

# INTRODUCTION TO LOGIC

## Lecture 8

### Identity and Definite Descriptions

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The analysis of the beginning would thus yield the notion of the unity of being and not-being—or, in a more reflected form, the unity of differentiatedness and non-differentiatedness, or the identity of identity and non-identity.

Hegel  
The Science of Logic

8.1 Qualitative and Numerical Identity

## The logicians' sense of 'identical'

In English, we use the words 'identity'/'identical' in a number of different ways.

### Wider uses of 'identity'/'identical'

- (1) Mancunians have a strong sense of cultural identity.
- (2) Dr. Jekyll has multiple identities.
- (3) Jedward are almost completely identical.

The sense of 'identity' used in (3) is sometimes called '**qualitative identity**'.

- (3) says that John and Edward are almost exactly similar in every respect.

None of these uses of 'identical' is the logicians' use.

## Outline

- (1) The language of predicate logic with identity:  $\mathcal{L}_=$ 
  - Syntax
  - Semantics
  - Proof theory
- (2) Formalisation in  $\mathcal{L}_=$ 
  - Numerical quantifiers
  - Definite descriptions

8.1 Qualitative and Numerical Identity

In logic, we always use 'identical' in the following strict sense

A is identical to B iff A is the very same thing as B  
i.e. A and B are one and the same thing.

This is sometimes called '**numerical identity**'

(Unless otherwise stated 'identity'/'identical' henceforth mean numerical identity/numerically identical.)

### Examples

- George Orwell is identical to Eric Arthur Blair
- Dr. Jekyll is identical to Mr. Hyde
- John is not identical to Edward

## A third formal language

The new language makes a single addition to  $\mathcal{L}_2$ .

### The language $\mathcal{L}_=$

The language  $\mathcal{L}_=$  of predicate logic with identity adds a single binary predicate letter to the language of predicate logic  $\mathcal{L}_2$ .

- $\mathcal{L}_=$  adds the identity predicate  $=$  to  $\mathcal{L}_2$

$=$  differs from the other predicate letters in several way.

- $P, R^2$ , etc., are non-logical expressions.  
Different  $\mathcal{L}_2$ -structures interpret them differently.
- $=$  is treated as a logical expression.  
It always has the same interpretation in any structure.
- Minor difference: we write  $a = b$  (rather than  $=ab$ ).

## Syntax

We make a slight change to the definition of atomic formula.

### Definition (atomic formulae of $\mathcal{L}_=$ )

All atomic formulae of  $\mathcal{L}_2$  are atomic formulae of  $\mathcal{L}_=$ .

Furthermore, if  $s$  and  $t$  are variables or constants, then  $s=t$  is an atomic formula of  $\mathcal{L}_=$ .

The definition of formula and sentence is otherwise just like the definition for  $\mathcal{L}_2$ .

### Examples

- Atomic  $\mathcal{L}_=$ -formulae:  $c=a, x=y_3, x=a, R^2ax$ .
- Complex  $\mathcal{L}_=$ -formulae:  $\neg x=y, \forall x(Rxy_2 \rightarrow y_2=x)$ .

## Semantics

The definition of structure is just the same as before.

### Definition: $\mathcal{L}_=$ -structure

An  $\mathcal{L}_=$ -structure is simply an  $\mathcal{L}_2$ -structure.

Why no change?

- Structures interpret non-logical expressions like  $P$  and  $a$ .
- Structures do not interpret logical expressions like  $\neg$  and  $\forall x$ .
- The fixed interpretation of logical expressions is specified in the definition of satisfaction.  
e.g.  $|\neg\phi|_{\mathcal{A}}^{\alpha} = \text{T}$  iff  $|\phi|_{\mathcal{A}}^{\alpha} = \text{F}$
- Similarly  $=$  is treated as a logical expression, which is not assigned a semantic value by the structure.
- The fixed interpretation of  $=$  is specified in the definition of satisfaction.

