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HT08

Lecture 8

Plane Strain and Measurement of Strain

Plane stress versus plane strain.
Transformation equations.
Principal strains and maximum shear strains.
Mohr's circle for plane strain.
Measurement of strain and strain rosettes.

Plane stress versus plane strain

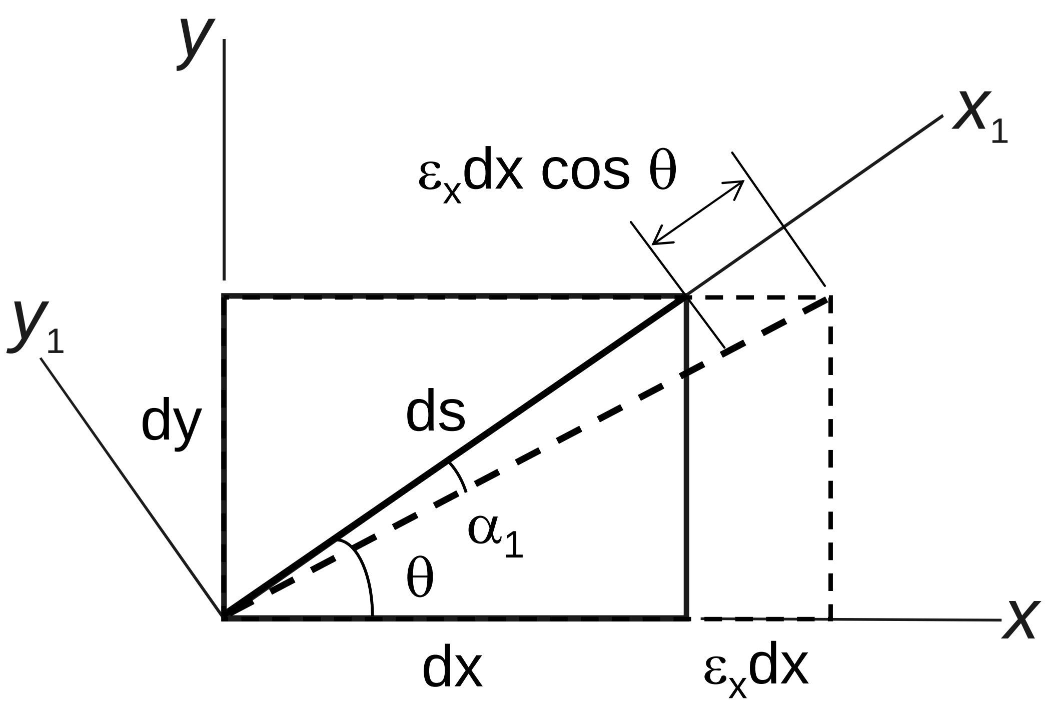
	Plane Stress	Plane Strain
Stresses	σ_z=0, τ_{xz}=0, τ_{yz}=0 σ_x, σ_y, τ_{xy} may be non-zero.	τ_{xz}=0, τ_{yz}=0 σ_x, σ_y, σ_z, τ_{xy} may be non-zero.
Strains	γ_{xz}=0, γ_{yz}=0 ε_x, ε_y, ε_z, γ_{xy} may be non-zero.	ε_z=0, γ_{xz}=0, γ_{yz}=0 ε_x, ε_y, γ_{xy} may be non-zero.

Plane stress and plane strain do not ordinarily occur simultaneously. One exception is when σ_z = 0 and σ_x = -σ_y, since Hooke's Law gives ε_z = 0.

Transformation Equations for Plane Strain

We want to derive equations for the normal strains ε_{x1} and ε_{y1} and the shear strain γ_{x1y1} associated with the x_1y_1 axes, which are rotated counter-clockwise through an angle θ from the xy axes.

Consider the change in length and orientation of the diagonal of a rectangular element in the xy plane after strains ε_x, ε_y, and γ_{xy} are applied.

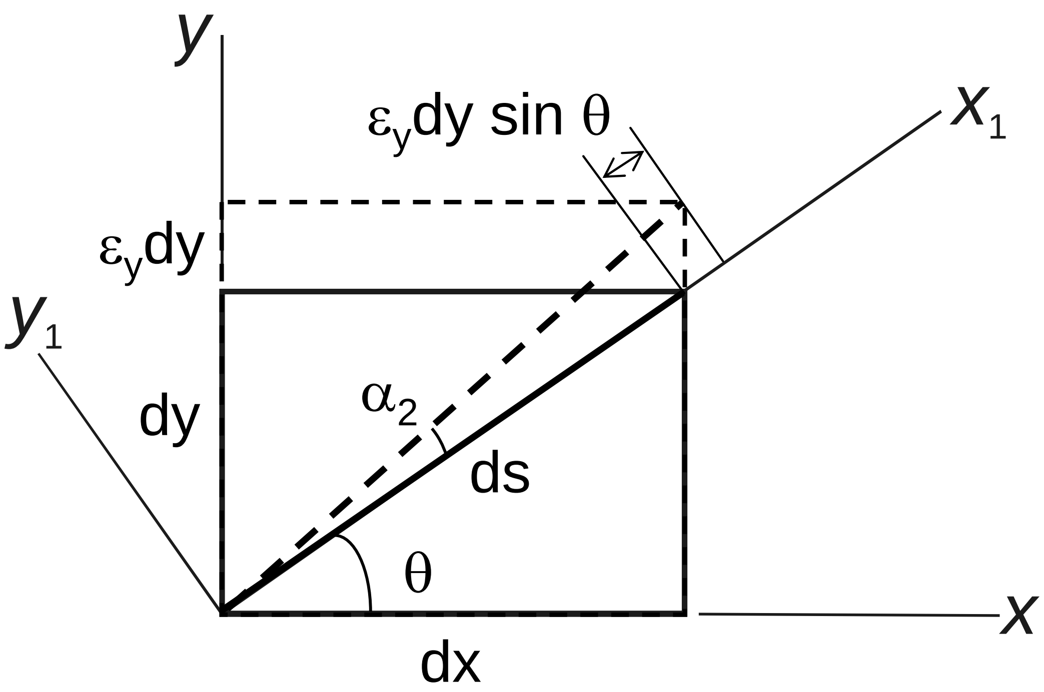


Diagonal increases in length in the x_1 direction by $\varepsilon_x dx \cos\theta$.

Diagonal rotates clockwise by α_1.

$$\alpha_1 ds = \varepsilon_x dx \sin\theta$$

$$\alpha_1 = \varepsilon_x \frac{dx}{ds} \sin\theta$$

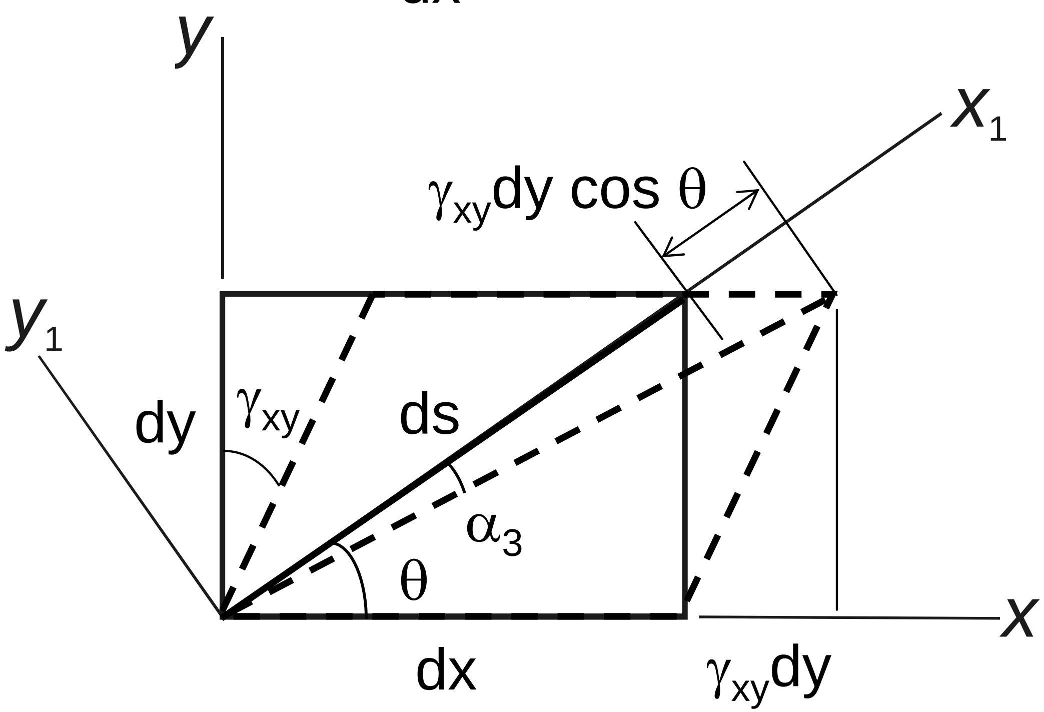


Diagonal increases in length in the x_1 direction by $\varepsilon_y dy \sin\theta$.

