P4 Stress and Strain Dr. A.B. Zavatsky
MT07

Lecture 2

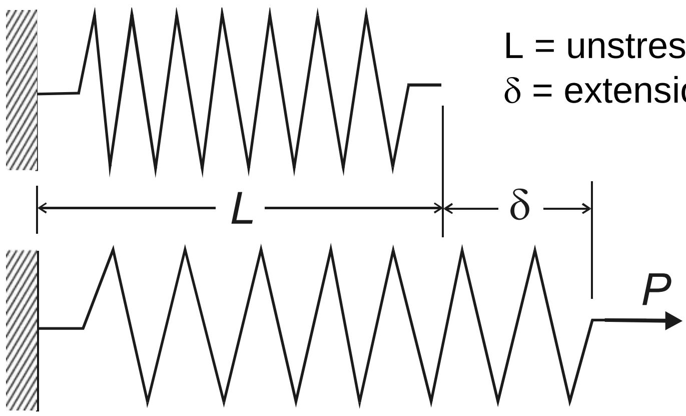
Axially-loaded Members

Stiffness and flexibility
Factor of safety, allowable stresses and loads
Changes in length under non-uniform conditions
(intermediate axial loads, prismatic segments, continuously varying loads or dimensions)
Elasto-plastic analysis

Axially-loaded Members

Structural components subjected only to tension or compression: coil springs, solid bars with straight longitudinal axes, cables, etc.

Coil springs – act primarily in shear or torsion, but overall stretching or shortening is analogous to a bar in tension or compression.



L = unstressed or relaxed length
δ = extension of spring under a load P

Stiffness and Flexibility

If the material of the spring is linearly elastic, the load *P* and elongation δ are proportional, or $P = k\,\delta$.

$k = P / \delta$ is the stiffness (or "spring constant") with units N/m
$f = \delta / P$ is the flexibility (or "compliance") with units m/N

For a prismatic bar,

$$\sigma = E\varepsilon \qquad \left(\frac{P}{A_o}\right) = E\left(\frac{\delta}{L_o}\right) \qquad \delta = \frac{PL_o}{EA_o}$$

$$k = \frac{P}{\delta} = \frac{EA_o}{L_o} \qquad f = \frac{\delta}{P} = \frac{L_o}{EA_o}$$

k and *f* play an important role in computational analysis of large structures, where they are assembled into stiffness and flexibility matrices for the entire structure.

Example

A rod made of aluminium alloy (*E* = 72 GPa) has length 500 mm and diameter 10 mm. What are its tensile stiffness and flexibility?

$$k = \frac{P}{\delta} = \frac{EA_o}{L_o} = \frac{\left(72\times10^9\right)\left(\pi\,(0.01/2)^2\right)}{0.5} = 11.3\times10^6\ \frac{\text{N}}{\text{m}}$$

$$\text{units}\ \frac{\left(\text{N/m}^2\right)\left(\text{m}^2\right)}{\text{m}} = \frac{\text{N}}{\text{m}}$$

$$f = \frac{1}{k} = 88.4\times10^{-9}\ \frac{\text{m}}{\text{N}}$$

Factor of Safety

The actual strength of a structure must exceed its required strength.

$$\text{Factor of safety } n = \frac{\text{Actual Strength}}{\text{Required Strength}}$$

$n > 1$ to avoid failure.

Usually n is chosen to be between 1 and 10.

Commercial aircraft: $1.2 < n < 1.5$
Military aircraft : $n < 1.1$ (but the crews wear parachutes!)
Missiles : $n = 1$ (not expected to return!)

For aircraft, small factors of safety are necessary to keep weight low and are justified by sophisticated modelling, testing of the actual materials used, extensive testing of prototype designs, and rigourous in-service inspections.
(*Machine Design: An Integrated Approach*, RL Norton, Prentice-Hall, 1996)

Estimating the 'required strength' can be challenging.

What loads will the wheel and frame of a bicycle experience?

Consider:
- age and weight of rider
- recklessness of the rider
- use on- or off-road
etc.

To what loads will a bridge be subjected?

Consider:
- weight and frequency of traffic
- wind loading
- water loading on piers
etc.

Are power stations flood-proof?
By Nick Higham, BBC News
Thursday, 15 November 2007, 08:29 GMT
http://news.bbc.co.uk/1/hi/uk/7095736.stm

…..
The Environment Agency - which maintains many of the seawalls along the Thames and Medway estuaries - seems equally confident.

