Dr. A.B. Zavatsky
MT07

Lecture 3

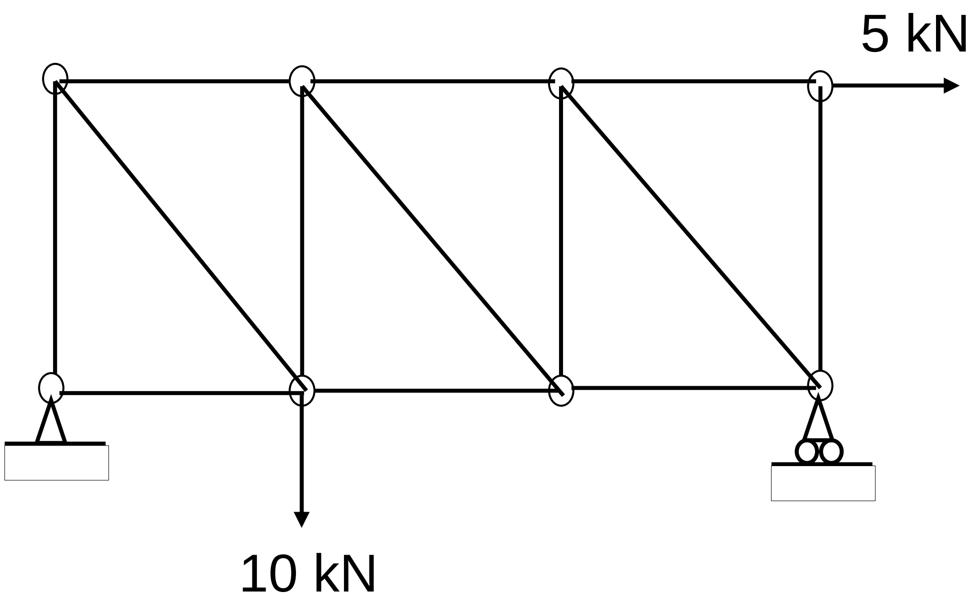
Statically Indeterminate Structures

Statically determinate structures.
Statically indeterminate structures (equations of equilibrium, compatibility, and force-displacement; use of displacement diagrams)
Bolts and turnbuckles.
Temperature effects.
Misfits and pre-strains.

Statically Determinate Structures

• Reactions and internal forces **can** be determined solely from free-body diagrams and equations of equilibrium.

• Results are **independent** of the material from which the structure has been made.



Unknowns
= reaction forces + bar forces
= (2 + 1) + 13 = 16

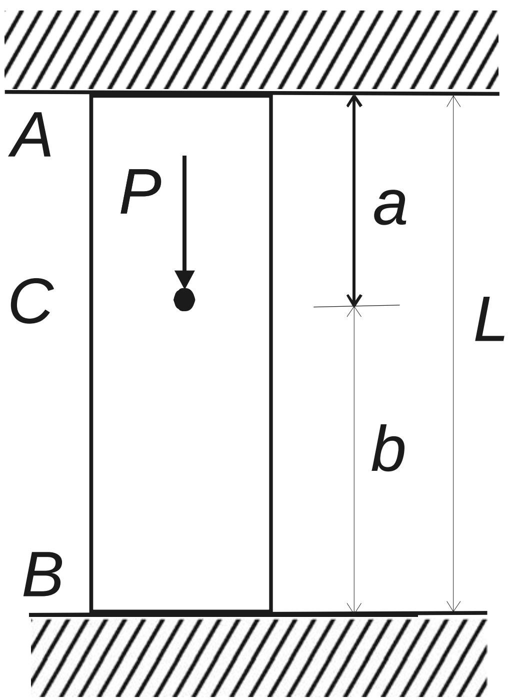
Independent equations
[equilibrium in x & y directions at each joint]
= 2 (number of joints)
= 2 (8) = 16

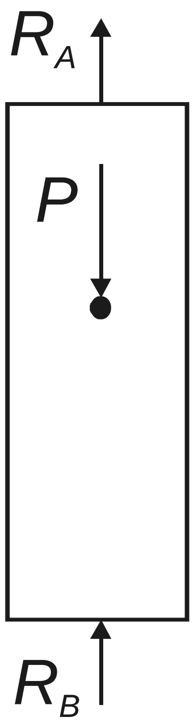
Double-check structure for internal mechanisms, etc.

Statically Indeterminate Structures

• Reactions and internal forces **cannot** be found by statics alone (more unknown forces than independent equations of equilibrium).

• Results are **dependent** on the material from which the structure has been made.





