

# INTRODUCTION TO LOGIC

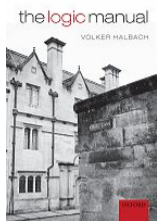
## 1 Sets, Relations, and Arguments

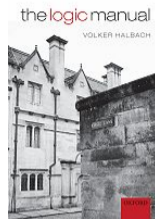
Volker Halbach

Pure logic is the ruin of the spirit.

*Antoine de Saint-Exupéry*

- *The Logic Manual*

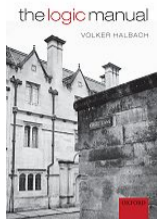




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web page for the book: <http://users.ox.ac.uk/~logicman/index.html>

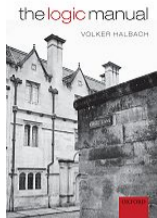
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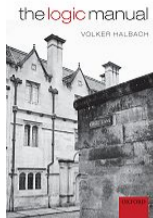
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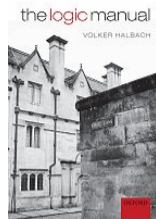
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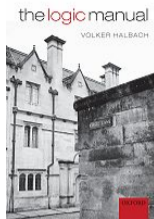
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Mark Sainsbury: *Logical Forms: An Introduction to Philosophical Logic*, Blackwell, second edition, 2001



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- Used in linguistics, mathematics, computer science,...
- Helps us make fine-grained conceptual distinctions.
- Allows us establish claims about theories such as consistency, independence and completeness.

# Arguments

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Sentences that are true or false are called **declarative sentences**.

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An **argument** consists of a set of declarative sentences (the **premisses**) and a declarative sentence (the **conclusion**) somehow marked as the concluded sentence.

## Example

I'm not dreaming if I can see the computer in front of me. I can see the computer in front of me. Therefore I'm not dreaming.

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'I'm not dreaming' is the conclusion, which is marked by 'therefore'.

Occasionally the conclusion precedes the premisses or is found between premisses. The conclusion needn't be marked as such by 'therefore' or a similar phrase.



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**I'm not dreaming.** For, if I can see the computer in front of me, I'm not dreaming; and I can see the computer in front of me.

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### Example

I'm not dreaming, if I can see the computer in front of me. Thus, **I'm not dreaming.** This is because I can see the computer in front of me.

The point of ‘good’ arguments is that the truth of the premisses guarantees the truth of the conclusion. Many arguments with this property exhibit certain patterns.

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$\mathcal{M}$  is recursively saturated, if  $\mathcal{M}$  can be expanded to a model of CT.  $\mathcal{M}$  can be expanded to a model of CT. Therefore  $\mathcal{M}$  is recursively saturated.

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### general form of both arguments

$A$  if  $B$ .  $B$ . Therefore  $A$ .

Logicians are interested in the patterns of ‘good’ arguments that cannot take one from true premisses to a false conclusion.

## Characterisation

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## Example

Socrates is a man. All men are mortal. Therefore Socrates is mortal.

## Example

Volker is a logician. All logicians own an SUV. Therefore Volker owns an SUV.

## Example

Volker is a logician. Some logicians are German. Therefore Volker is German.

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Both premisses and the conclusion are true. However, the argument is not logically valid. We just need to interpret 'Volker' as 'James':

## Example

James Studd is a logician. Some logicians are German. Therefore James is German.

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Only the subject-specific vocabulary can be re-interpreted, not the logical vocabulary such as 'all', 'some', 'and', or 'not'.

## Features of logically valid arguments:

- The truth of the conclusion follows from the truth of the premisses independently of what the subject-specific expressions mean. Whatever tortoises are, whoever Zeno is, whatever exists: if the premisses of the argument are true the conclusion will be true.

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- It's not possible that the premisses of a logically valid argument are true and its conclusion is false.
- In a logically valid argument the conclusion can be false (in that case at least one of its premisses is false).

The following argument isn't logically valid:

### Example

Every EU citizen can enter the US without a visa. Max is a citizen of Sweden.  
Therefore Max can enter the US without a visa.

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However, one can transform it into a logically valid argument by adding a premiss:

### Example

Every EU citizen can enter the US without a visa. Max is a citizen of Sweden. Every citizen of Sweden is a EU citizen. Therefore Max can enter the US without a visa.

## Characterisation (consistency)

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### Characterisation (logical truth)

A sentence is **logically true** if and only if it is true under all interpretations.

‘All metaphysicians are metaphysicians.’

## Characterisation (contradiction)

A sentence is a **contradiction** if and only if it is false under all interpretations.

‘Some metaphysicians who are also ethicists aren’t metaphysicians.’

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I’ll make these notions precise for the formal languages of propositional and predicate logic.

# Sets

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A set is a collection of objects.

The objects in the set are the **elements** of the set.

There is a set that has exactly all books as elements.

There is a set that has Volker Halbach as its only element.

Sets are identical if and only if they have the same elements.