It says the Thames defences were designed in the 1970s to cope, up until 2030, with the kind of floods that happen **once every 1000 years**. And the 1970s designers in fact **over-estimated** the rate of sea level rise, so the defences should still meet that standard well beyond 2030.
…
Last week the Met Office said that storm surges were likely to become higher and more frequent as the century progressed, thanks to climate change - and that floods that currently occur **once in every 100 years** on the East Coast could happen **once every 10 years** by the end of the century.
…
Ambiental, a company that specialises in flood modelling and risk assessment, says there is a residual risk that even defences built to cope with the kind of floods that occur **once every 200 or 250 years** can be overtopped or breached. **Human error - a floodgate left open, perhaps - can never be ruled out.**

Table 1-3 Factors Used to Determine a Safety Factor for Ductile Materials

Information	Quality of Information	Factor
		F1
	The actual material used was tested	1.3
Material property data available from tests	Representative material test data are available	2
	Fairly representative material test data are available	3
	Poorly representative material test data are available	5+
		F2
	Are identical to material test conditions	1.3
Environmental conditions in which it will be used	Essentially room-ambient environment	2
	Moderately challenging environment	3
	Extremely challenging environment	5+
		F3
	Models have been tested against experiments	1.3
Analytical models for loading and stress	Models accurately represent system	2
	Models approximately represent system	3
	Models are crude approximations	5+

For a ductile material, *n* can be estimated as max(F1, F2, F3).
Note that these are only guidelines!

(*Machine Design: An Integrated Approach*, RL Norton, Prentice-Hall, 1996)

Allowable Stresses and Loads

To avoid permanent deformation of a structure when the loads are removed, we try to load the structure only in the elastic region.

Hence, we can calculate an allowable stress based on the yield stress.

$$\text{Allowable stress (or working stress)} = \frac{\text{Yield Strength}}{\text{Factor of safety}} = \frac{\sigma_Y}{n}$$

For prismatic bars in direct tension or compression (no buckling), allowable loads or required areas can be found once allowable stresses are calculated.

$$P_{allow} = \sigma_{allow}\, A_o$$

Example

A steel bar with diameter 30 mm functions in tension as part of a truss. We do not want the bar to yield. An experienced design engineer recommends a safety factor of 2.5 for this application. What is the allowable load?

$$\sigma_{allow} = \frac{\sigma_Y}{n}$$

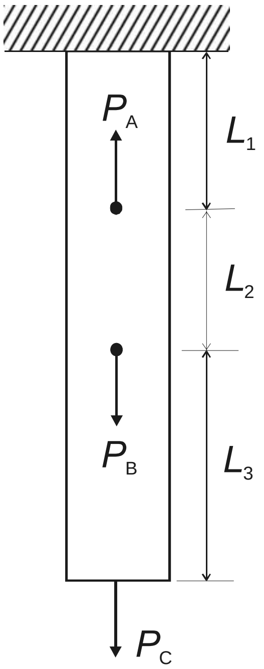
$$P_{allow} = \sigma_{allow}\, A_o = \frac{\sigma_Y A_o}{n}$$

σ_Y for steel $= 240$ MPa (HLT, page 41)

$$P_{allow} = \frac{\left(240 \times 10^6\right)\left(\pi\,(0.03/2)^2\right)}{2.5} = 67.9 \times 10^3\ \text{N} = 67.9\ \text{kN}$$

Changes in length of prismatic bars under non-uniform conditions:

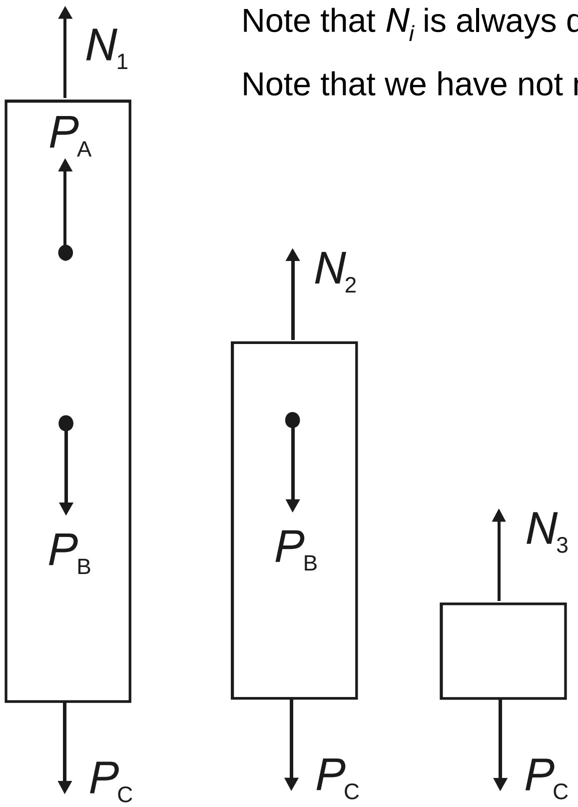
Bars with intermediate axial loads



The change in length of the bar δ can be determined by adding algebraically the elongations and shortenings of the individual segments.

