-modules;* $W \ncong W'$ *and* $W \ncong \mathbf{w} \cdot W$ *as* $\mathbb{C}G$*-modules;* $\mathrm{Aut}_H(V) \cong \mathbb{C} \setminus 0$.

This essential difference is due to the fact that **w**-invariant bilinear and sesquilinear forms behave differently under scaling.

Hermitian Representations

Relation between irreducible representations of $\mathbb{C}G$ and $\mathbb{C}\widehat{G}$.

Let V be an simple $\mathbb{C}\widehat{G}$-module.

Let W be an simple submodule of $V\downarrow_{\mathbb{C}G}$.

$V\downarrow$	W	$W \oplus \mathbf{w} \cdot W$
$W\uparrow$	$V \oplus (V \otimes \pi)$	V
$W \cong \mathbf{w} \cdot W$?	Yes	No
$V \cong V \otimes \pi$?	No	Yes

Hermitian Representations

The claim is that in the best way we can hope for; that hermitian representations are the same as $\mathbb{C}\widehat{G}$-modules.

Theorem

The following pairs of ∞-categories are equivalent:

(i) $[\![\mathrm{Iso}(\mathcal{R}(G))]\!]$ *and* $[\![\mathrm{Iso}(\mathcal{H}(G))]\!]$,

(ii) $[\![\mathrm{Mono}(\mathcal{R}(G))]\!]$ *and* $[\![\mathcal{H}(G)]\!]$.

Here $\mathcal{R}(G)$ and $\mathcal{H}(G)$ are the categories of $\mathbb{C}\widehat{G}$-modules and hermitian representations respectively.

Hermitian Representations

Example

Consider $G = 1$.

There are two irreducible representations of C_2.

A **w**-invariant **w**-symmetric sesquilinear form is just a hermitian inner product.

So the correspondence just recovers the familiar fact that any hermitian inner product over $\mathbb{C}$ is congruent to some

$$\begin{pmatrix} \mathbf{I_m} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I_n}. \end{pmatrix}$$

Further Directions

Thank you for listening.
Any questions?

References

D. Rumynin and J. Taylor (2020)
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