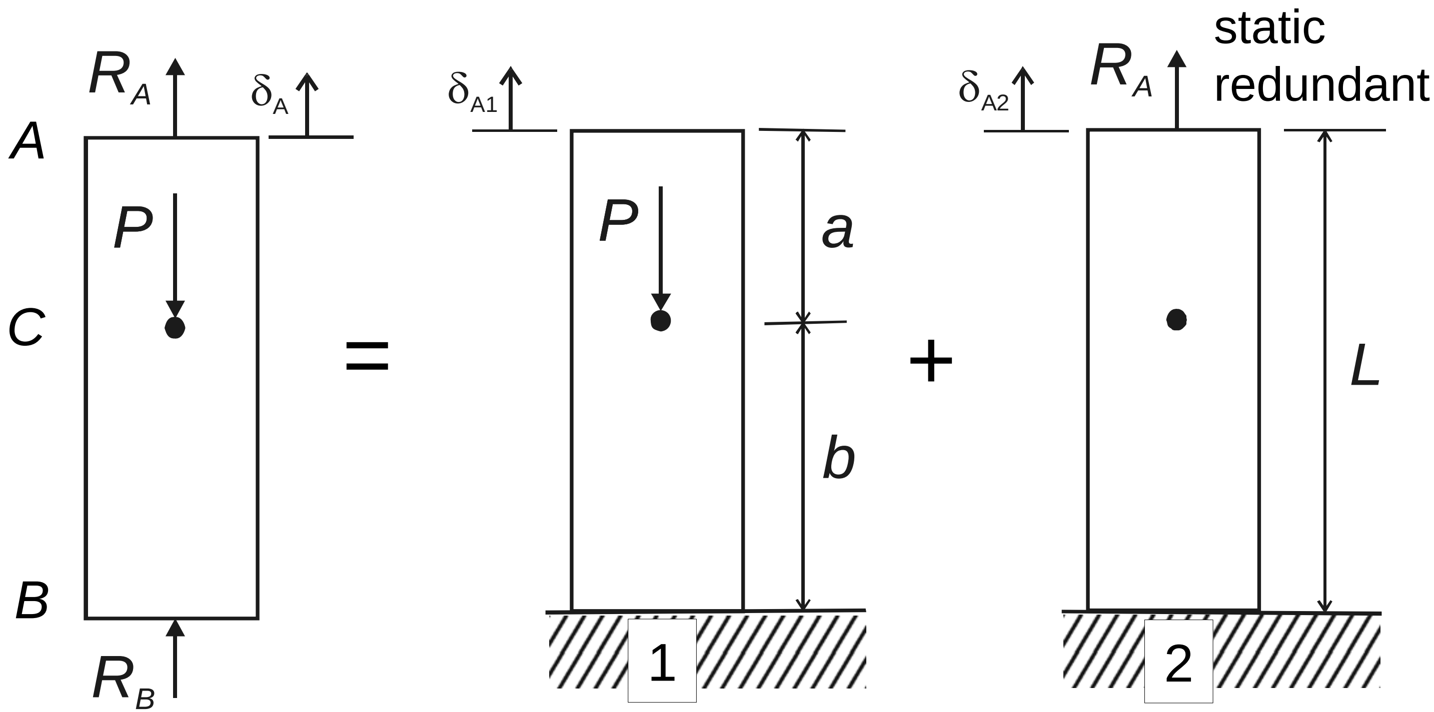
2 unknown forces

Only 1 useful equation of equilibrium

$$R_A - P + R_B = 0$$

Need to find another equation

"Flexibility" or "Force" Method



"released" structure

Equation of compatibility – expresses the fact that the change in length of the bar must be compatible with the conditions at the supports

$$\delta_A = \delta_{A1} + \delta_{A2} = 0$$

Write the force-displacement relations. These take the mechanical properties of the material into account.

$$\delta_{A1} = \frac{-Pb}{EA} \quad \text{and} \quad \delta_{A2} = \frac{R_A L}{EA}$$

Substituting into the equation of compatibility gives:

$$\delta_A = \frac{-Pb}{EA} + \frac{R_A L}{EA} = 0$$

$$R_A = \frac{Pb}{L}$$

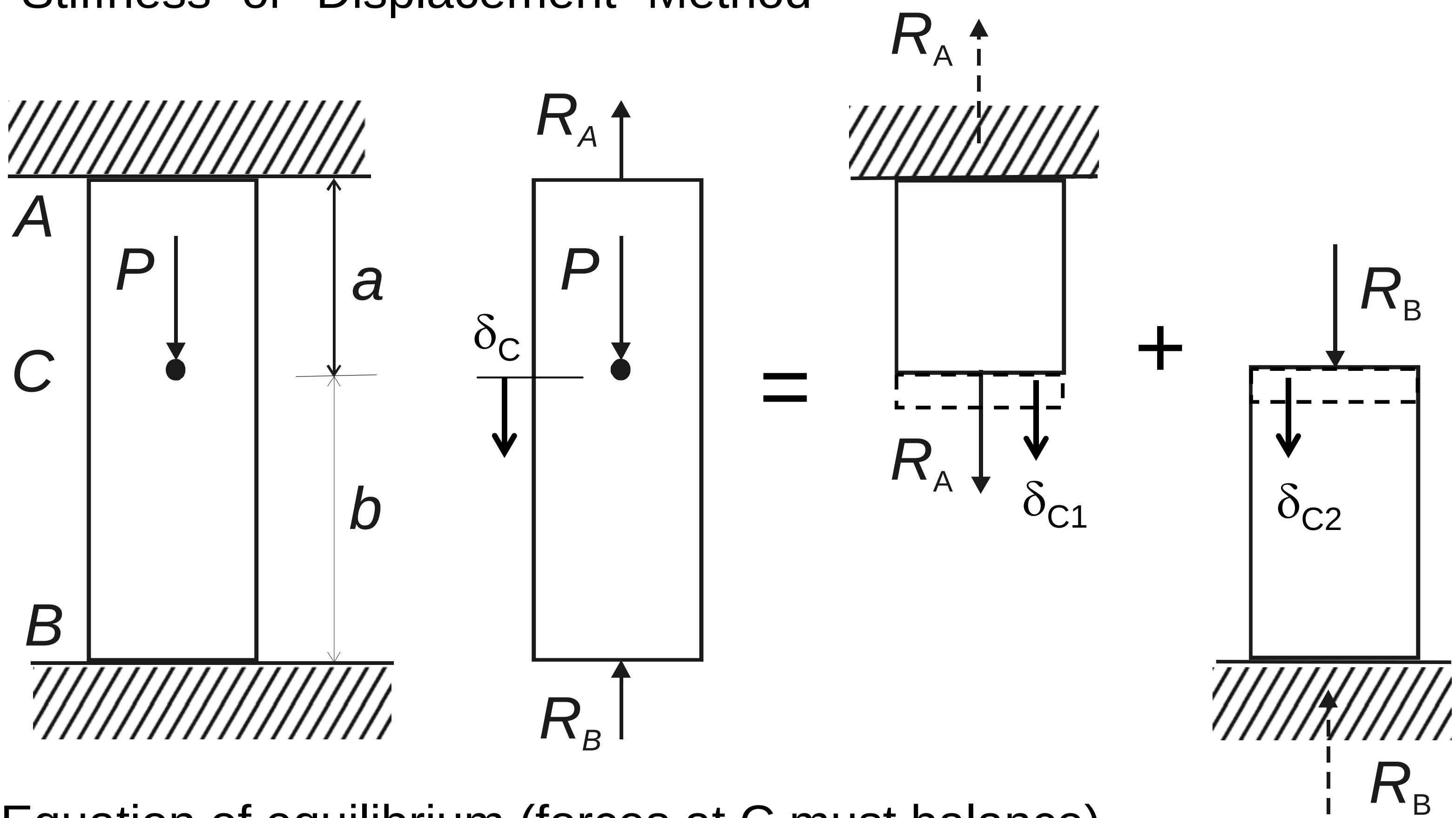
*Note that flexibilities (*b*/*EA*) and (*L*/*EA*) appear in this equation. Hence, this approach is called the “flexibility” method.