Identify segments of the bar (i).

Draw free-body diagrams and determine internal axial forces N_i.



Note that N_i is always drawn as if the bar were in **tension**.

Note that we have not needed to include the reaction force.

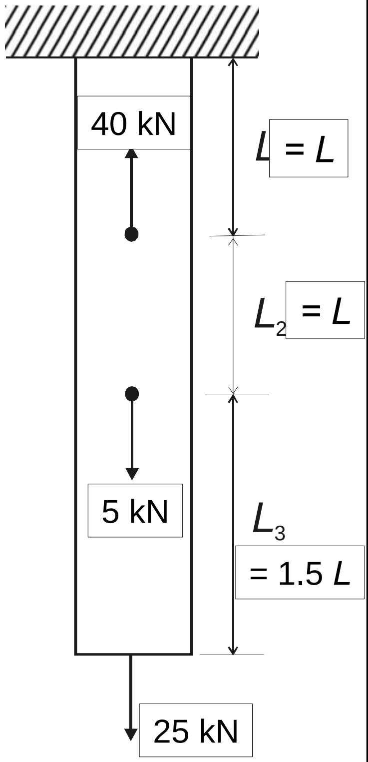
$$\sum F_1 = N_1 + P_A - P_B - P_C = 0$$
$$N_1 = P_B + P_C - P_A$$

$$\sum F_2 = N_2 - P_B - P_C = 0$$
$$N_2 = P_B + P_C$$

$$\sum F_3 = N_3 - P_C = 0$$
$$N_3 = P_C$$

Determine change in length δ_i for each segment.

$$\delta_1 = \frac{N_1 L_1}{EA} \qquad \delta_2 = \frac{N_2 L_2}{EA} \qquad \delta_3 = \frac{N_3 L_3}{EA}$$



Add the δ_i's algebraically to determine the overall change in length.

$$\delta = \delta_1 + \delta_2 + \delta_3$$

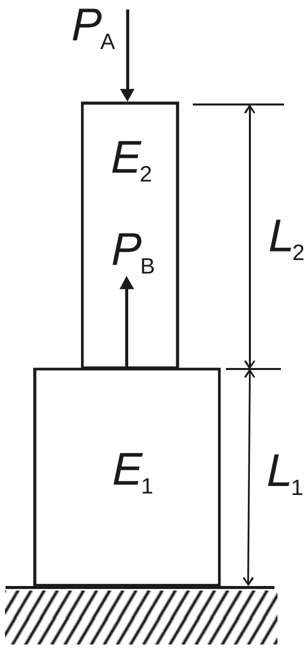
If P_A = 40 kN, P_B = 5 kN, P_C = 25 kN,
then N_1 = -10 kN, N_2 = 30 kN, N_3 = 25 kN.

If $L_1 = L_2 = (3/2)\ L_3 = L$,
then δ_1 = -10 r kN, δ_2 = 30 r kN, δ_3 = 37.5 r kN,
where $r = (L/AE)$.

So δ = 57.5 kN (L/AE)

Changes in length of prismatic bars under non-uniform conditions:

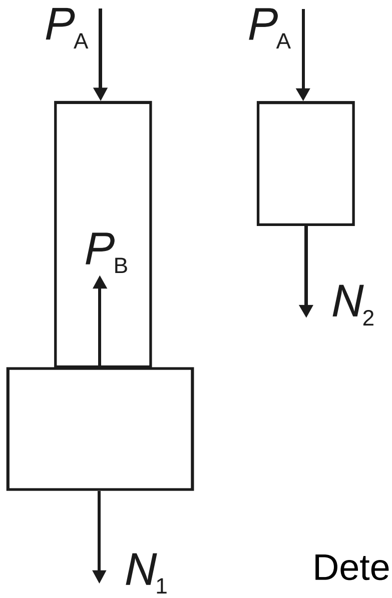
Bars consisting of prismatic segments



The same approach used for intermediate axial loads can be used here. Each segment of the bar has its own forces, dimensions, and material properties.