Diagonal rotates counter-clockwise by α_2.

$$\alpha_2 ds = \varepsilon_y dy \cos\theta$$

$$\alpha_2 = \varepsilon_y \frac{dy}{ds} \cos\theta$$



Diagonal increases in length in the x_1 direction by $\gamma_{xy} dx \cos\theta$.

Diagonal rotates clockwise by α_3.

$$\alpha_3 ds = \gamma_{xy} dy \sin\theta$$

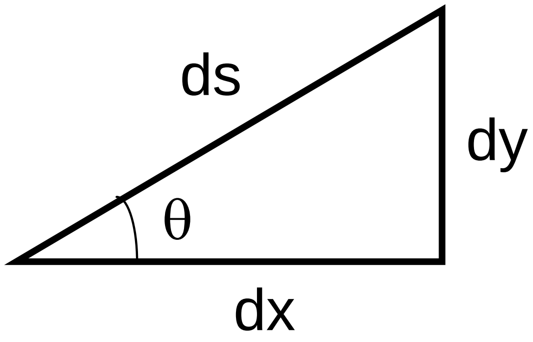
$$\alpha_3 = \gamma_{xy} \frac{dy}{ds} \sin\theta$$

The total increase in the length of the diagonal is:

$$\Delta(ds) = \varepsilon_x dx\ \cos\theta + \varepsilon_y dy\ \sin\theta + \gamma_{xy} dy\ \cos\theta$$

The normal strain ε_{x1} is the change in length over the original length:

$$\varepsilon_{x1} = \frac{\Delta(ds)}{ds} = \varepsilon_x \frac{dx}{ds}\ \cos\theta + \varepsilon_y \frac{dy}{ds}\ \sin\theta + \gamma_{xy} \frac{dy}{ds}\ \cos\theta$$



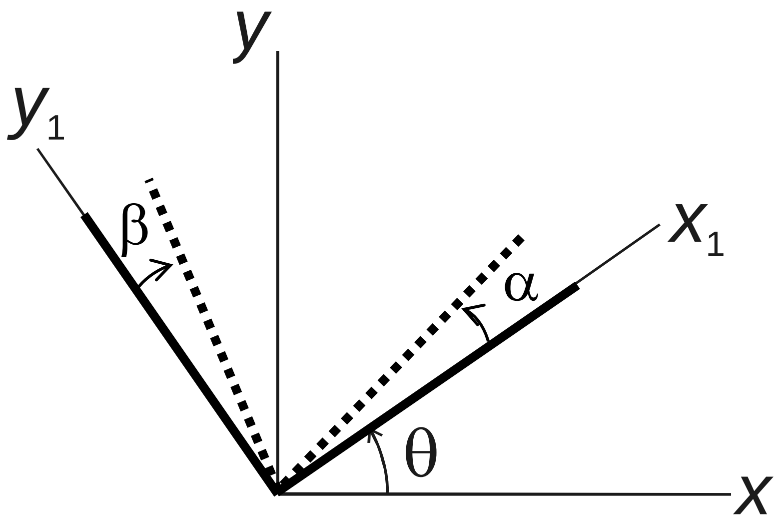
$$\frac{dx}{ds} = \cos\theta \qquad \frac{dy}{ds} = \sin\theta$$

So, the normal strain ε_{x1} is:

$$\varepsilon_{x1} = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy} \sin\theta\cos\theta$$

The normal strain ε_{y1} can be found by substituting θ+90° into the equation for ε_{x1}.

To find the shear strain γ_{x1y1}, we must find the decrease in angle of lines in the material that were initially along the $x_1 y_1$ axes.



$$\gamma_{x1y1} = \alpha + \beta$$

To find α, we just sum α_1, α_2, and α_3, taking the direction of the rotation into account.

$$\alpha = -\alpha_1 + \alpha_2 - \alpha_3$$

$$\alpha = -\varepsilon_x \frac{dx}{ds} \sin\theta + \varepsilon_y \frac{dy}{ds} \cos\theta - \gamma_{xy} \frac{dy}{ds} \sin\theta$$

$$\alpha = -\varepsilon_x \sin\theta\cos\theta + \varepsilon_y \sin\theta\cos\theta - \gamma_{xy} \sin^2\theta$$

$$\alpha = -(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta - \gamma_{xy} \sin^2\theta$$

To find the angle β, we can substitute θ+90 into the equation for α, but we must insert a negative sign, since α is counterclockwise and β is clockwise.

$$\beta = (\varepsilon_x - \varepsilon_y)\sin(\theta + 90)\cos(\theta + 90) + \gamma_{xy}\sin^2(\theta + 90)$$

$$\beta = -(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta + \gamma_{xy}\cos^2\theta$$

So, the shear strain γ_{x1y1} is:

$$\gamma_{x1y1} = \alpha + \beta$$

$$\gamma_{x1y1} = -(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta - \gamma_{xy}\sin^2\theta - (\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta - \gamma_{xy}\cos^2\theta$$

$$\gamma_{x1y1} = -2(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta - \gamma_{xy}(\sin^2\theta - \cos^2\theta)$$

$$\frac{\gamma_{x1y1}}{2} = -(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta - \frac{\gamma_{xy}}{2}(\sin^2\theta - \cos^2\theta)$$

Using trigonometric identities for sinθ cosθ, $\sin^2\theta$, and $\cos^2\theta$ gives the strain transformation equations …

$$\boxed{\varepsilon_{x1}} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\boxed{\frac{\gamma_{x1y1}}{2}} = -\frac{\left(\varepsilon_x - \varepsilon_y\right)}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

Now, compare the strain transformation equations to the stress transformation equations:

$$\boxed{\sigma_{x1}} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\boxed{\tau_{x1y1}} = -\frac{\left(\sigma_x - \sigma_y\right)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

The equations have the same form, but with different variables:

$$\boxed{\varepsilon_{x1} \Leftrightarrow \sigma_{x1}} \qquad \boxed{\frac{\gamma_{x1y1}}{2} \Leftrightarrow \tau_{x1y1}}$$

$$\varepsilon_x \Leftrightarrow \sigma_x$$

$$\varepsilon_y \Leftrightarrow \sigma_y \qquad \frac{\gamma_{xy}}{2} \Leftrightarrow \tau_{xy}$$

So, the all the equations that we derived based on the stress transformation equations can be converted to equations for strains if we make the appropriate substitutions.