Substituting into the equilibrium equation gives:

$$R_B = P - R_A = \frac{Pa}{L}$$

*Note that we have solved for forces. Hence, this approach is also called the “force” method.

“Stiffness” or “Displacement” Method



Equation of equilibrium (forces at C must balance)

$$R_A + R_B = P$$

Equation of compatibility (at point C) $\delta_C = \delta_{C1} = \delta_{C2}$

Write the force-displacement relations and solve for the forces.

$$\delta_{C1} = \frac{R_A a}{EA} \quad \text{and} \quad \delta_{C2} = \frac{R_B b}{EA}$$

$$R_A = \frac{\delta_{C1} EA}{a} \quad \text{and} \quad R_B = \frac{\delta_{C2} EA}{b}$$

Substituting into the equilibrium equation gives:

$$\frac{\delta_{C1} EA}{a} + \frac{\delta_{C2} EA}{b} = P$$

*Note that stiffnesses (EA/a) and (EA/b) appear in this equation. Hence, this approach is called the “stiffness” method.

Using the compatibility condition (displacements equal) gives:

$$\delta_C = \frac{Pab}{EA(a+b)} = \frac{Pab}{EAL}$$

*Note that we have solved for displacement. Hence, this approach is also called the “displacement” method.

Finally, substituting into the expressions for forces gives:

$$R_A = \left(\frac{Pab}{EAL}\right)\left(\frac{AE}{a}\right) = \frac{Pb}{L}$$

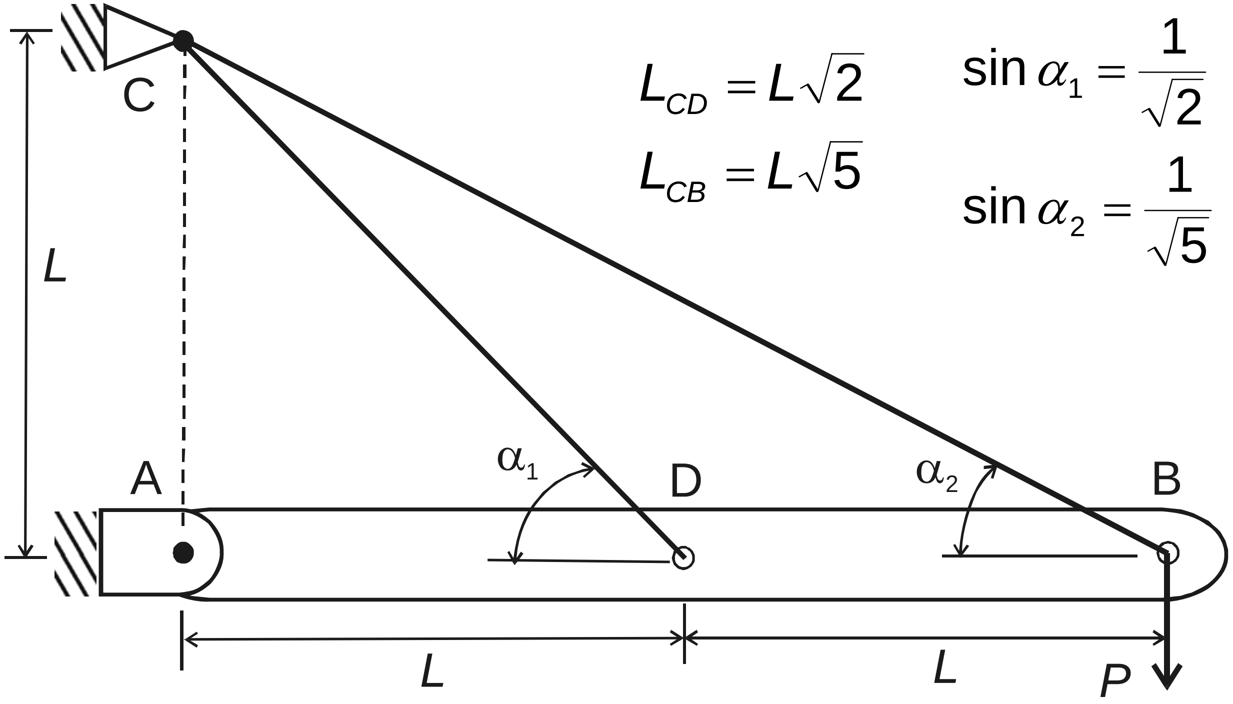
$$R_B = \left(\frac{Pab}{EAL}\right)\left(\frac{AE}{b}\right) = \frac{Pa}{L}$$

So, both the flexibility method and the stiffness method give the same result.

The choice of approach will depend on the problem being solved.

Use of Displacement Diagrams in Statically Indeterminate Problems

(based on Example 3, page 70, Gere & Timoshenko)



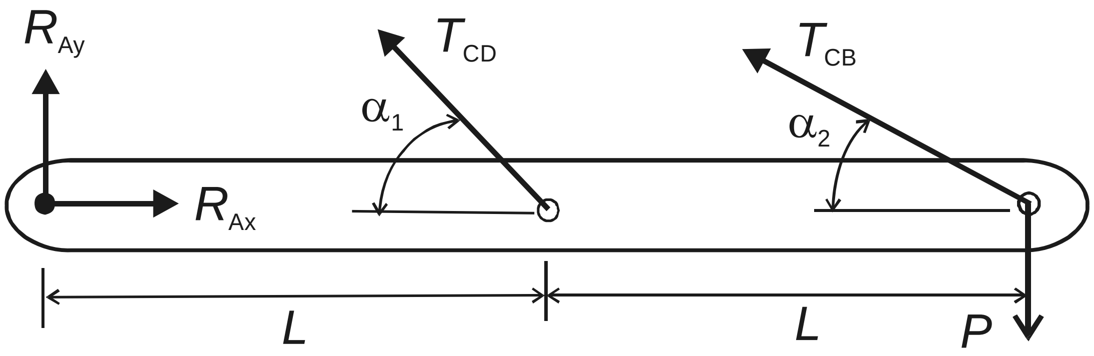
$$L_{CD} = L\sqrt{2}$$

$$L_{CB} = L\sqrt{5}$$

$$\sin\alpha_1 = \frac{1}{\sqrt{2}}$$

$$\sin\alpha_2 = \frac{1}{\sqrt{5}}$$

Bar ADB is supported by two wires, CD and CB. A load *P* is applied at B. The wires have axial rigidity *EA*. Disregarding the weight of the bar, find the forces in the wires.