Let  $\mathcal{A}$  be an  $\mathcal{L}_=$ -structure (i.e. an  $\mathcal{L}_2$ -structure).

Truth in  $\mathcal{A}$  is defined just as before with one addition:

### Definition: satisfaction of identity statements

(ix)  $|s=t|_{\mathcal{A}}^{\alpha} = \text{T}$  if and only if  $|s|_{\mathcal{A}}^{\alpha} = |t|_{\mathcal{A}}^{\alpha}$ .

**Note:**  $=$  is used in both  $\mathcal{L}_=$  and the metalanguage.

The other definitions from Chapter 5 carry over directly to  $\mathcal{L}_=$ .

- Valid
- Logical truth
- Contradiction
- Logically equivalent
- Semantically consistent

These are defined just as before replacing ' $\mathcal{L}_2$ ' with ' $\mathcal{L}_=$ '.

**Worked example**

$\forall x \forall y x = y$  isn't logically true.

Counterexample: let  $\mathcal{A}$  be an  $\mathcal{L}_=$ -structure with domain  $\{1, 2\}$ .

Proof. Let  $\alpha$  be an assignment over  $\mathcal{A}$ .

Sufficient to prove (STP:)  $\forall x \forall y x = y$  is false in  $\mathcal{A}$  under  $\alpha$ .

**Now:**  $|\forall x \forall y x = y|_{\mathcal{A}}^{\alpha} = \text{T}$  iff  $|\forall y x = y|_{\mathcal{A}}^{\beta} = \text{T}$  for every  $\beta$  differing from  $\alpha$  at most in  $x$ .  $|\forall x \forall y x = y|_{\mathcal{A}}^{\alpha} = \text{F}$  iff  $|\forall y x = y|_{\mathcal{A}}^{\beta} = \text{F}$  for **some**  $\beta$  differing from  $\alpha$  at most in  $x$ .

**STP:**  $|\forall y x = y|_{\mathcal{A}}^{\beta} = \text{F}$  for some assignment  $\beta$  differing from  $\alpha$  at most in  $x$ .

**But:**  $|\forall y x = y|_{\mathcal{A}}^{\beta} = \text{T}$  iff  $|x = y|_{\mathcal{A}}^{\gamma} = \text{T}$  for every  $\gamma$  differing from  $\beta$  at most in  $y$ .  $|\forall y x = y|_{\mathcal{A}}^{\beta} = \text{F}$  iff  $|x = y|_{\mathcal{A}}^{\gamma} = \text{F}$  for **some**  $\gamma$  differing from  $\beta$  at most in  $y$ .

**STP:**  $|x = y|_{\mathcal{A}}^{\gamma} = \text{F}$  for some  $\gamma$  differing from  $\beta$  in at most  $y$ , differing from  $\alpha$  in at most  $x$  and  $y$ .

**So:** Let  $\gamma$  assign  $x$  to 1 and  $y$  to 2 (otherwise agreeing with  $\alpha$ )

**=Elim**

If  $s$  and  $t$  are constants, the result of appending  $\phi[t/v]$  to a proof of  $\phi[s/v]$  and a proof of  $s = t$  or  $t = s$  is a proof of  $\phi[t/v]$ .

$$\frac{\begin{array}{c} \vdots \\ \phi[s/v] \end{array} \quad \begin{array}{c} \vdots \\ s = t \end{array}}{\phi[t/v]} = \text{Elim} \qquad \frac{\begin{array}{c} \vdots \\ \phi[s/v] \end{array} \quad \begin{array}{c} \vdots \\ t = s \end{array}}{\phi[t/v]} = \text{Elim}$$

**Proof theory**

Natural Deduction for  $\mathcal{L}_=$  has the same rules as Natural Deduction for  $\mathcal{L}_2$  with the addition of rules for  $=$ .