Principal Strains and Principal Angles

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad \tan\, 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

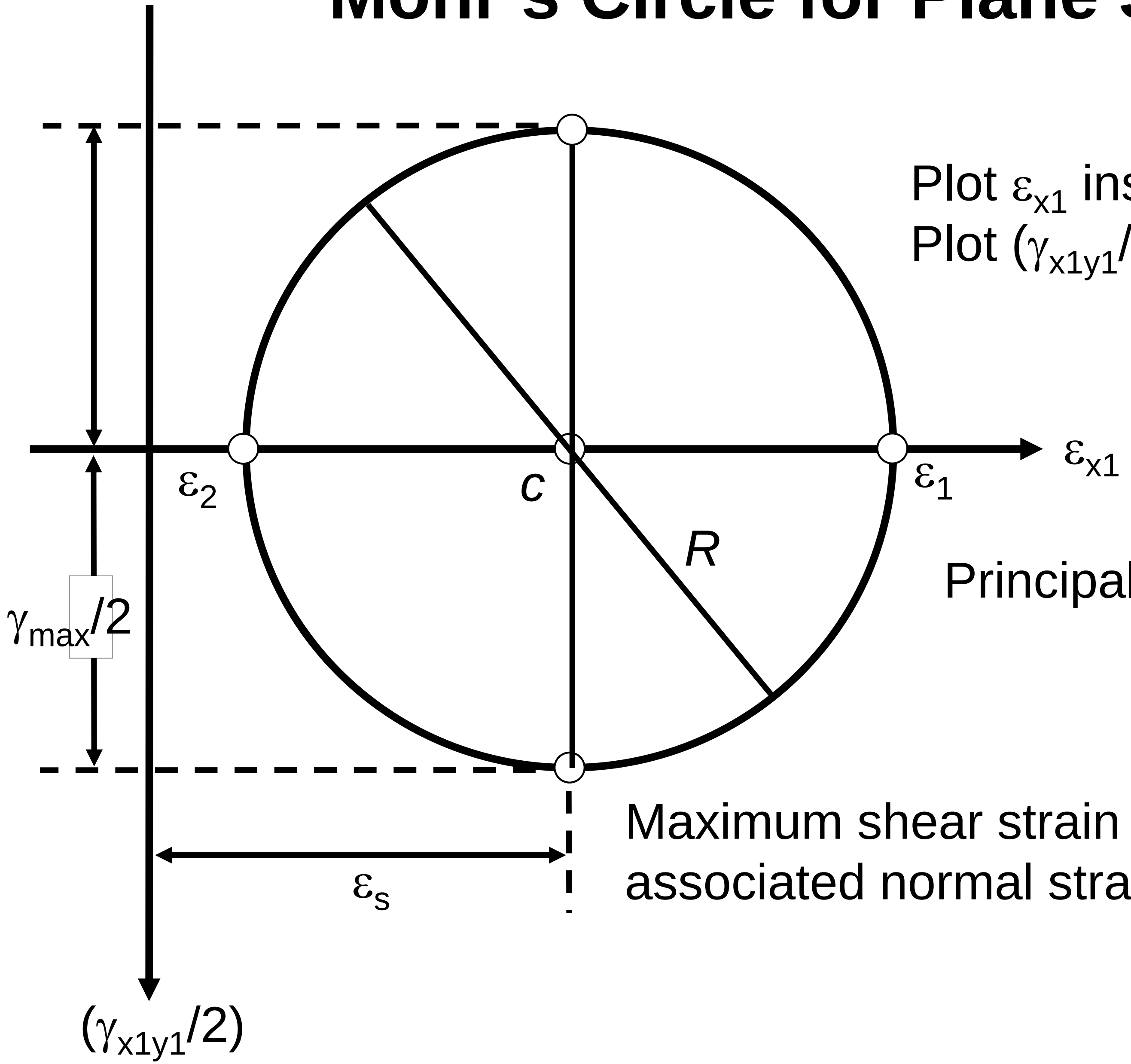
Remember that $\varepsilon_z = 0$ (plane strain).
Shear strains are zero on the principal planes.
Principal stresses and principal strains occur in the same directions.

Maximum Shear Strains

$$\frac{\gamma_{\max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad \tan 2\theta_s = -\left(\frac{\varepsilon_x - \varepsilon_y}{\gamma_{xy}}\right)$$

The maximum shear strains are associated with axes at 45° to the directions of the principal strains.

Mohr's Circle for Plane Strain



Plot ε_{x1} instead of σ_{x1}.
Plot $(\gamma_{x1y1}/2)$ instead of τ_{x1y1}.

Principal strains ε_1, ε_2

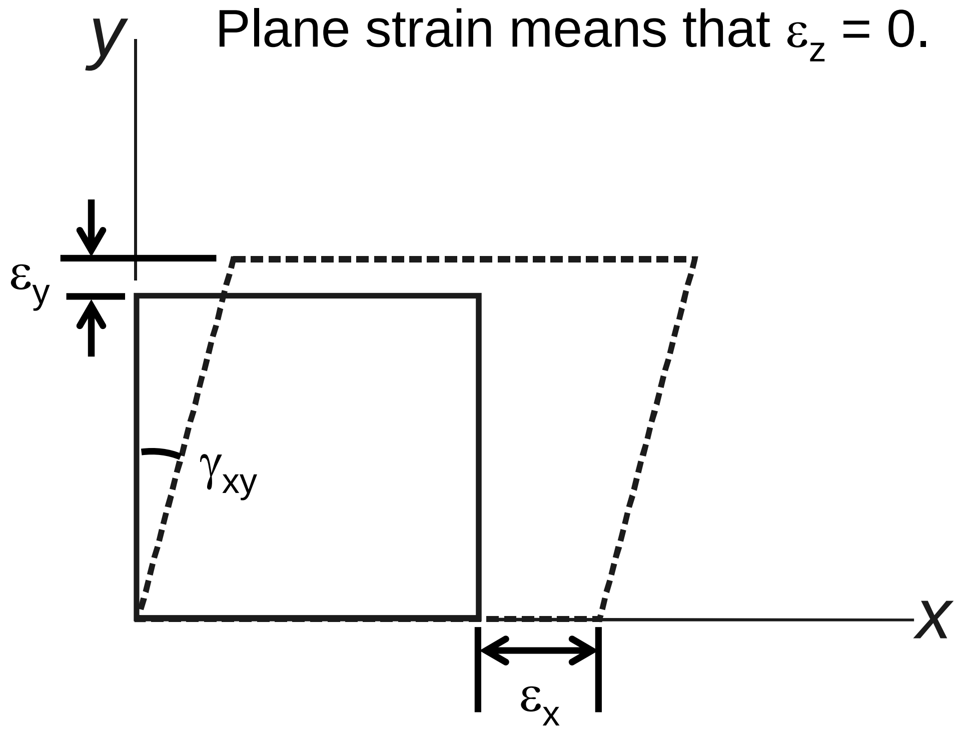
Maximum shear strain γ_{max} with associated normal strain ε_s

Example

An element of material in plane strain has
$\varepsilon_x = 340 \times 10^{-6}$, $\varepsilon_y = 110 \times 10^{-6}$, $\gamma_{xy} = 180 \times 10^{-6}$

Find the principal strains, the (in-plane) maximum shear strains, and the strains on an element oriented at an angle θ=30°.

Plane strain means that $\varepsilon_z = 0$.



(based on Gere & Timoshenko, p 439)

Equations give
$\varepsilon_1 = 371 \times 10^{-6}$
$\varepsilon_2 = 79 \times 10^{-6}$
θ_p = 19.0° and 109.0°
$\gamma_{max} = 290 \times 10^{-6}$
θ_s = -26.0° and 64.0°

The transformation equations with θ=30° give
$\varepsilon_{x1} = 360 \times 10^{-6}$
$\gamma_{x1y1} = -110 \times 10^{-6}$

Using $\varepsilon_x + \varepsilon_y = \varepsilon_{x1} + \varepsilon_{y1}$
gives $\varepsilon_{y1} = 90 \times 10^{-6}$