Identify segments of the bar (i).

Draw free-body diagrams and determine internal axial forces N_i.



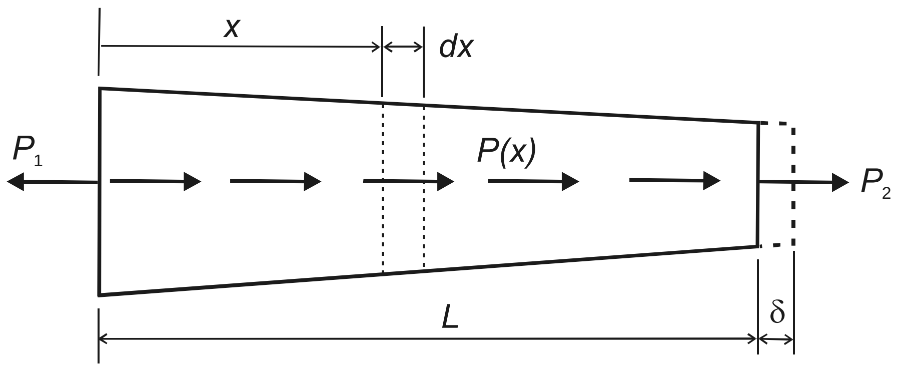
Write equilibrium equations and determine internal forces.

Determine change in length δ_i for each segment.

Add the δ_i's algebraically to determine the overall change in length.

Changes in length of prismatic bars under non-uniform conditions:

Bars with continuously varying loads and/or dimensions



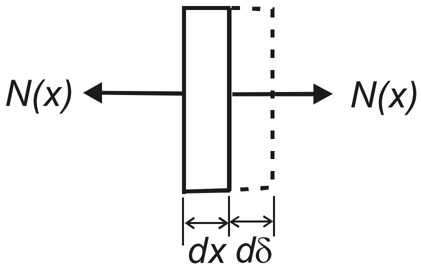
The bar has a cross-sectional area $A(x)$ that varies **gradually** along its length.

The bar is subjected to concentrated loads at its ends and a variable external load $P(x)$ distributed along its length (e.g. weight of a vertical bar or friction forces on the surface of the bar).

The bar is in equilibrium, so

$$\sum F = -P_1 + P_2 + \int_o^L P(x)\,dx = 0$$

Consider a differential element of the bar.



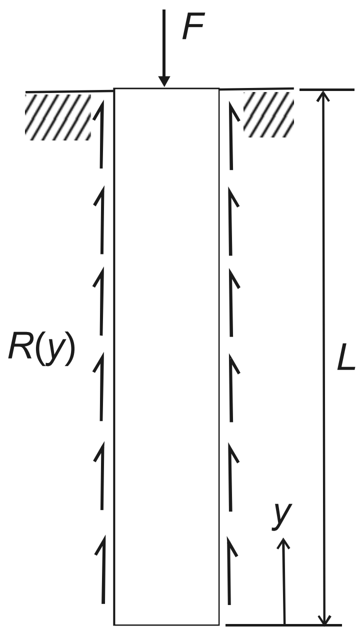
$$\sigma = \frac{N(x)}{A(x)} \quad \text{and} \quad \varepsilon = \frac{d\delta}{dx}$$

$$\sigma = E\,\varepsilon \quad \text{so} \quad \frac{N(x)}{A(x)} = E\,\frac{d\delta}{dx}$$

$$d\delta = \frac{N(x)\,dx}{A(x)\,E} \quad \text{and} \quad \delta = \int_0^L \frac{N(x)\,dx}{A(x)\,E}$$

Example

(based on Q4-24 in *Introduction to Mechanics of Solids*, EP Popov)



A uniform timber pile which has been driven to a depth L in clay carries an applied load F at the top. This load is resisted **entirely** by the friction force $R(y) = k\,y^2$ along the pile.

Determine the overall shortening of the pile in terms of F, L, A, and E.

Calculate the amount of shortening when F = 425 kN, L = 12 m, $A = 65 \times 10^{-3}$ m^2, E = 10 GPa.