Equilibrium

Horizontal Direction $\sum F_x = R_{Ax} - T_{CD} \cos \alpha_1 - T_{CB} \cos \alpha_2 = 0$

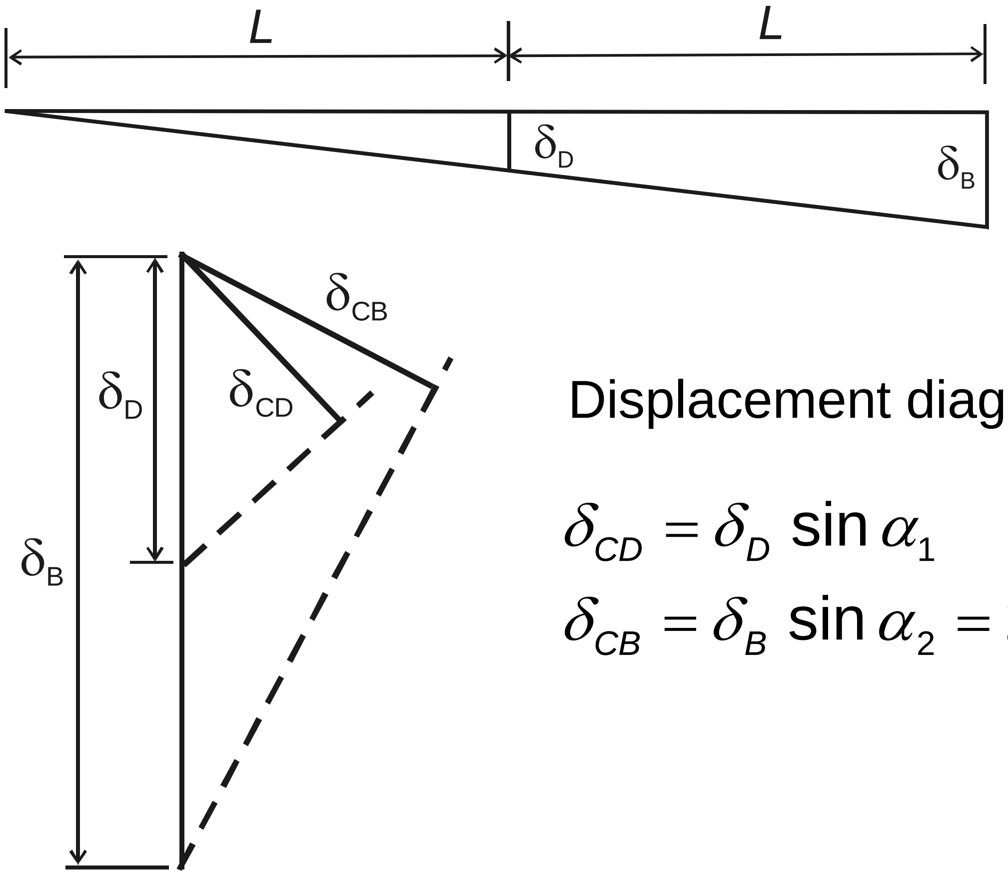
Vertical Direction $\sum F_y = R_{Ay} + T_{CD} \sin \alpha_1 + T_{CB} \sin \alpha_2 - P = 0$

Moments about A (ccw +)

$$\sum M_A = \left(T_{CD} \sin \alpha_1\right)L + \left(T_{CB} \sin \alpha_2\right)2L - P\left(2L\right) = 0$$

4 unknown forces, only 3 equations

Compatibility
We can relate the tensions in the two wires by considering the extensions of the wires.



$$\delta_B = 2\delta_D$$

Displacement diagram

$$\delta_{CD} = \delta_D \sin\alpha_1$$

$$\delta_{CB} = \delta_B \sin\alpha_2 = 2\delta_D \sin\alpha_2$$

Force-displacement relations

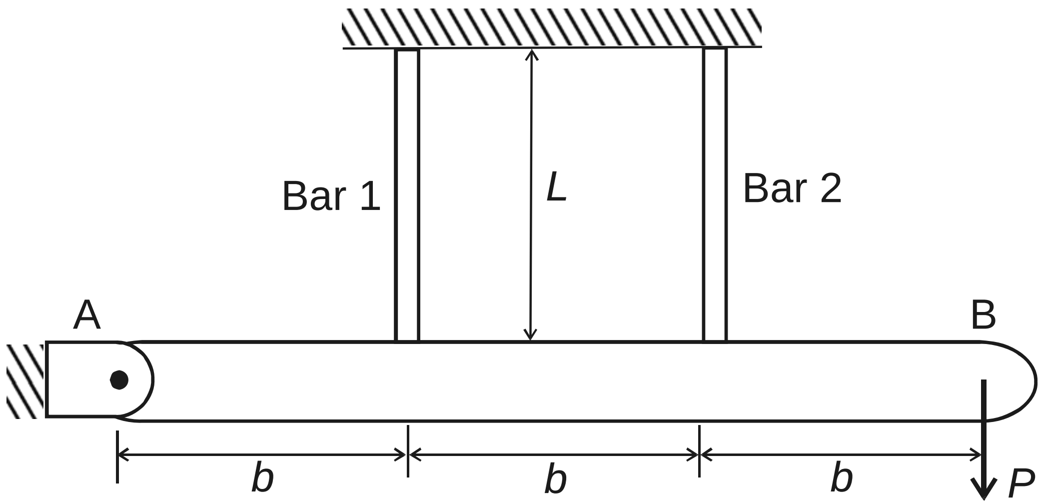
$$\delta_{CD} = \frac{T_{CD} L_{CD}}{AE} \quad \text{and} \quad \delta_{CB} = \frac{T_{CB} L_{CB}}{AE}$$

Using the equilibrium moment equation, the compatibility equation, and the force-displacement relations, it is possible to solve for the forces in the wires. We find that

$T_{CB} = 1.125\ P$ and $T_{CD} = 1.406\ P$

Elasto-plastic Analysis of a Statically Indeterminate Structure

(based on Example 2-19 from Gere)