Mohr's Circle

$$c = \varepsilon_{avg} = \frac{\varepsilon_x + \varepsilon_y}{2}$$

$$c = \frac{340 + 110}{2} = 225$$

$$R = \sqrt{(340 - 225)^2 + (180/2)^2}$$

$$R = \sqrt{115^2 + 90^2} = 146$$

Principal Strains

$$\varepsilon_{1,2} = c \pm R = 225 \pm 146$$

$$\varepsilon_{1,2} = 371, 79$$

$$\tan 2\theta_{p1} = \frac{90}{340 - 225}$$

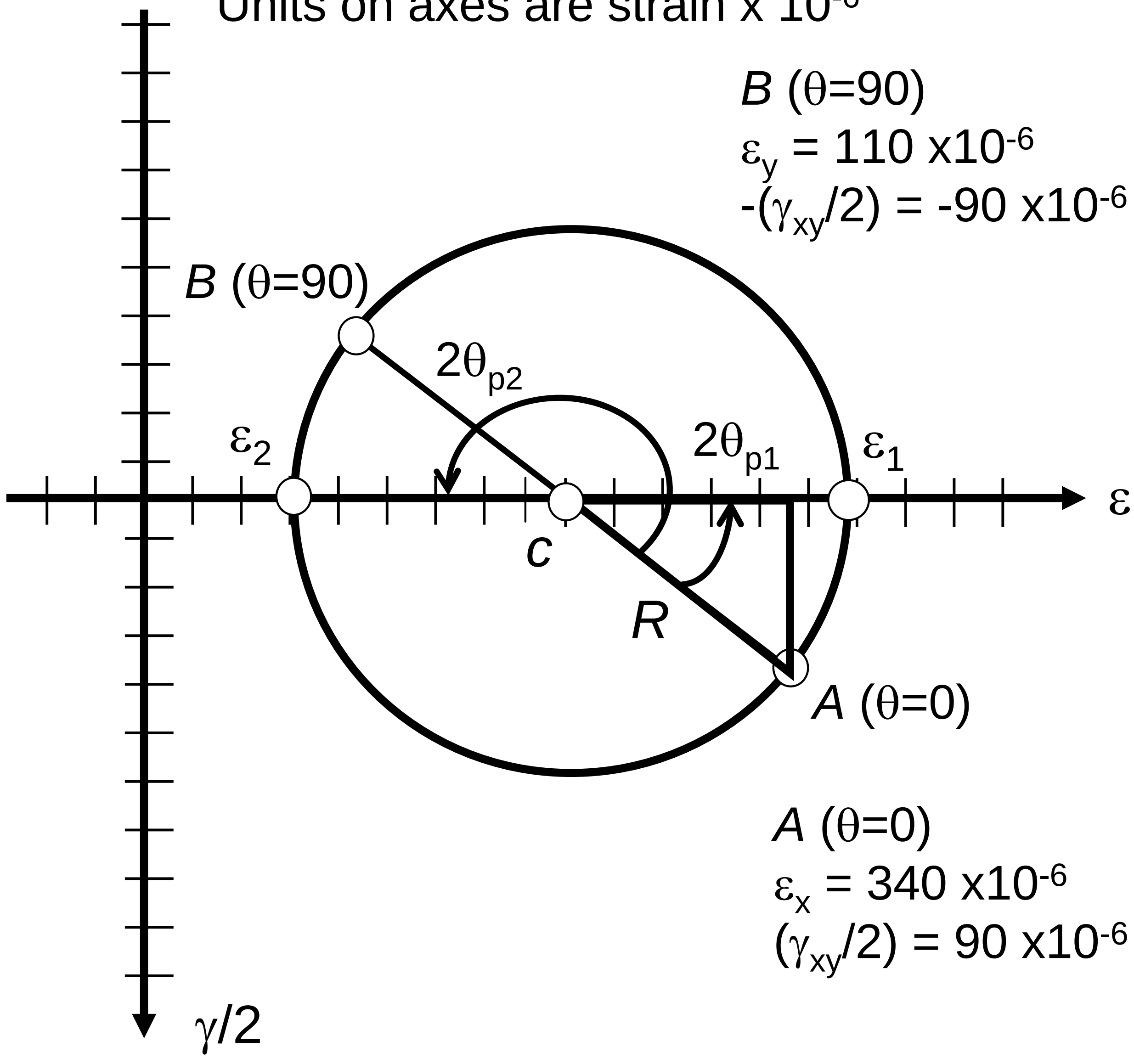
$$2\theta_{p1} = 38.05°$$

$$\theta_{p1} = 19.0°$$

$$2\theta_{p2} = 2\theta_{p1} + 180°$$

$$\theta_{p2} = 109.0°$$

Units on axes are strain x 10^{-6}



B (θ=90)
ε_y = 110 x10^{-6}
-(γ_{xy}/2) = -90 x10^{-6}

A (θ=0)
ε_x = 340 x10^{-6}
(γ_{xy}/2) = 90 x10^{-6}

Units on axes are strain x 10^{-6}

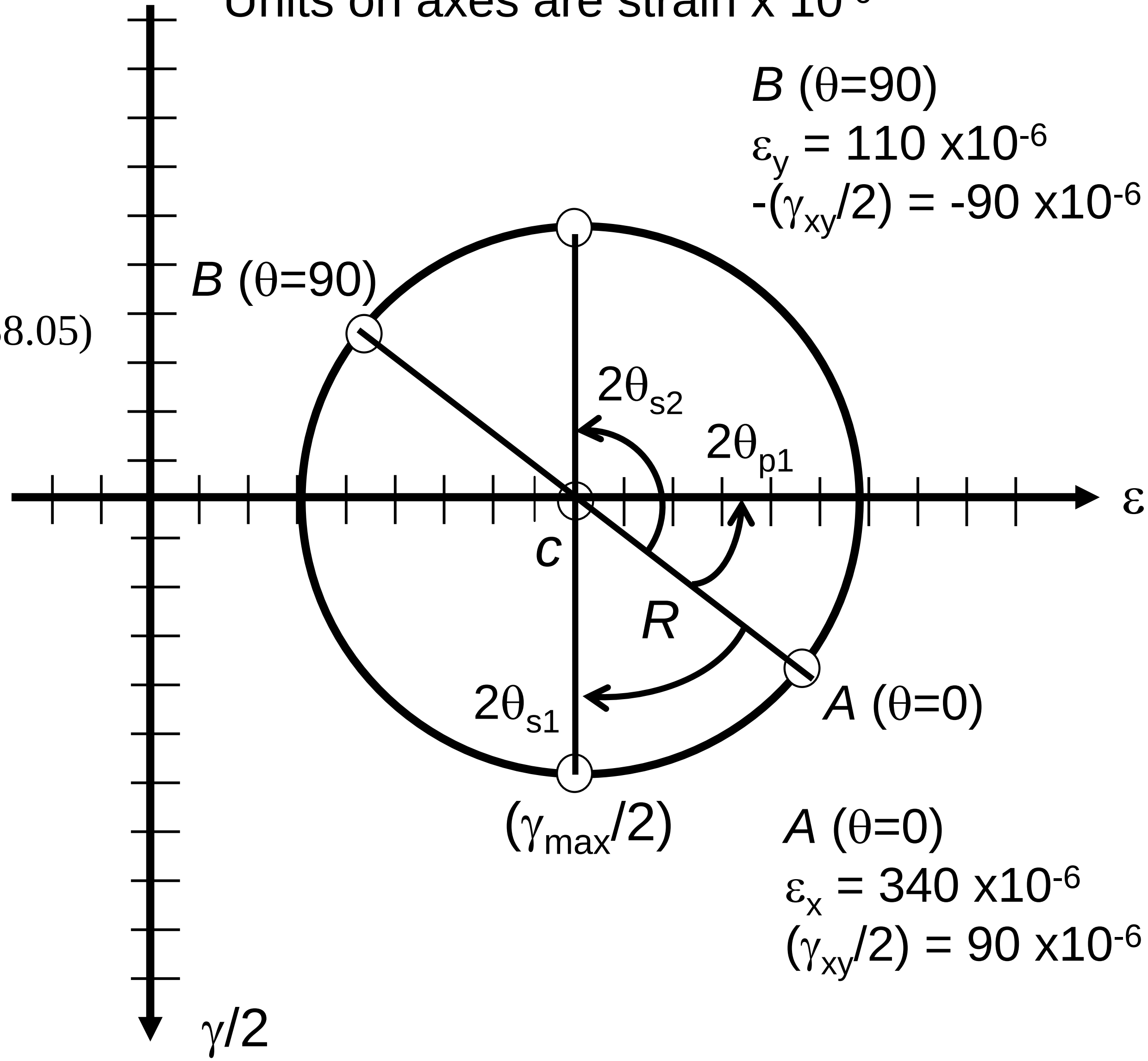
Maximum Shear

$$(\gamma_{\max}/2) = R = 146$$
$$\gamma_{\max} = 292$$

$$2\theta_{s1} = -(90 - 2\theta_{p1}) = -(90 - 38.05)$$
$$2\theta_{s1} = -51.95$$
$$\theta_{s1} = -26.0°$$