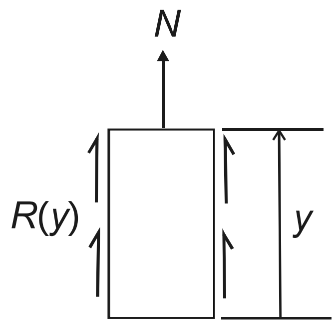
Use overall equilibrium of the timber pile to find the constant *k*.

$$\int_0^L R(y)\,dy - F = 0$$

$$\int_o^L ky^2\,dy - F = 0$$

$$F = \frac{kL^3}{3} \quad \text{and} \quad k = \frac{3F}{L^3}$$

Consider equilibrium at a section through the pile:



$$N + \int_0^y R(y)\,dy = 0$$

$$N + \int_0^y ky^2\,dy = 0$$

$$N = -\frac{ky^3}{3}$$

Integrate to find the change in length of the pile:

$$\delta = \int_0^L \frac{N(y)\,dy}{AE} = \int_0^L \left(\frac{-ky^3}{3}\right)\left(\frac{dy}{AE}\right)$$

$$\delta = \frac{-FL}{4AE}$$ (negative sign indicates compression)

Substituting numerical values for F,L,A,E gives:

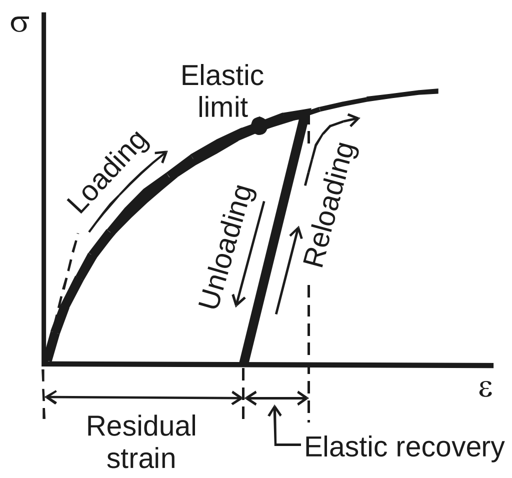
$$\delta = \frac{FL}{4AE} = \frac{\left(425\times10^3\right)(12)}{4\left(65\times10^{-3}\right)\left(10\times10^9\right)} = 1.96\times10^{-3}\text{ m} = 1.96\text{ mm}$$

Double-checking to see how the loads compare with the failure load:

$$\sigma = \frac{F}{A} = \frac{425\times10^3}{65\times10^{-3}} = 6.54\times10^6\text{ Pa} = 6.54\text{ MPa}$$

$$\sigma_{\text{failure}}\text{ (compression)} = 20-80\text{ MPa (HLT)}$$

Elasto-plastic analysis of bars

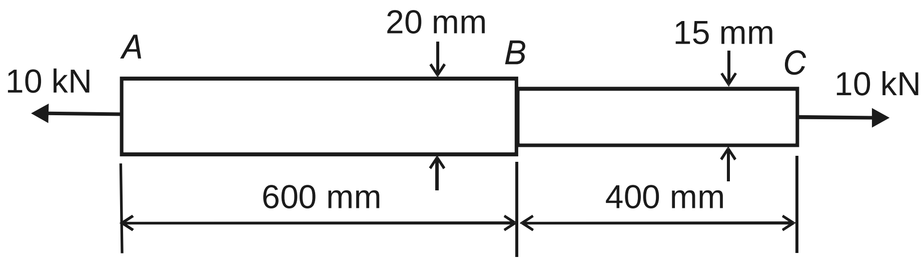


After unloading, there is a certain amount of **elastic recovery** and some **residual strain**, that is, a permanent elongation of the specimen. Upon reloading, the unloading curve is followed.

Example

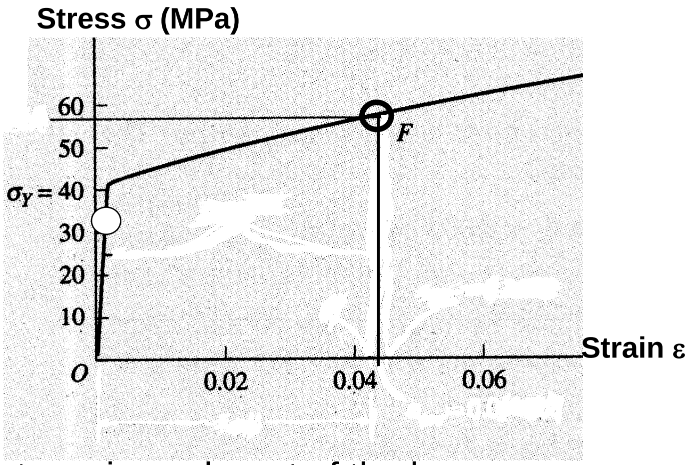
(based on Example 9.3 in *Statics and Mechanics of Materials*, RC Hibbeler)

The aluminium rod shown is subjected to an axial load. Each segment has a circular cross-section.