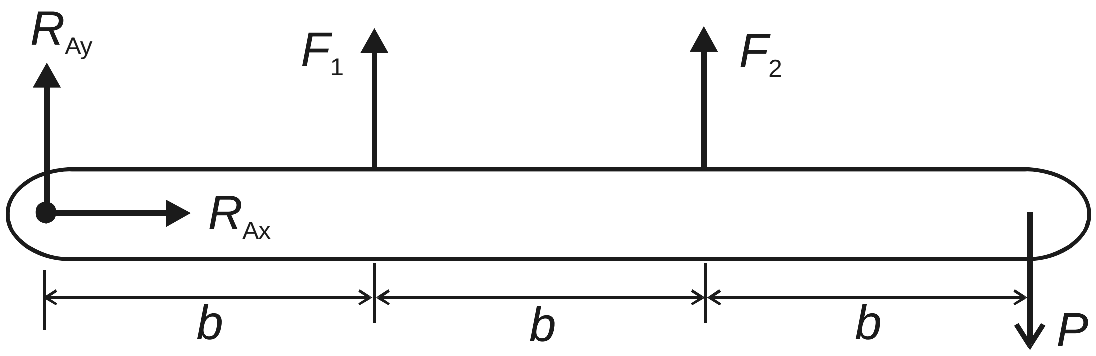


Horizontal beam AB is rigid. Supporting bars 1 and 2 are made of an elastic perfectly plastic material with yield stress σ_Y, yield strain ε_Y, and Young's modulus $E = \sigma_Y / \varepsilon_Y$. Each bar has cross-sectional area A.

(a) Find the yield load P_Y and the corresponding yield displacement Δ_{BY} at point B.

(b) Find the plastic load P_P and the corresponding plastic displacement Δ_{BP} at point B.

(c) Draw a load-displacement diagram relating the load P to the displacement Δ_B of point B.



Moment Equilibrium (ccw +)

$$\sum M_A = (F_1)b + (F_2)2b - P(3b) = 0$$

$$F_1 + 2F_2 = 3P$$

Compatibility

$$\delta_2 = 2\delta_1$$

Force-displacement

$$\delta_1 = \frac{F_1 L}{AE} \quad \text{and} \quad \delta_2 = \frac{F_2 L}{AE}$$

$$\Rightarrow$$

$$F_1 = \frac{3P}{5}$$

$$F_2 = \frac{6P}{5}$$

Bar 2 will yield first, since $F_2 > F_1$.

$$F_2 = \frac{6P_Y}{5} = \sigma_Y A$$

$$P_Y = \frac{5\sigma_Y A}{6} \quad \text{(a)}$$

The corresponding elongation of bar 2 is:

$$\delta_2 = \frac{F_2 L}{AE} = \left(\frac{F_2}{A}\right)\left(\frac{L}{E}\right) = \frac{\sigma_Y L}{E}$$

The downward displacement of the bar at point B is:

$$\Delta_{BY} = \frac{3\delta_2}{2} = \frac{3\sigma_Y L}{2E} \quad \text{(a)}$$

When the plastic load P_P is reached, both bars will be stretched to the yield stress, and $F_1 = F_2 = \sigma_Y A$. From equilibrium,

$$F_1 + 2F_2 = 3P$$

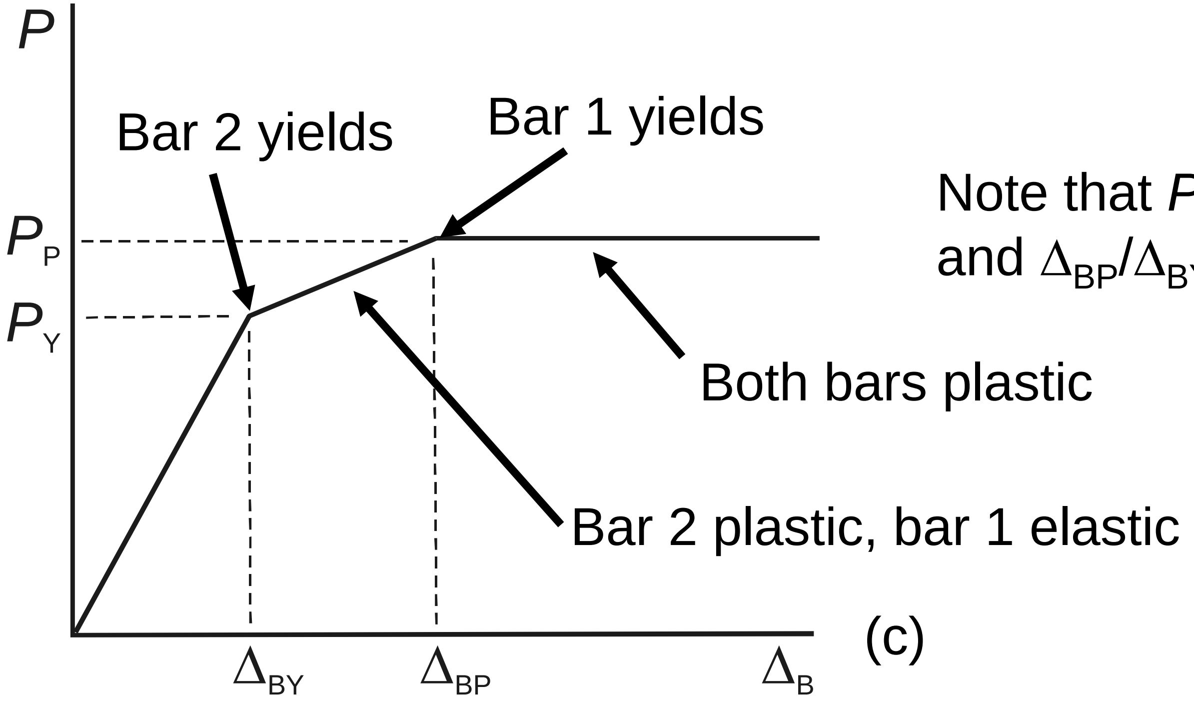
$$P_P = \sigma_Y A \quad \text{(b)}$$

The corresponding elongation of bar 1 (which has just reached yield) is:

$$\delta_1 = \frac{F_1 L}{AE} = \left(\frac{F_1}{A}\right)\left(\frac{L}{E}\right) = \frac{\sigma_Y L}{E}$$

The downward displacement of the bar at point B is:

$$\Delta_{BP} = 3\delta_1 = \frac{3\sigma_Y L}{E} \quad \text{(b)}$$



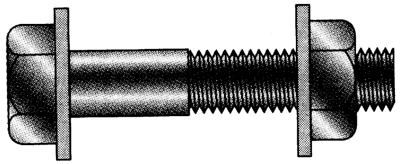
Note that $P_P/P_Y = 6/5$ and $\Delta_{BP}/\Delta_{BY} = 2$

(c)

Bolts and Turnbuckles

The simplest way to produce a change in length.