$$2\theta_{s2} = 2\theta_{p1} + 90°$$
$$\theta_{s2} = 64.0°$$

$$\varepsilon_s = c = 225$$



B (θ=90)
ε_y = 110 x10^{-6}
-(γ_{xy}/2) = -90 x10^{-6}

A (θ=0)
ε_x = 340 x10^{-6}
(γ_{xy}/2) = 90 x10^{-6}

Strains when θ = 30°

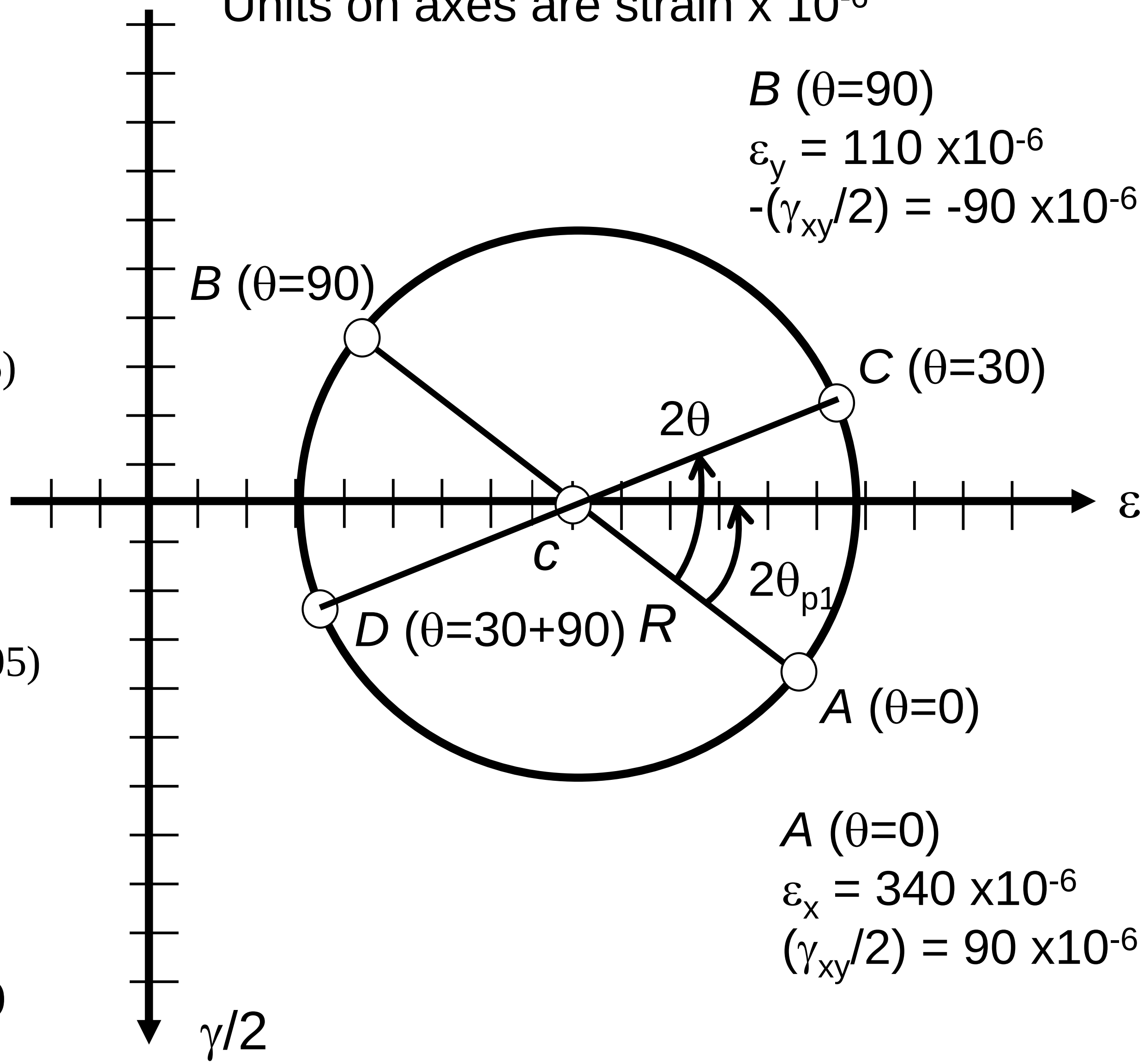
$\theta = 30°$ $\quad 2\theta = 60°$

$$\varepsilon_{x1} = c + R\cos(2\theta - 2\theta_{p1})$$
$$\varepsilon_{x1} = 225 + 146\cos(60 - 38.05)$$
$$\varepsilon_{x1} = 360$$

$$(\gamma_{x1y1}/2) = -R\sin(2\theta - 2\theta_{p1})$$
$$(\gamma_{x1y1}/2) = -146\sin(60 - 38.05)$$
$$(\gamma_{x1y1}/2) = -55$$
$$\gamma_{x1y1} = -110$$

$$\varepsilon_{y1} = c - R\cos(2\theta - 2\theta_{p1})$$
$$\varepsilon_{y1} = 225 - 146\cos(60 - 38.05)$$
$$\varepsilon_{y1} = 90$$

