EXERCISES BOOKLET

for the
Logic Manual

2024/2025

There are no changes to the exercises
from last year's edition

Volker Halbach

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PREFACE

The most recent version of this *Exercises Booklet* can be downloaded from `http://logicmanual.philosophy.ox.ac.uk/index.html`, the web page of the Logic Manual. I have also uploaded some files with partial truth tables tables, proofs in Natural Deduction, past papers with solutions and lecture slides. Peter Fritz has supplied a full set of exercises with solutions. For self-study I recommend Peter's exercises, while the version you are looking at is intended for use in tutorials and classes.

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1 Sets, Relations and Arguments

EXERCISE 1.1. Consider the following relations:

(i) {⟨Hydrogen, Oxygen⟩, ⟨Oxygen, Hydrogen⟩, ⟨Hydrogen, Hydrogen⟩}

(ii) {⟨Mercury, Oxygen⟩, ⟨Oxygen, Nitrogen⟩, ⟨Mercury, Nitrogen⟩}

(iii) {⟨Mercury, Mercury⟩, ⟨Oxygen, Oxygen⟩, ⟨Nitrogen, Nitrogen⟩}

(iv) ∅, that is the set without elements

Let S be the set with the chemical elements Hydrogen, Oxygen, Mercury and Nitrogen as (set-theoretic) elements. Determine for each of the relations (i)–(iv)

(a) whether it is reflexive on S,

(b) whether it is symmetric,

(c) whether it is transitive, and

(d) whether it is a function.

EXERCISE 1.2. Specify a relation that is symmetric but not transitive. Try to find such a relation with a minimal number of elements.

EXERCISE 1.3. Specify a relation and a set S such that the relation is reflexive on S and asymmetric.

EXERCISE 1.4. Is the relation {⟨Paris, London⟩, ⟨London, Rome⟩, ⟨London, the capital of Italy⟩} a function?

EXERCISE 1.5. Consider the relation containing the ordered pairs ⟨Germany, Italy⟩, ⟨Germany, Germany⟩, ⟨Italy, Italy⟩, ⟨France, France⟩ but no other pairs.
  (a) Is this relation reflexive on the set {Germany, Italy, France}?
  (b) Is this relation transitive on {Germany, Italy, France}?
  (c) Is this relation symmetric on {Germany, Italy, France}?
  (d) Is it an equivalence relation on {Germany, Italy, France}?
  (e) Is it an equivalence relation on {Germany, France}?

EXERCISE 1.6. Consider the following relations, where d and e are persons:

  (i) the set of all ordered pairs ⟨d, e⟩ such that d is taller than e
  (ii) $\{⟨d, e⟩ : d \text{ loves } e\}$
  (iii) the relation with all ordered pairs ⟨d, e⟩ as members such that d is the father of e
  (iv) the relation with all ordered pairs ⟨d, e⟩ as members such that e is the father of d
  (v) the relation of *being of a similar age*

Determine for each of these relations whether it is symmetric, whether it is transitive, and whether it is a function.

EXERCISE 1.7. Identify premisses and conclusions in the following arguments. Are the arguments logically valid?

  (i) All men are mortal, Socrates is a man. Thus, Socrates is mortal.
  (ii) Houses become cheaper only if interest rates are falling. Now houses are actually becoming cheaper, although interest rates are not falling. So the Prime Minister will become the king of France.
  (iii) Tom will move to Edinburgh. This is because he got a job there and he can't find another job where he is living now.
  (iv) Alfred can see the house. So he must have at least one eye.
  (v) If the mind is immortal, it's not identical with the body. So if the mind is identical to the body, the mind is not immortal.
  (vi) This must be a Manx cat: it hasn't got a tail.

EXERCISE 1.8. Identify the premisses and the conclusion in the following argument:

> Many students will be either in Hegel's or in Schopenhauer's lectures, if they are scheduled at the same time. And of course Schopenhauer will schedule them at the same time as Hegel's. If Hegel's lectures are entertaining, then many students will go to them. That means of course many students will go to Hegel's but not many will go to Schopenhauer's lectures. For if Schopenhauer's lectures are entertaining, Hegel's must be entertaining as well; and of course many students will only come to Schopenhauer's lectures if they are entertaining.