Nut and Bolt

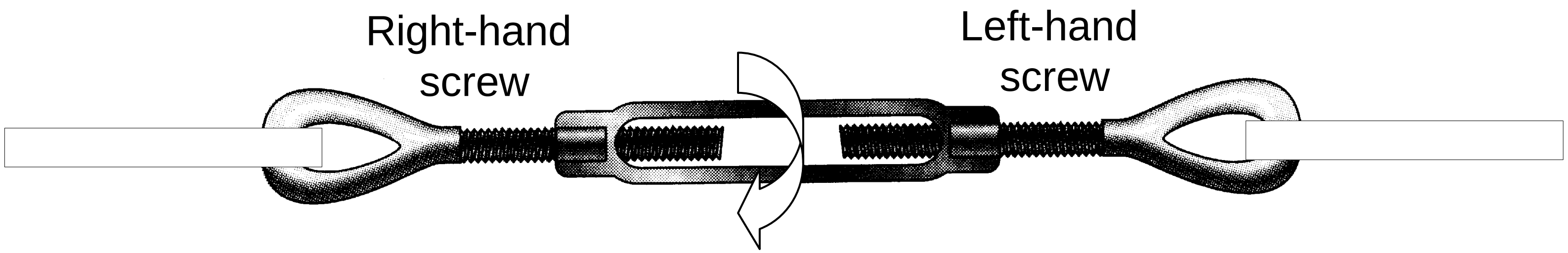


Distance δ travelled by the nut = $n\ p$

n = number of turns (not necessarily an integer)

p = pitch of the screw (units mm / turn)

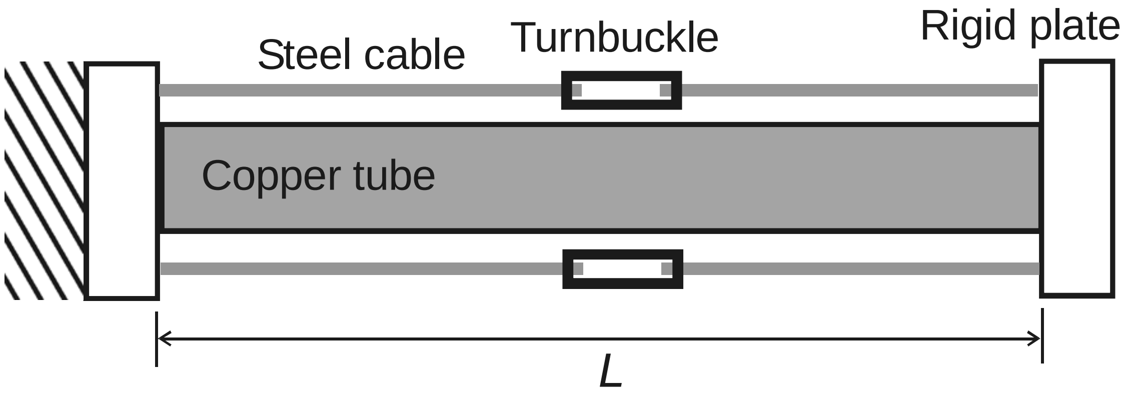
Turnbuckle



Distance δ travelled = $2\ n\ p$

Often used to tension cables

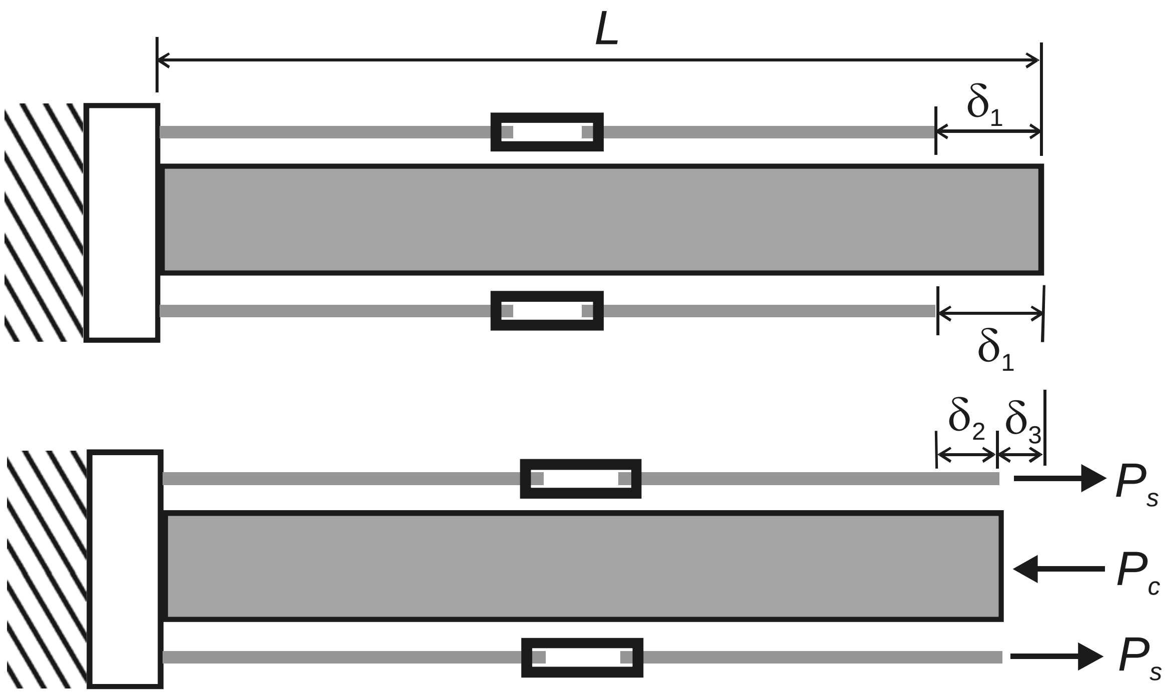
Based on Gere, Example 2-9



The slack is removed from the cables by rotating the turnbuckles until the assembly is snug but with no initial stresses (do not want to stretch the cables and compress the tube).