Units on axes are strain x 10^{-6}

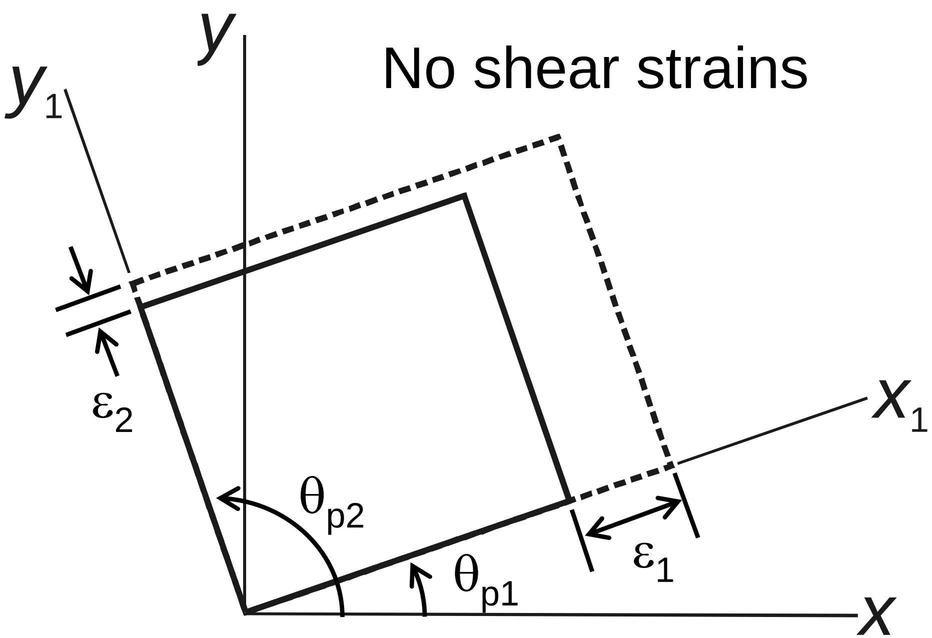


B (θ=90)
ε_y = 110 x10^{-6}
-(γ_{xy}/2) = -90 x10^{-6}

A (θ=0)
ε_x = 340 x10^{-6}
(γ_{xy}/2) = 90 x10^{-6}

Principal Strains

Maximum Shear Strain

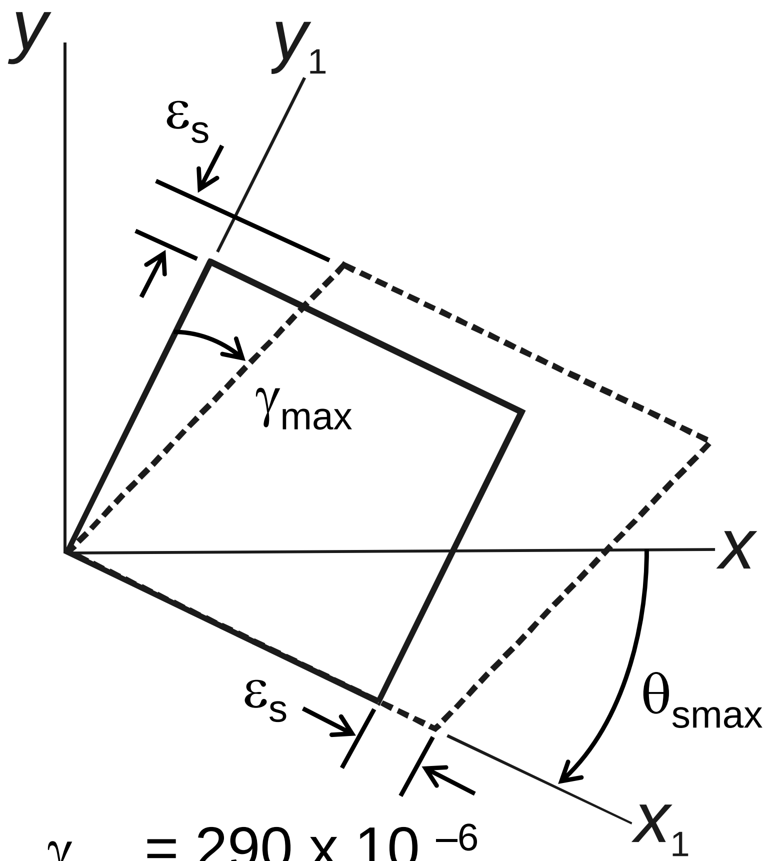


ε_1 = 371 x 10^{-6}

ε_2 = 79 x 10^{-6}

θ_{p1} = 19.0°

θ_{p2} = 109.0°

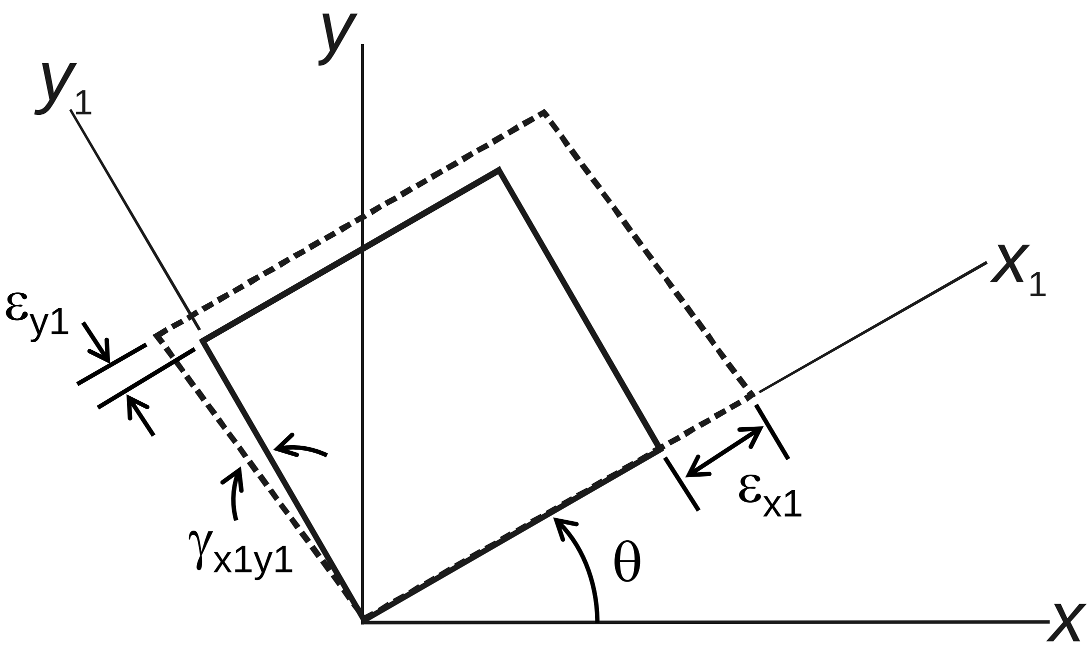


γ_{max}= 290 x 10^{-6}

θ_{smax} = -26.0°

ε_s= 225 x 10^{-6}

Strains when $\theta = 30°$



$\varepsilon_{x1} = 360 \times 10^{-6}$
$\varepsilon_{y1} = 90 \times 10^{-6}$
$\gamma_{x1y1} = -110 \times 10^{-6}$

Measurement of Strain

- It is very difficult to measure normal and shear stresses in a body, particularly stresses at a point.

- It is relatively easy to measure the strains on the surface of a body (normal strains, that is, not shear strains).

- From three independent measurements of normal strain at a point, it is possible to find principal strains and their directions.

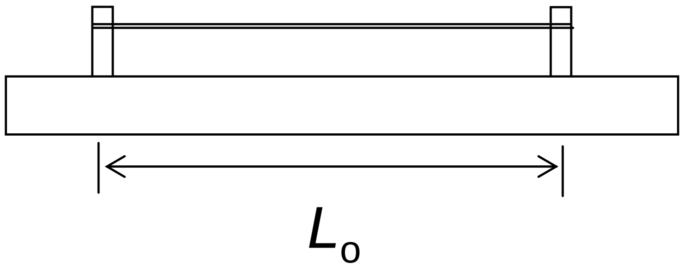
- If the material obeys Hooke's Law, the principal strains can be used to find the principal stresses.

- Strain measurement can be **direct** (using electrical-type gauges based on resistive, capacitive, inductive, or photoelectric principles) or **indirect** (using optical methods, such as photo-elasticity, the Moiré technique, or holographic interferometry).

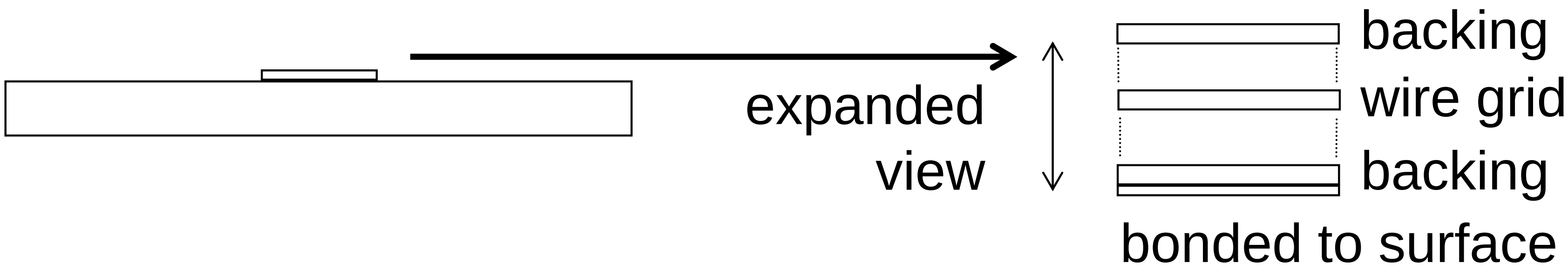
Resistance Strain Gauges

• Based on the idea that the resistance of a metal wire changes when the wire is subjected to mechanical strain (Lord Kelvin, 1856). When a wire is stretched, a **longer** length of **smaller** sectioned conductor results.

• The earliest strain gauges were of the “unbonded” type and used pillars, separated by the gauge length, with wires stretched between them.

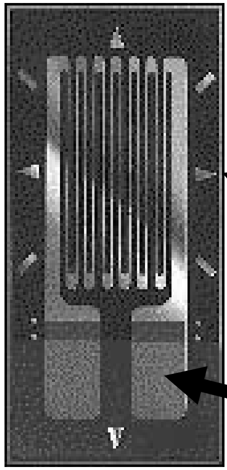


• Later gauges were “bonded”, with the resistance element applied directly to the surface of the strained member.



During the 1950s, foil-type gauges began to replace the wire-type. The foil-type gauges typically consist of a metal film element on a thin epoxy support and are made using printed-circuit techniques.

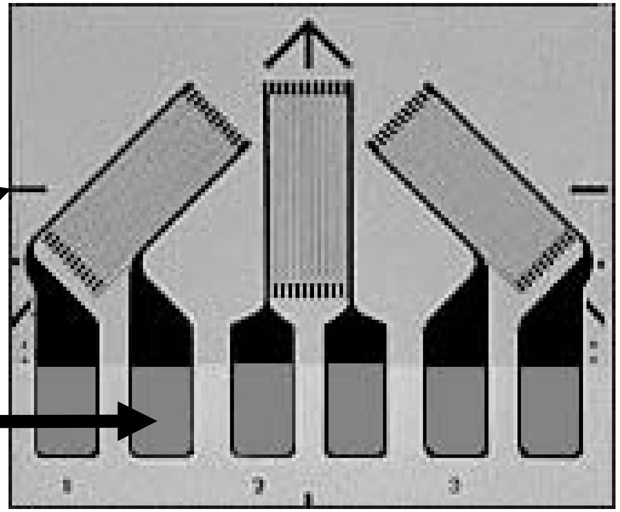
Foil-type gauges can be made in a number of configurations (examples from www.vishay.com):



single element

alignment marks

solder tabs for wires



planar three-element rosette (0°- 45°- 90°)

Gauge length is typically around 1 mm.