**=Intro**

Any assumption of the form  $t = t$  where  $t$  is a constant can and must be discharged.

A proof with an application of =Intro looks like this:

$$\frac{[t = t]}{\vdots}$$

**Example:** prove  $\vdash \forall z (z = z)$

**Worked example:** prove the following.

$\vdash \forall x \forall y (Rxy \rightarrow (x = y \rightarrow Ryx))$

$$\frac{\begin{array}{c} \vdots \\ \phi[s/v] \end{array} \quad \begin{array}{c} \vdots \\ s = t \end{array}}{\phi[t/v]} = \text{Elim} \qquad \frac{\begin{array}{c} \vdots \\ \phi[s/v] \end{array} \quad \begin{array}{c} \vdots \\ t = s \end{array}}{\phi[t/v]} = \text{Elim}$$

## Adequacy

Soundness and Completeness still hold.

Let  $\Gamma$  be a set of  $\mathcal{L}_=$ -sentences and  $\phi$  an  $\mathcal{L}_=$ -sentence.

### Theorem (adequacy)

$\Gamma \vdash \phi$  if and only if  $\Gamma \models \phi$ .

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Identity can also be used to formalise numerical quantifiers.

Dictionary:  $P$ : ... is a perfect being.

### Formalise

(1) There are at least two perfect beings.

Incorrect formalisation:  $\exists x \exists y (Px \wedge Py)$ .

Correct formalisation:  $\exists x \exists y (Px \wedge Py \wedge \neg x = y)$ .

(2) There is at most one perfect being.

Formalisation:  $\neg \exists x \exists y (Px \wedge Py \wedge \neg x = y)$ .

Alternative formalisation:  $\forall x \forall y ((Px \wedge Py) \rightarrow x = y)$ .

(3) There is exactly one perfect being.

Formalisation:  $\exists x Px \wedge \forall x \forall y ((Px \wedge Py) \rightarrow x = y)$ .

Alternative formalisation:  $\exists x (Px \wedge \forall y (Py \rightarrow y = x))$ .

## Formalisation with identity

Using  $=$  one can formalise ‘is [identical to]’ in English.

### Formalise:

William II is Wilhelm II.

Formalisation:  $a = b$ .

Dictionary:  $a$ : William II.  $b$ : Wilhelm II.

**Note:** don’t confuse the ‘is’ of identity with the ‘is’ of predication.

### Formalise:

Wilhelm II is an emperor.

Formalisation:  $Ea$ .

Dictionary:  $a$ : Wilhelm.  $E$ : ... is an emperor.

Here ‘is’ forms part of the predicate ‘is an emperor.’

## Definite descriptions

### Examples of definite descriptions:

- ‘the Queen’
- ‘Bellerophon’s winged horse’
- ‘the author of Ulysses’

In  $\mathcal{L}_2$ : the best we can do is to formalise definite descriptions as constants.

But this isn’t perfect...

**Example** **Not valid**

Bellerophon's winged horse isn't real; so there is something that is Bellerophon's winged horse.

The obvious formalisation with constants is valid.

Formalisation: premiss:  $\neg Rb$ . Conclusion:  $\exists x(x = b)$ .

Dictionary:  $R$ : ... is real.  $b$ : Bellerophon's winged horse.

(In fact: the conclusion is a logical truth.)

Source of the trouble:

- $\mathcal{L}_=$ -constants always refer to an object in a  $\mathcal{L}_=$ -structure.
- definite descriptions may fail to pick out a unique object.

**Formalise:**

Bellerophon's winged horse isn't real.

$R$ : ... is real.  $B$ : ... is a winged horse belonging to Bellerophon.

On Russell's view this can have two readings.

Paraphrase 1: (i) there is exactly one winged horse belonging to Bellerophon and (ii) it is not real.

Formalisation 1:  $\exists x(Bx \wedge \forall y(By \rightarrow y = x) \wedge \neg Rx)$ .