Find the forces in the tube and cables when the turnbuckles are tightened by *n* turns, and determine the shortening of the tube.

If the turnbuckles are rotated through *n* turns, the cables will shorten by a distance $\delta_1 = 2\,n\,p$.



The tensile forces in the cables P_s and the compressive force in the tube P_c must be such that the final lengths of the cables and tube is the same.

Equilibrium (forces must balance) $2P_s - P_c = 0$

Compatibility (shortening of tube must equal shortening of cable)

$$\delta_3 = \delta_1 - \delta_2$$

Force-displacement

$$\delta_1 = 2np$$

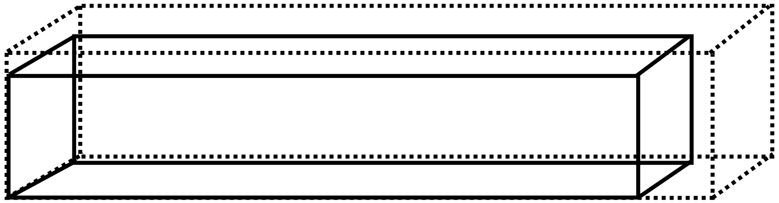
$$\delta_2 = \frac{P_s L}{E_s A_s}$$

$$\delta_3 = \frac{P_c L}{E_c A_c}$$

With these equations, we can solve for the forces in the tube and cables and for the shortening of the tube.

Temperature Effects

Changes in temperature produce expansion or contraction of structural materials.



When heated, the block expands in all three directions: x, y, z.

For most structural materials, $\varepsilon_T = \alpha\ (\Delta T)$

ε_T = thermal strain
α = coefficient of thermal expansion (HLT, units 1/K or 1/°C)
ΔT = change in temperature

The change in length of the block in ANY direction can be found using $\delta_T = \varepsilon_T\, L = \alpha\ (\Delta T)\ L$, where L is one of the block's dimensions.

Expansion is positive; contraction is negative.

A relatively modest change in temperature produces about the same magnitude of strain as that caused by ordinary working stresses. This shows that temperature effects can be important in engineering design.

For mild steel, $\alpha = 11 \times 10^{-6}\ K^{-1}$ (HLT, page 39)

Thermal strain:
$\varepsilon_T = \alpha\ (\Delta T) = 11 \times 10^{-6}\ K^{-1}\ (40\ K) = 0.00044 = 440\ \mu\varepsilon$

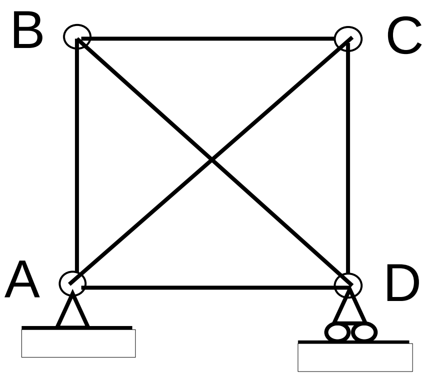
Yield strain:
$\varepsilon_Y = \sigma_Y / E = (240 \times 10^6\ Pa)/(210 \times 10^9\ Pa) = 0.00114 = 1140\ \mu\varepsilon$
Ordinary working stress might be 50% of the yield stress, giving 570 $\mu\varepsilon$

Thermal strains are USUALLY reversible.

Free expansion or contraction occurs when an object rests on a frictionless surface or hangs in open space. Then no stresses are produced by a uniform temperature change, but there are strains.

In statically determinate structures, **uniform** temperature changes in the members produce thermal strains (and corresponding changes in length) without producing any corresponding stresses.

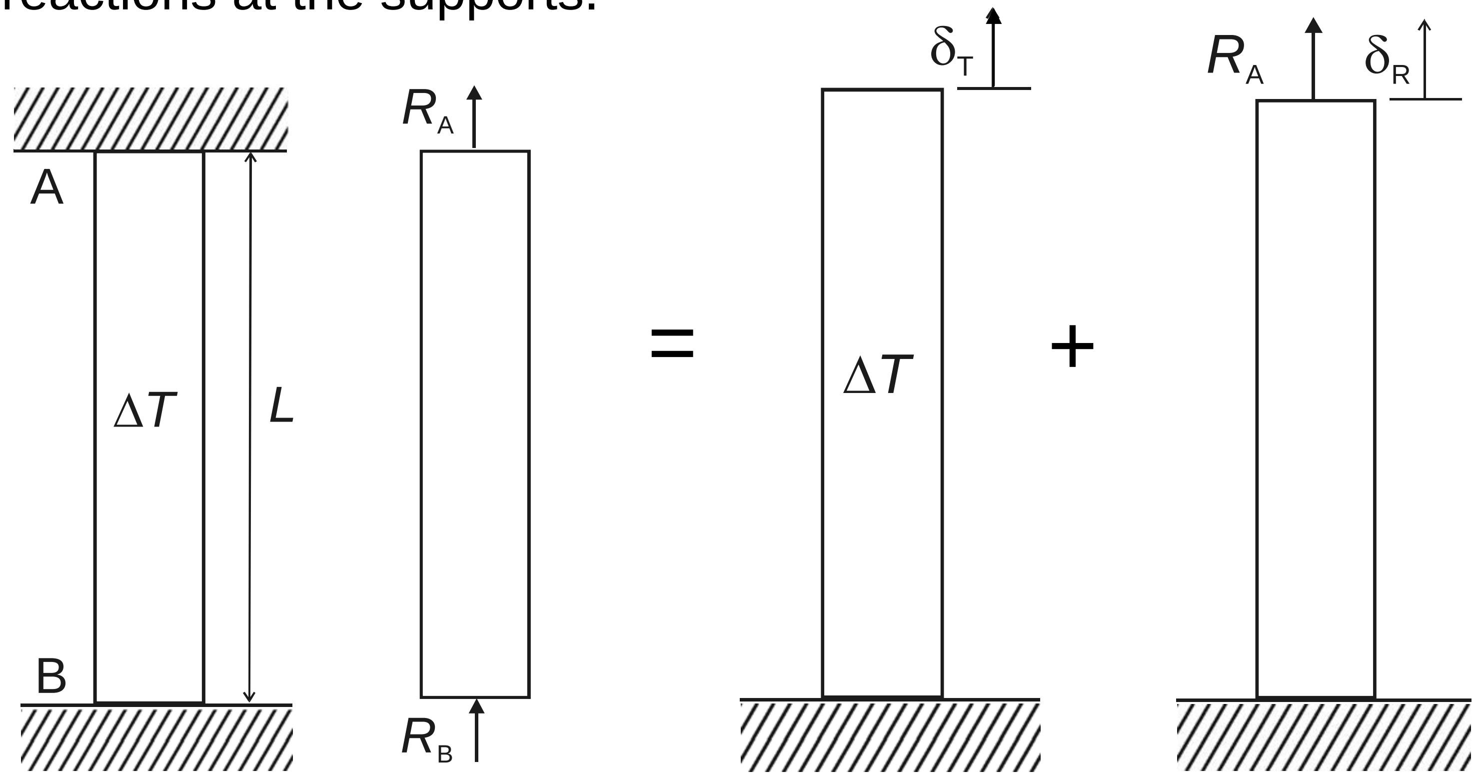
A statically indeterminate structure **may or may not** develop thermal stresses, depending on the character of the structure and the nature of the temperature changes.