The stress-strain curve for the material is given (next slide).
E = 70 GPa and σ_Y = 40 MPa.

Determine the elongation of the rod when the load is applied.



Calculate the stress in each part of the bar:

$$\sigma_{AB} = \frac{P}{A_{AB}} = \frac{10 \times 10^3}{\pi\,(0.020/2)^2} = 31.83 \times 10^6 \text{ Pa} = 31.83 \text{ MPa}$$

$$\sigma_{BC} = \frac{P}{A_{BC}} = \frac{10 \times 10^3}{\pi\,(0.015/2)^2} = 56.59 \times 10^6 \text{ Pa} = 56.59 \text{ MPa}$$

Region AB is strained elastically, since σ_Y = 40 MPa > 31.83 MPa
Region BC is strained plastically, since σ_Y = 40 MPa < 56.59 MPa

The elongation of region AB is:

$$\varepsilon_{AB} = \frac{\sigma_{AB}}{E} = \frac{31.83 \times 10^6}{70 \times 10^9} = 454.71 \times 10^{-6}$$

$$\delta_{AB} = \varepsilon_{AB} L_{AB} = \left(454.71 \times 10^{-6}\right)(0.6) = 272.83 \times 10^{-6}\,\text{m} = 0.273\,\text{mm}$$

The elongation of region BC must be found with reference to the stress-strain diagram. When σ = 56.6 MPa, ε_{BC} = 0.0450.

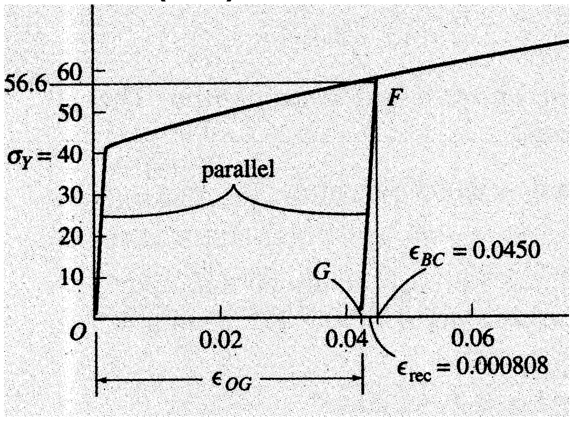
$$\delta_{BC} = \varepsilon_{BC} L_{BC} = (0.0450)(0.4) = 18.00 \times 10^{-3}\,\text{m} = 18.00\,\text{mm}$$

So, the overall change in length of the bar is

$$\delta = \delta_{AB} + \delta_{BC} = 0.27\,\text{mm} + 18.00\,\text{mm} = 18.27\,\text{mm}$$

If the load is now removed, what is the permanent elongation of the rod?

Stress σ (MPa)



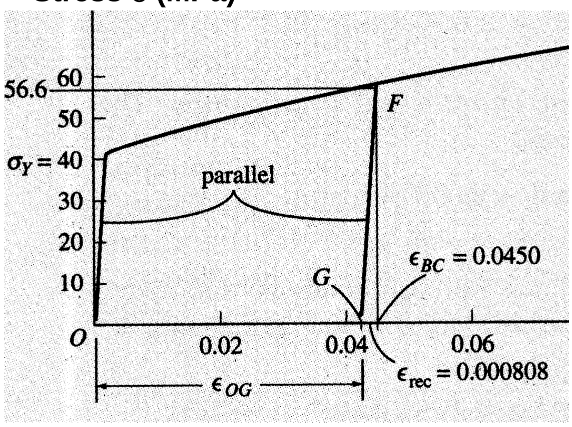
Strain ε

Because segment AB was loaded elastically, it return to its original length when the load is removed.

Because segment BC was loaded into the plastic region, it will recover along line FG. The elastic strain recovered is

$$\varepsilon_{rec} = \frac{\sigma_{BC}}{E} = \frac{56.59 \times 10^6}{70 \times 10^9} = 0.000808$$

Stress σ (MPa)



Strain ε

The remaining plastic strain in BC is then

$$\varepsilon_{OG} = 0.0450 - 0.000808 = 0.0442$$

So, when the load is removed, the rod remains elongated by

$$\delta_{permanent} = \varepsilon_{OG}\, L_{BC} = 0.0442\,(400\text{ mm}) = 17.7\text{ mm}$$