2 Syntax and Semantics of Propositional Logic

EXERCISE 2.1. Add quotation marks to the following sentences so that
true English sentences are obtained. In some cases there is more than
one solution. Try to find all solutions.

(i) Potassium designates a chemical element.
(ii) Snow is white is true if and only if snow is white.
(iii) John, Jane and Jeremy all have J as their first letter.
(iv) George is the quotation of George.
(v) Tom is monosyllabic and Reginald is polysyllabic.

EXERCISE 2.2. Check whether the following expressions are sentences of
$\mathcal{L}_1$.

(i) $(((P_1 \to P_1) \to P_1) \lor Q)$
(ii) $(((P_2 \land R)) \to Q_4)$
(iii) $(P \to \neg P)$
(iv) $(P \neg \to P)$
(v) $(\neg P \to P)$
(vi) $(P \to \neg\neg\neg(R \lor \neg R))$
(vii) $\neg((P \to (P \to \neg Q)) \leftrightarrow \neg\neg(R_2 \leftrightarrow \neg(P \lor R_7)))$

No bracketing conventions are applied in the expressions.

EXERCISE 2.3. The following expressions are abbreviations of $\mathcal{L}_1$-sentences.
Restore the brackets that have been dropped in accordance with the Brack-
eting Conventions of Section 2.3.

(i) $\neg P \land Q$
(ii) $P \land \neg Q \land R \leftrightarrow \neg P_3 \lor P \lor R_5$
(iii) $\neg\neg\neg(P \to Q) \leftrightarrow P$

EXERCISE 2.4. Drop as many brackets as possible from the following $\mathcal{L}_1$-sentences by applying the Bracketing Conventions from Section 2.3.

  (i) $(((\neg P \rightarrow \neg Q) \vee Q_2) \wedge P)$
  (ii) $(((\neg P \rightarrow \neg Q) \wedge Q_2) \wedge P)$
(iii) $\neg(((P \wedge (P \rightarrow \neg Q)) \wedge Q_1) \wedge P)$

EXERCISE 2.5. Show that the following sentences are tautologies. You may use partial truth tables. Examples of calculations of partial truth tables can be found on WebLearn.

(i)	$P \wedge (P \rightarrow Q) \rightarrow Q$	(modus ponens)
(ii)	$\neg Q \wedge (P \rightarrow Q) \rightarrow \neg P$	(modus tollens)
(iii)	$P \vee \neg P$	(law of excluded middle)
(iv)	$\neg(P \wedge \neg P)$	(law of contradiction)
(v)	$(\neg P \rightarrow P) \rightarrow P$	(consequentia mirabilis)
(vi)	$(P \rightarrow Q) \wedge (\neg P \rightarrow Q) \rightarrow Q$	(classical dilemma)
(vii)	$\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$	(de Morgan-law)
(viii)	$\neg(P \vee Q) \leftrightarrow (\neg P \wedge \neg Q)$	(de Morgan-law)
(ix)	$P \wedge \neg P \rightarrow Q$	(ex falso quodlibet)

EXERCISE 2.6. Classify the following $\mathcal{L}_1$-sentences as tautologies, contradictions or as sentences that are neither.