Dubious: this is true only if there are non-real things .

Paraphrase 2: It's not the case that ((i) there is exactly one winged horse belonging to Bellerophon and (ii) it is real).

Formalisation 2:  $\neg \exists x(Bx \wedge \forall y(By \rightarrow y = x) \wedge Rx)$ .

**Russell's theory of descriptions.**

There's a better way to formalise definite descriptions in  $\mathcal{L}_=$ .

**Formalise:**

The author of Ulysses wrote Dubliners.

Russell analyses this as the conjunction of two claims.

- (i) **There is exactly one author of Ulysses**
- (ii) **and it wrote Dubliners.**

Dictionary:  $A$ : ... is an author of Ulysses.

$W$ : ... wrote Dubliners.

Formalisation:  $\exists x(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge Wx)$

**Example** **Not valid**

Bellerophon's winged horse isn't real; so there is something that is Bellerophon's winged horse.

We can capture its non-validity by using the second formalisation of the premiss.

Dictionary:  $R$ : ... is real.

$B$ : ... is a winged horse belonging to Bellerophon.

**Formalisation** **Not valid**

Premiss:  $\neg \exists x(Bx \wedge \forall y(By \rightarrow y = x) \wedge Rx)$ .

Conclusion:  $\exists xBx$ .

The structure  $\mathcal{A}$  is a counterexample to this argument.

$$D_{\mathcal{A}} = \{x : x \text{ is a horse}\}; |B|_{\mathcal{A}} = \emptyset.$$

(It doesn't matter what the extension of  $R$  is here.)

## Multiple descriptions

We deal with these much like multiple quantifiers.

### Formalise

The author of Ulysses likes the author of the Odyssey

Dictionary: U: ... is an author of Ulysses

O: ... is an author of the Odyssey. L: ... likes ...

It's helpful to break this into two steps.

### Partial formalisation:

$$\exists x_1 (Ux_1 \wedge \forall y_1 (Uy_1 \rightarrow y_1 = x_1) \\ \wedge x_1 \text{ likes the author of the Odyssey})$$

It remains to formalise 'x<sub>1</sub> likes the author of the Odyssey'.

x<sub>1</sub> likes the author of the Odyssey

Paraphrase: the author of the Odyssey is liked by x<sub>1</sub>.

Formalisation:  $\exists x_2 (Ox_2 \wedge \forall y_2 (Oy_2 \rightarrow y_2 = x_2) \wedge Lx_1x_2)$ .

Finally, we put this together with what we had before.

**The author of Ulysses likes the author of the Odyssey**

$$\exists x_1 (Ux_1 \wedge \forall y_1 (Uy_1 \rightarrow y_1 = x_1) \\ \wedge x_1 \text{ likes the author of the Odyssey}).$$

$$\exists x_1 (Ux_1 \wedge \forall y_1 (Uy_1 \rightarrow y_1 = x_1) \\ \wedge \exists x_2 (Ox_2 \wedge \forall y_2 (Oy_2 \rightarrow y_2 = x_2) \wedge Lx_1x_2)).$$

Closing

Closing

## Logical constants

$\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists$  and  $=$  are our **only** logical expressions.<sup>45</sup>

This raises two questions:

**Q1** What's special about these expressions?

**A1** Alfred Tarski proposes to analyse topic neutrality in terms of 'permutation invariance'

- Roughly: logical expressions are the ones whose meaning is insensitive to which object is which.
- See Tarski 'What are Logical Notions?' *History and Philosophy of Logic* 7, 143–154.

**Q2** What happens if we add more logical constants?

**A2** This is the business of philosophical logic.

**Extension of  $\mathcal{L}_2$**     **New logical expressions**

Generalised quantifiers	more than half infinitely many, etc.
Modal logic	It is necessarily the case that It is possibly the case that
Deontic logic	It is obligatory that It is permissible that

See the finals paper 127: Philosophical Logic.