Performance of bonded metallic strain gauges depends on: grid material and configuration, backing material, bonding material and method, gauge protection, and associated electrical circuitry.

It is possible to derive an equation relating strain ε and the change in resistance of the gauge Δ*R*:

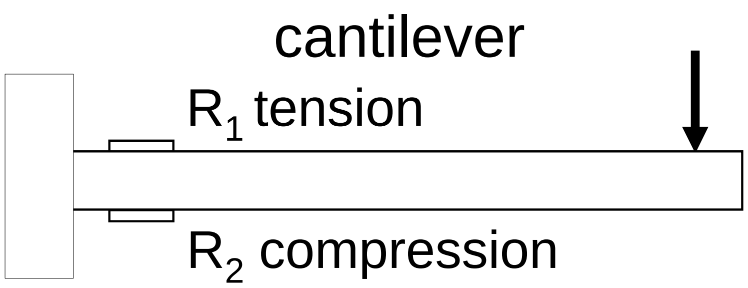
$$\varepsilon = \frac{1}{F}\frac{\Delta R}{R}$$

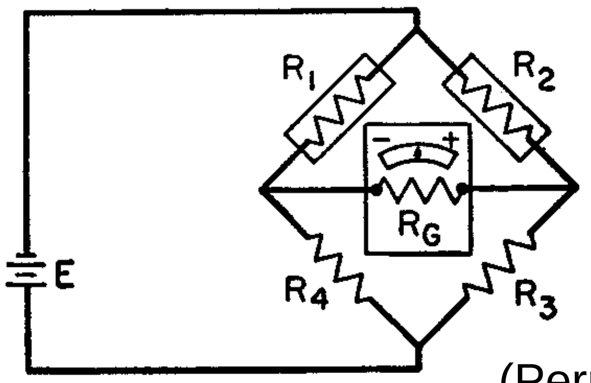
F = gauge factor (related to Poisson's ratio and resistivity)
R = resistance of the gauge

A typical strain gauge might have *F* = 2.0 and *R* = 120 Ω and be used to measure microstrain (10^{-6}).

$$\Delta R = \varepsilon\, F\, R = (10^{-6})(2.0)(120) = 0.00024\ \Omega$$

This is a resistance change of 0.0002%, meaning that something more sensitive than an ohmmeter is required to measure the resistance change. Some form of bridge arrangement (such as a Wheatstone bridge) is most widely used.

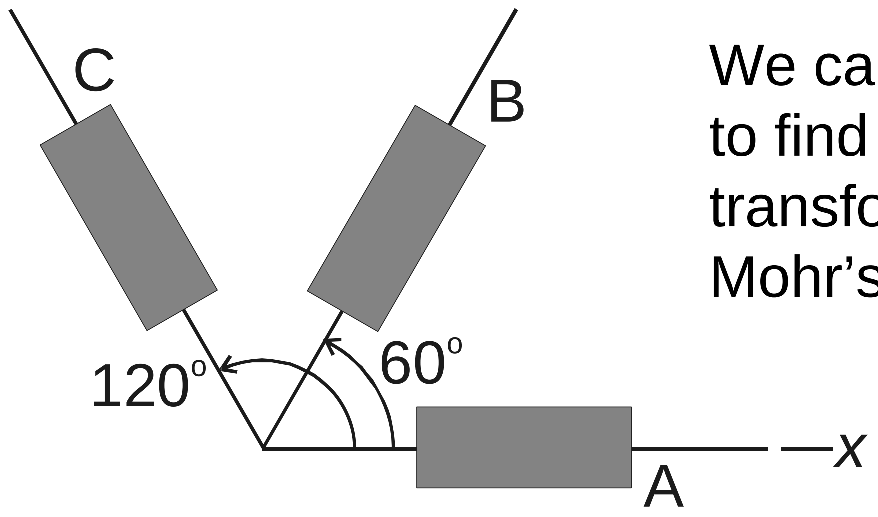




(Perry & Lissner)

Strain Rosettes and Principal Strains and Stresses

A “0°-60°-120°” strain gauge rosette is bonded to the surface of a thin steel plate. Under one loading condition, the strain measurements are $\varepsilon_A = 60\ \mu\varepsilon$, $\varepsilon_B = 135\ \mu\varepsilon$, $\varepsilon_C = 264\ \mu\varepsilon$. Find the principal strains, their orientations, and the principal stresses.



We can use more than one approach to find the principal stresses: transformation equations alone, Mohr’s circle alone, or a combination.

(Based on Hibbeler, ex. 15.20 & 15.21)

Transformation equations

From the measured strains, find ε_x, ε_y, and γ_{xy}.

ε_A = 60 $\mu\varepsilon$, θ_A = 0°
ε_B = 135 $\mu\varepsilon$, θ_B = 60°
ε_C = 264 $\mu\varepsilon$, θ_C = 120°

$$\varepsilon_A = 60 = \varepsilon_x \cos^2 0° + \varepsilon_y \sin^2 0° + \gamma_{xy} \sin 0° \cos 0°$$

$$\boxed{\varepsilon_A = 60 = \varepsilon_x}$$

$$\varepsilon_B = 135 = \varepsilon_x \cos^2 60° + \varepsilon_y \sin^2 60° + \gamma_{xy} \sin 60° \cos 60°$$

$$\boxed{\varepsilon_B = 135 = 0.25\varepsilon_x + 0.75\varepsilon_y + 0.433\gamma_{xy}}$$

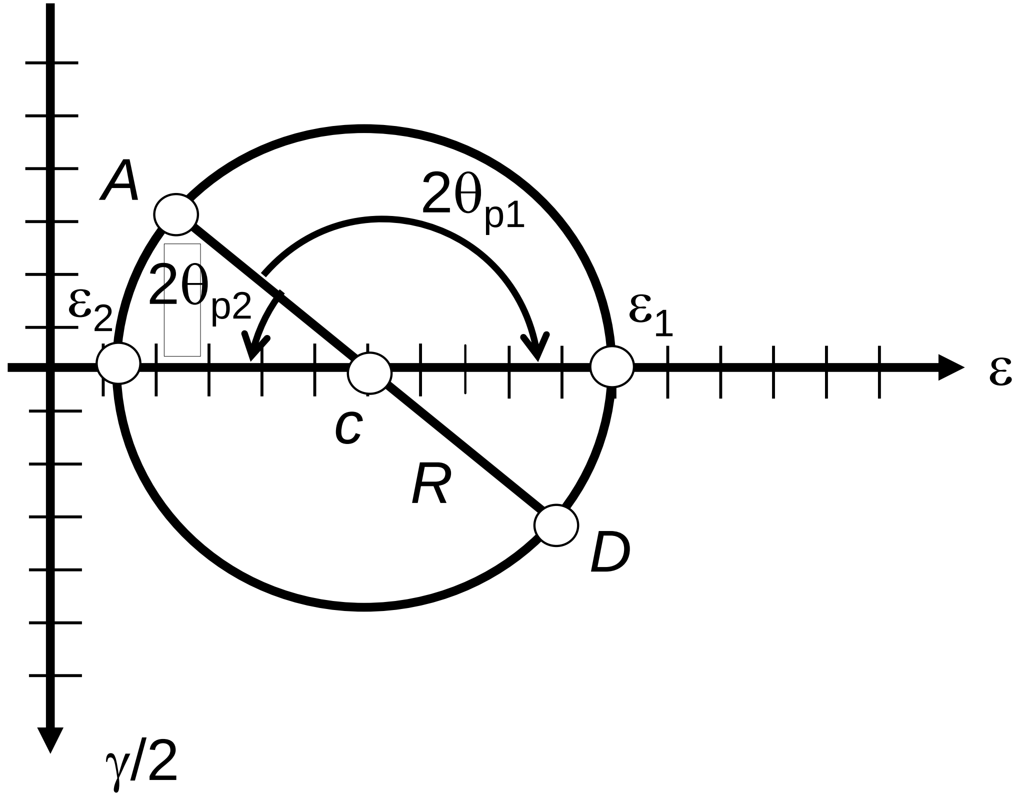
$$\varepsilon_C = 264 = \varepsilon_x \cos^2 120° + \varepsilon_y \sin^2 120° + \gamma_{xy} \sin 120° \cos 120°$$

$$\boxed{\varepsilon_C = 264 = 0.25\varepsilon_x + 0.75\varepsilon_y - 0.433\gamma_{xy}}$$

3 equations, 3 unknowns
Solve to find ε_x = 60 $\mu\varepsilon$, ε_y = 246 $\mu\varepsilon$, γ_{xy} = -149 $\mu\varepsilon$

Use ε_x, ε_y, and γ_{xy} in the equations for principal strains to find ε_1 = 272 $\mu\varepsilon$, θ_{p1} = -70.6°, ε_2 = 34 $\mu\varepsilon$, θ_{p2} = 19.4°.