  (i) $P \wedge P$
  (ii) $((P \rightarrow Q) \rightarrow R) \leftrightarrow (P \rightarrow (Q \rightarrow R))$
(iii) $(P \leftrightarrow (Q \leftrightarrow R)) \leftrightarrow ((P \leftrightarrow Q) \leftrightarrow R)$
(iv) $\neg(P \rightarrow Q) \leftrightarrow (P \wedge \neg Q)$

EXERCISE 2.7. In the definition of truth in an $\mathcal{L}_1$-interpretation I have specified conditions under which a sentence is true in an $\mathcal{L}_1$-interpretation. These conditions also determine when a sentence is false because a sentence of $\mathcal{L}_1$ is false if and only if it is not true. Write down analogous clauses that indicate the conditions under which non-atomic sentences are false. The first two clauses (for $\neg$ and $\wedge$) are as follows:

(i) $|\neg\phi|_{\mathcal{A}} = F$ if and only if $|\phi|_{\mathcal{A}} = T$.
(ii) $|\phi \wedge \psi|_{\mathcal{A}} = F$ if and only if $|\phi|_{\mathcal{A}} = F$ or $|\psi|_{\mathcal{A}} = F$.

Complete the list with clauses for $\vee$, $\rightarrow$, and $\leftrightarrow$.

EXERCISE 2.8. Prove Theorem 2.12, that is, prove the following claim assuming that ϕ and all elements of Γ are $\mathcal{L}_1$-sentences:

> $\Gamma \vDash \phi$ if and only if the set containing all sentences in Γ and $\neg\phi$ is semantically inconsistent.

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3 Formalisation in Propositional Logic

EXERCISE 3.1. Discuss whether the following argument is propositionally valid.

> If Jones arrives at the airport after the scheduled departure time, the plane will wait for him. Therefore, if Jones arrives at the airport after the scheduled departure time and nobody notices that he arrives at the airport after the scheduled departure time, the plane will wait for Jones.

EXERCISE 3.2. Determine the scopes of the underlined occurrences of connectives in the following sentences, which have been abbreviated in accordance with the bracketing conventions.

 (i) $P \to \underline{\neg}(P_{44} \lor \neg(Q_3 \land \neg Q_3))$
 (ii) $P\underline{\land}Q \land \neg R \land Q$
 (iii) $P \to Q \land \neg R_2\underline{\land}\neg(P_2 \leftrightarrow P_1)$

EXERCISE 3.3. Draw truth tables for the following English expressions in the style of the truth table for 'A because B' in Section 3.1 of the Manual. That is, determine for (i)–(iv) below whether substituting a true sentence for A yields only true sentences or only false sentences or true and false sentences. Then check the result of substituting false sentences. Proceed in a similar way for phrases (v)–(vi), which contain A and B.

 (i) Robin believes that A
 (ii) Robin knows that A
 (iii) Robin knows that A, but it's not true that A
 (iv) The infallible clairvoyant believes that A
 (v) A, but B
 (vi) Suppose A; then B

EXERCISE 3.4. Formalise the following sentences as accurately as possible using the arrow →.

  (i) If God can create the soul without the body, then soul and body are different.
 (ii) The rise in interest rates is a sufficient reason for a house price crash.
(iii) The boy and the general are the same person, only if the general can remember what he did as a boy.
 (iv) My believing that the wall is yellow is a necessary condition for my knowing that the wall is yellow.

EXERCISE 3.5. Formalise the following sentences in the language of propositional logic. Your formalisations should be as detailed as possible.

  (i) Russell and Whitehead wrote *Principia Mathematica*.
 (ii) The traffic light turned green, and Bill pulled away.
(iii) Ben, who hates logic, is a philosophy student.

EXERCISE 3.6. Show that the following argument becomes propositionally valid after adding assumptions upon which the speaker might naturally be expected to be relying. Note any difficulties or points of interest.

> Many students will be either in Hegel's or in Schopenhauer's lectures, if they are scheduled at the same time. And of course Schopenhauer will schedule them at the same time as Hegel's. If Hegel's lectures are entertaining, then many students will go to them. That means of course many students will go to Hegel's but not many will go to Schopenhauer's lectures. For if Schopenhauer's lectures are entertaining, Hegel's must be entertaining as well; and of course many students will only come to Schopenhauer's lectures if they are entertaining.

4 The Syntax of Predicate Logic

EXERCISE 4.1. Determine whether the following expressions are formulae of $\mathcal{L}_2$ and say which of those are also sentences of $\mathcal{L}_2$. Add the omitted arity indices to all predicate letters and mark all free occurrences of variables. Bracketing conventions are not applied.