Since D can move horizontally, no stresses are developed when the entire structure is heated **uniformly**.

If only some bars are heated, thermal stresses will develop.

Example: The temperature of the bar is raised by ΔT. Find the reactions at the supports.



Equilibrium: $R_A + R_B = 0$

Compatibility: $\delta_T + \delta_R = 0$ or $\delta_T = -\delta_R$

Using temperature-displacement and force-displacement relations:

$$\delta_T = \alpha(\Delta T)L$$

$$\delta_R = \frac{R_A L}{AE}$$

Substituting in to the compatibility condition gives:

$$\alpha(\Delta T)L = \frac{-R_A L}{AE}$$

$$R_A = -EA\,\alpha(\Delta T) \qquad \text{(compression)}$$

Substituting into the equilibrium equation gives:

$$R_B = -R_A = EA\,\alpha(\Delta T)$$

Misfits and Pre-strains

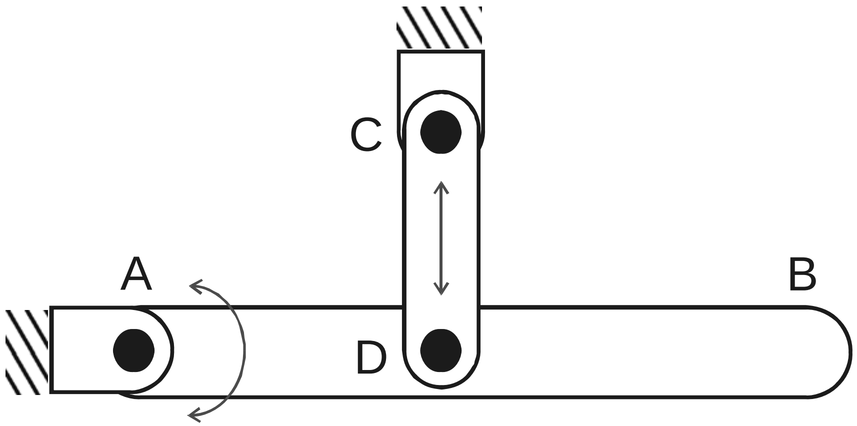
When a member is manufactured with a length slightly different from its prescribed length, the member will not fit into the structure as planned and the geometry of the structure will be different from what was planned. Such situations are misfits.

Some misfits are created intentionally to introduce strains into the structure at the time it is built. Because these strains exist before any loads are applied, they are called pre-strains. Along with the pre-strains are usually pre-stresses.

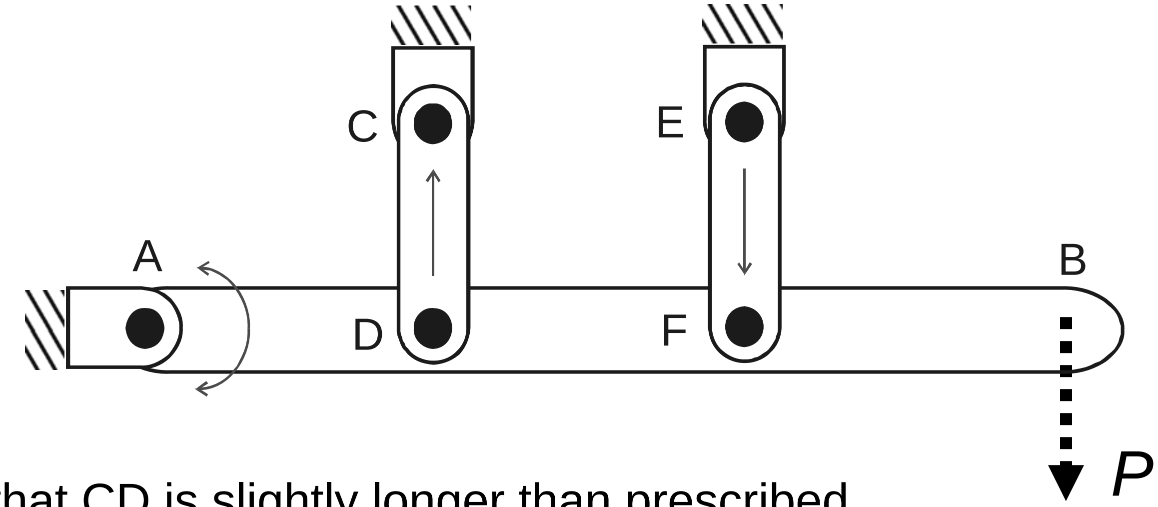
Examples: spokes in bicycle wheels, pre-tensioned faces of tennis racquets, shrink-fitted machine parts, pre-stressed concrete beams

If a structure is statically determinate, small misfits in one or more members will not produce strains or stresses, although the initial configuration will depart from the theoretical.

Here, having CD longer than expected will not induce pre-strains or pre-stresses. AB can rotate to accommodate the length change.



In a statically indeterminate structure, misfits cannot be accommodated without pre-stresses.



Assume that CD is slightly longer than prescribed.

Then to assemble the structure, CD must be compressed by external forces (or EF must be stretched by external forces).

The bars can then be fitted into place and the external loads released. As a result, the beam AB will deform and rotate.

If CD is put into compression, EF will be in tension. So, pre-strains will exist and the structure will be pre-stressed, although no external loads are acting.

When a load *P* is added, additional stresses and strains will result.