Alternatively, use ε_x, ε_y, and γ_{xy} to construct the Mohr's circle for (in-plane) strains and find principal strains and angles.



c = (60+246)/2 = 153 $\mu\varepsilon$

A: ε_x = 60 $\mu\varepsilon$
(γ_{xy}/2)= -74.5 $\mu\varepsilon$

D: ε_y = 246 $\mu\varepsilon$
(γ_{xy}/2)= +74.5 $\mu\varepsilon$

R = 119 $\mu\varepsilon$

To find the principal stresses, use Hooke's Law for plane stress ($\sigma_z = 0$)

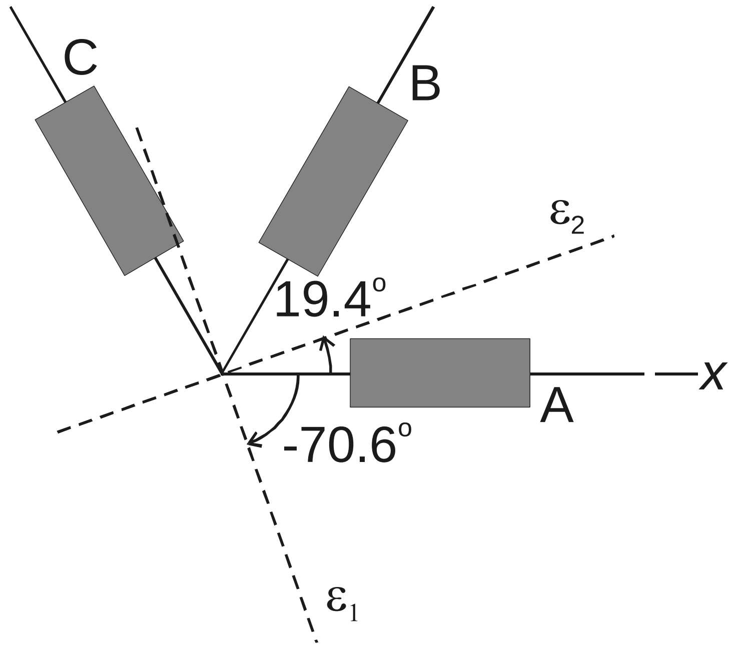
$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)$$

$\varepsilon_x = \varepsilon_1 = 272 \times 10^{-6}$
$\varepsilon_y = \varepsilon_2 = 34 \times 10^{-6}$

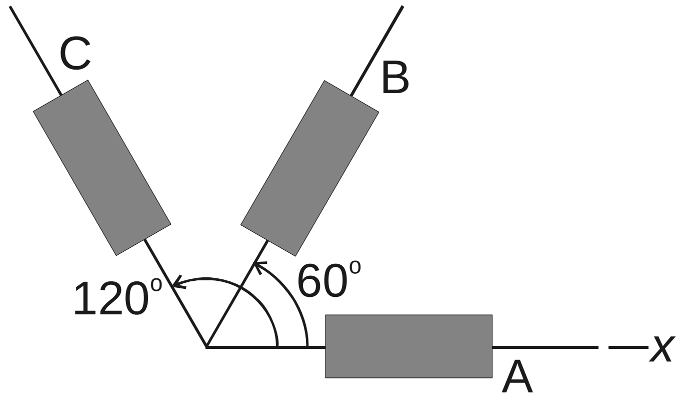
E = 210 GPa
$\nu = 0.3$

So, the principal stresses are:
$\sigma_x = \sigma_1 = 65$ MPa
$\sigma_y = \sigma_2 = 26$ MPa



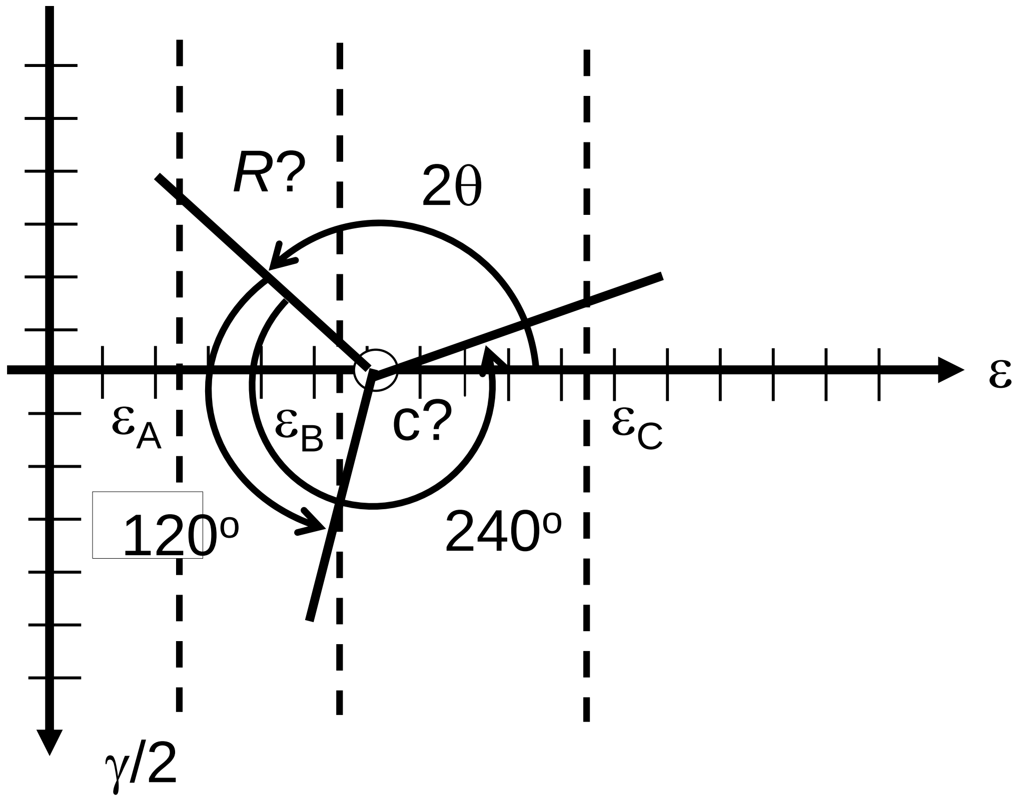
Mohr's Circle

$\varepsilon_A = 60\ \mu\varepsilon$, $\varepsilon_B = 135\ \mu\varepsilon$, $\varepsilon_C = 264\ \mu\varepsilon$.



$$\varepsilon_A = 60 = c + R\cos 2\theta$$
$$\varepsilon_B = 135 = c + R\cos 2(\theta+60)$$
$$\varepsilon_C = 264 = c + R\cos 2(\theta+120)$$

3 equations, 3 unknowns



Solve the equations to get c = 153, R = 119, and 2θ = 141.3°
When you solve for 2θ, you may get –38.7°. But we have drawn the diagram above such that 2θ is positive, so you should take 2θ = -38.7° + 180° = 141.3°.

Next, draw the Mohr's circle and find principal strains as before.
Finally, find principal stresses using Hooke's Law.