(i) $\forall x \, (P_1 x \to Q y)$
(ii) $\exists x \, \neg(\neg\neg\exists y \, P y \land \neg\neg\neg\neg\neg R x a)$
(iii) P
(iv) $\forall x \, \exists y \, \exists z \, (R_{25} x y z)$
(v) $\forall x \, \exists x \, Q x x$
(vi) $\neg(\neg(\exists x \, P x \land \exists y \, Q y))$
(vii) $\forall x \, (\exists y \, (P x y \land P x) \lor Q x y x)$

EXERCISE 4.2. The following expressions are abbreviations of formulae of $\mathcal{L}_2$. Supply all brackets and indices that have been omitted according to the notational conventions and mark all free occurrences of variables.

(i) $\forall x \, \forall y \, (P_4 x y \to P_4 y x \land R x)$
(ii) $\forall x \, R x x z \land \exists y \, R x z x$
(iii) $\neg \forall z_2 \, R x z$
(iv) $\forall x \, \neg\neg(P x y \lor R y x \lor R z y)$

EXERCISE 4.3. Provide $\mathcal{L}_2$-formalisations for the following English sentences. Make them as detailed as possible.

(i) London is big and ugly.
(ii) Culham is a large village.
(iii) A city has a city hall.
(iv) Material objects are divisible.
(v) Tom owns at least one car.

(vi) Tom owns at least one car and he won't sell it.
(vii) One man has visited every country.

EXERCISE 4.4. Translate the $\mathcal{L}_2$-sentences below into English using the following dictionary:

$$
\begin{array}{rl}
a: & \text{Tom} \\
P^1: & \ldots \text{ is a person} \\
Q^1: & \ldots \text{ acts freely}
\end{array}
$$

(i) Qa
(ii) $(Qa \vee \neg Pa)$
(iii) $\forall x\,(Px \rightarrow Qx)$
(iv) $\forall x\,(Px \leftrightarrow Qx)$
(v) $\neg \exists z_1\, Qz_1$

EXERCISE 4.5. Translate the $\mathcal{L}_2$-sentences below into English using the following dictionary:

$$
\begin{array}{rl}
P^1: & \ldots \text{ is a set} \\
R^2: & \ldots \text{ is an element of} \ldots
\end{array}
$$

(i) $\neg \exists z\, Pz$
(ii) $\neg \forall x\,(Px \rightarrow \exists y\, Ryx)$
(iii) $\exists x\,(Px \wedge \neg \exists y\, Ryx)$
(iv) $\neg \exists z\,(Pz \wedge \forall x\, Rxz)$

5 The Semantics of Predicate Logic

EXERCISE 5.1. Consider an $\mathcal{L}_2$-structure $\mathcal{S}$ with the domain $D_\mathcal{S}$ and the following semantical values of a, b, P, and R:

$$D_\mathcal{S} = \{1, 2, 3\}$$
$$|a|_\mathcal{S} = 1$$
$$|b|_\mathcal{S} = 3$$
$$|P^1|_\mathcal{S} = \{2\}$$
$$|R^2|_\mathcal{S} = \{\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 1, 3 \rangle\}$$

Are the following sentences true or false in this structure? Sketch proofs of your answers.

  (i) Pa
 (ii) Rab
(iii) Rba
 (iv) $Rab \leftrightarrow Rba$
  (v) $Rbb \vee (\neg Pa \wedge \neg Raa)$
 (vi) $\exists x\, Rax$
(vii) $\exists x (Rax \wedge Rxb)$
(viii) $Pb \vee \exists x\, Rxx$
 (ix) $\forall x\, \exists y\, Rxy$
  (x) $\forall x (Px \rightarrow (\exists y\, Ryx \wedge \exists y\, Rxy))$
 (xi) $\forall x (Px \rightarrow \exists y\, (Ryx \wedge Rxy))$

As an example I will show that (viii) is false in $\mathcal{S}$:

First I show that $|P^1b|_\mathcal{S}$ = F:

$$3 \notin \{2\}$$
$$|b|_\mathcal{S} \notin |P^1|_\mathcal{S}$$
$$|Pb|_\mathcal{S} = F$$

In the next step I prove $|\exists x\, Rxx|_\mathcal{S}$ = F. Let α be a variable assignment over $\mathcal{S}$. Then $|x|_\mathcal{S}^\alpha$ is 1 or 2 or 3. But neither $\langle 1, 1\rangle$ nor $\langle 2, 2\rangle$ nor $\langle 3, 3\rangle$ is in $|R|_\mathcal{S}$, that is, in $\{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 1, 3\rangle\}$. Therefore, there following holds:

$$\langle |x|_\mathcal{S}^\alpha, |x|_\mathcal{S}^\alpha\rangle \notin |R^2|_\mathcal{S}$$
$$|Rxx|_\mathcal{S}^\alpha = F$$
$$|\exists x\, Rxx|_\mathcal{S} = F$$

The last line holds because $\exists x\, Rxx$ is false if and only if Rxx is satisfied by no variable assignment.

Since $|Pb|_\mathcal{S}$ = F and $|\exists x\, Rxx|_\mathcal{S}$ = F it follows that $|Pb \vee \exists x\, Rxx|_\mathcal{S}$ = F.

EXERCISE 5.2. Justify the following claims by providing counterexamples. You do not have to prove that your structures are actually counterexamples, that is, you do not have to prove that the premises are true and the conclusions false in the respective structures.

  (i) $Pa \nvDash \exists x\, (Px \wedge Qx)$
 (ii) $\forall y\, (Py \rightarrow \exists x\, Ryx) \nvDash \forall x(Px \rightarrow \exists y\, Ryy)$
(iii) $\forall y\, Ryy \nvDash \forall x\, Rax$

EXERCISE 5.3. Prove the following claim assuming that ϕ and all elements of Γ are sentences of the language $\mathcal{L}_2$:

> $\Gamma \vDash \phi$ if and only if the set containing $\neg\phi$ and all elements of Γ is semantically inconsistent.

EXERCISE 5.4.   (i) Provide a sentence that contains no other than unary predicate letters and that is true in some structure with a domain containing at least three elements, but that is not true in any structure with a domain containing less than three elements.

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(ii) Provide a sentence containing no constants and predicate letters other than R^2 that is true in some structure with a domain containing at least two objects but that is not true in any structure with a domain containing only one object.

(iii) Provide a sentence containing no constants and predicate letters other than R^2 that is true in some structure with a domain containing at least three objects but that is not true in any structure with a domain containing less than three objects.

(iv) Provide a sentence that is true in some structure with an infinite domain but not in any structure with a finite domain.

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6 Natural Deduction

Further examples of proofs and hints for constructing proofs can be found on WebLearn.

Exercises 6.1–6.2 are on propositional logic only.

As there may be too many many exercises, I suggest that Exercises 6.2 and 6.5 and possibly some of the other exercises are postponed to the two remaining weeks.

EXERCISE 6.1. Establish the following claims by providing proofs in Natural Deduction.

(i) $\vdash P \to (R \lor P)$

(ii) $R \land Q \vdash Q \land R$

(iii) $P \to Q \vdash \neg Q \to \neg P$

(iv) $\vdash (P \to \neg P) \to \neg P$

(v) $P \leftrightarrow Q, \neg Q \vdash \neg P$

(vi) $P \land Q \to R \vdash P \to (Q \to R)$

(vii) $\neg(P \to Q) \vdash P$

EXERCISE 6.2. The solution to Exercise 3.6 consists in the formalisation of an English argument (plus additional premisses) in the language $\mathcal{L}_1$ of propositional logic and in a proof of the validity of the resulting argument in $\mathcal{L}_1$. There the task was to prove the validity of the sentence with an incomplete truth table. Alternatively one can show its validity by a proof in Natural Deduction. The English argument in Exercise 3.6 can be formalised as follows:

$$P \to (Q \lor R) \land \neg(Q \land R), P_1, P_1 \to P,$$
$$Q_1 \to Q, P_2 \to Q_1, R \to P_2 \vDash Q \land \neg R$$

Establish this claim by proving the following claim by means of a proof in Natural Deduction:

$$P \to (Q \vee R) \wedge \neg(Q \wedge R), P_1, P_1 \to P,$$
$$Q_1 \to Q, P_2 \to Q_1, R \to P_2 \vDash Q \wedge \neg R$$

EXERCISE 6.3. Prove the following claims.

(i) $\forall x\,(Px \to P_1 x), \neg P_1 a \vdash \neg Pa$
(ii) $\forall x\,(Px \to Qx), Pa \vdash \exists y\, Qy$
(iii) $\neg\forall x\, Qx \vdash \exists x\, \neg Qx$
(iv) $\exists x\, \neg Pxa \vdash \exists z\, \neg\forall y\, Pyz$
(v) $\exists x\, \exists y\, \forall z\, \forall x_1\, Pxyzx_1 \vdash \forall z\, \exists x\, \forall x_1\, \exists y\, Pxyzx_1$

EXERCISE 6.4. Formalise the following argument in $\mathcal{L}_2$:

> All philosophers who have studied logic know Gödel. Therefore, if all philosophers have studied logic they all know Gödel.

Show that the resulting argument in $\mathcal{L}_2$ is valid.

EXERCISE 6.5. Establish $\exists x\, \forall y\,(Rxy \leftrightarrow \neg Ryy) \vdash P$ by means of a proof in Natural Deduction (P is the sentence letter).

7 Formalisation in Predicate Logic

EXERCISE 7.1. Find the mistakes in the following proofs, that is, list all steps in the proof that are not licensed by a rule of the system of Natural Deduction. If possible, repair the proof by providing a correct proof. If the argument is not valid, provide a counterexample.

(i) $\exists x\, (Px \wedge Qx) \vdash \exists x\, Px \wedge \exists x\, Qx$

$$
\cfrac{
\cfrac{
\exists x(Px \wedge Qx) \qquad \cfrac{[Pa \wedge Qa]}{Pa}
}{
\cfrac{Pa}{\exists x Px}
}
\qquad
\cfrac{
\exists x(Px \wedge Qx) \qquad \cfrac{[Pa \wedge Qa]}{Qa}
}{
\cfrac{Qa}{\exists x Qx}
}
}{
\exists x\, Px \wedge \exists x\, Qx
}
$$

(ii) $\forall x\, \exists y\, Rxy \vdash \exists y\, \forall x\, Rxy$

$$
\cfrac{
\cfrac{
\cfrac{\forall x\, \exists y\, Rxy}{\exists y\, Ray} \qquad [Rab]
}{
\cfrac{\cfrac{Rab}{\forall x\, Rxb}}{\exists y\, \forall x\, Rxy}
}
}{}
$$

(iii) $\exists y\, (Py \rightarrow Qy) \vdash \forall x(Px \rightarrow Qx)$

$$
\cfrac{
\exists y\, (Py \rightarrow Qy) \qquad
\cfrac{
\cfrac{
\cfrac{[Pa] \qquad [Pa \rightarrow Qa]}{Qa}
}{
\cfrac{Pa \rightarrow Qa}{\forall x(Px \rightarrow Qx)}
}
}{}
}{
\forall x(Px \rightarrow Qx)
}
$$

EXERCISE 7.2. Formalise the following sentences in the language of $\mathcal{L}_2$. The formalisations should be as detailed as possible. Provide a dictionary for your translations.

 (i) Not every book author is famous.
 (ii) Some books are famous.
 (iii) A book is famous if and only if it's well written.
 (iv) Tom does not believe that not every book author is famous.

EXERCISE 7.3. Reveal the ambiguities in the following sentences by formalising each sentence in two (or more) different ways:

 (i) Ben despises a logician.
 (ii) Harry slanders Ron and his parents.
 (iii) A student is better than a tutor.
 (iv) Only rich Germans buy houses in Munich.
 (v) James likes a fast car.
 (vi) Some mistakes were made by everyone.

EXERCISE 7.4 (Russell's paradox). The following exercise deals with a paradox that shows that certain assumptions about the existence of sets and properties lead to a contradiction.

 (i) Using the dictionary

> R: … is an element of …

   translate the sentence $\exists x \, \forall y \, (Ryx \leftrightarrow \neg Ryy)$ into an English sentence.

 (ii) Using the dictionary

> R: … has … (as its property)

   translate the sentence $\exists x \, \forall y \, (Ryx \leftrightarrow \neg Ryy)$ into an English sentence.

 (iii) In Exercise 6.5 I asked for a proof of the following claim:

$$\exists x \, \forall y \, (Ryx \leftrightarrow \neg Ryy) \vdash P.$$

Show that any set of sentences containing the sentence

$$\exists x \, \forall y \, (Ryx \leftrightarrow \neg Ryy)$$

is syntactically inconsistent.

(iv) The expression $\{x : A\}$ is used as an abbreviation for 'the set of all x such that A', where A is a claim about x. What is the problem of defining sets in this way?

EXERCISE 7.5. Consider the following argument:

Everything has a cause. Therefore there is a cause of everything.

Is the argument valid in predicate logic? Substantiate your answer by proving or disproving the validity of the formalisation of the argument. Is the argument logically valid?

8 Identity and Definite Descriptions

EXERCISE 8.1. Add the brackets that have been omitted in accordance with the bracketing conventions to the following sentence:

$$\forall x \forall y \forall z (Px \wedge Py \wedge Pz \rightarrow x = y \vee y = z \vee x = z)$$

EXERCISE 8.2. Prove the following claims by means of counterexamples:

  (i) $Qab, Qba \not\models a = b$
  (ii) $\forall x \forall y (Px \rightarrow (Py \rightarrow \neg x = y)) \not\models \exists x \exists y \neg x = y$

You do not have to show that the premises are true and the conclusions are false in the models. Specifying the counterexample will suffice.

EXERCISE 8.3. The following sentence is to be formalised in $\mathcal{L}_=$:

> Paolo is a philosopher.

The following two formalisations are proposed:

  (i) Pa
  (ii) $\exists x (x = a \wedge Px)$

(ii) is arrived at by reasoning as follows: The 'is' in the original sentence can be replaced by 'is identical to', so the logical form of the sentence is:

> There is an x ((x is identical to Paolo) and (x is a philosopher))

Is there any reason to prefer one formalisation over the other?

EXERCISE 8.4. Establish the following claims by means of proofs in Natural Deduction.

  (i) $\vdash \exists y\, y = y$

(ii) $\exists x\, Px, \exists x\, \neg Px \;\vdash\; \exists x\, \exists y\, \neg x = y$

EXERCISE 8.5. Show that the following two sentences are logically equivalent in predicate logic with identity:

(i) $\exists x\, (\forall y(Py \to x = y) \wedge Px)$
(ii) $\exists x\, \forall y\, (Py \leftrightarrow x = y)$ Prove the equivalence by establishing the following two claims:

(i) $\exists x\, (\forall y\, (Py \to x = y) \wedge Px) \vdash \exists x\, \forall y\, (Py \leftrightarrow x = y)$
(ii) $\exists x\, \forall y\, (Py \leftrightarrow x = y) \vdash \exists x\, (\forall y\, (Py \to x = y) \wedge Px)$

EXERCISE 8.6. Formalise the following sentences as $\mathcal{L}_=$-sentences using the following dictionary:

P:	… is clever
Q:	… is a tutor
Q_1:	… is a philosophy student
R:	… is better than …

 (i) There are two philosophy students.
 (ii) The clever tutor is better than any philosophy student.
(iii) The philosophy student who is better than all tutors is clever.
(iv) There are fewer than three tutors.