### Example

The set of all animals with kidneys and the set of all animals with a heart are identical, because exactly those animals that have kidneys also have a heart and vice versa.

The claim ' $a$  is an element of  $S$ ' can be written as ' $a \in S$ '. One also says ' $S$  contains  $a$ ' or ' $a$  is in  $S$ '.

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The set  $\{\text{Oxford}, \emptyset, \text{Volker Halbach}\}$  has as its elements exactly three things: Oxford, the empty set  $\emptyset$ , and me.



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Here is another way to denote sets:

$$\{d : d \text{ is an animal with a heart}\}$$

is the set of all animals with a heart. It has as its elements exactly all animals with a heart.

## Example

$$\{\text{Oxford}, \emptyset, \text{Volker Halbach}\} = \{\text{Volker Halbach}, \text{Oxford}, \emptyset\}$$

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# Relations

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Ordered pairs are identical if and only if they agree in their first and second components, or more formally:

$$\langle d, e \rangle = \langle f, g \rangle \text{ iff } (d = f \text{ and } e = g)$$

The abbreviation ‘iff’ stands for ‘if and only if’.



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There are also triples (3-tuples) like  $\langle \text{London, Munich, Rome} \rangle$ , quadruples, 5-tuples, 6-tuples etc.

## Definition

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## Example

The relation of *being a bigger city than* is the set  $\{\langle \text{London, Munich} \rangle, \langle \text{London, Birmingham} \rangle, \langle \text{Paris, Milan} \rangle \dots\}$ .

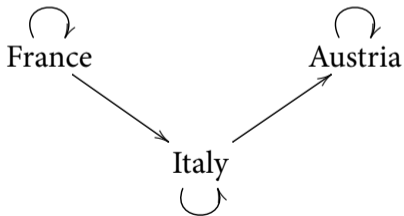
The following set is a binary relation:

$$\{\langle \text{France, Italy} \rangle, \langle \text{Italy, Austria} \rangle, \langle \text{France, France} \rangle, \\ \langle \text{Italy, Italy} \rangle, \langle \text{Austria, Austria} \rangle\}$$

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Some relations can be visualised by diagrams. Every pair corresponds to an arrow:



I'll mention only some properties of relations.

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## Definition

A binary relation  $R$  is **symmetric** iff for all  $d, e$ : if  $\langle d, e \rangle \in R$  then  $\langle e, d \rangle \in R$ .



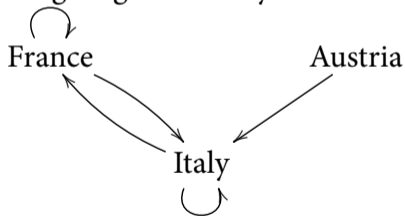
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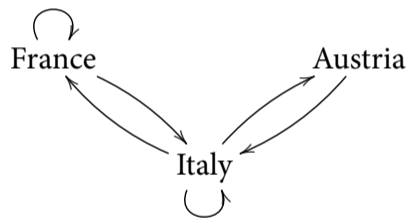
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The relation with the following diagram isn't symmetric:



The pair  $\langle \text{Austria}, \text{Italy} \rangle$  is in the relation, but the pair  $\langle \text{Italy}, \text{Austria} \rangle$  isn't.

The relation with the following diagram is symmetric.



## Definition

A binary relation is transitive iff for all  $d, e, f$ : if  $\langle d, e \rangle \in R$  and  $\langle e, f \rangle \in R$ , then also  $\langle d, f \rangle \in R$

In the diagram of a transitive relation there is for any two-arrow way from an point to a point a direct arrow.

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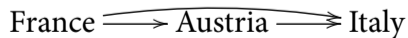
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This is the diagram of a relation that's not transitive:



This is the diagram of a relation that is transitive:



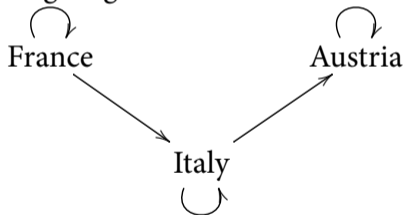
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A binary relation  $R$  is **reflexive on a set  $S$**  iff for all  $d$  in  $S$  the pair  $\langle d, d \rangle$  is an element of  $R$ .

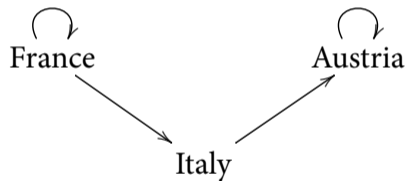
## Definition

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The relation with the following diagram is reflexive on the set  $\{\text{France, Austria, Italy}\}$ .



The relation with the following diagram is not reflexive on  $\{\text{France, Austria, Italy}\}$ , but reflexive on  $\{\text{France, Austria}\}$ :



# Functions

## Definition

A binary relation  $R$  is a **function** iff for all  $d, e, f$ : if  $\langle d, e \rangle \in R$  and  $\langle d, f \rangle \in R$  then  $e = f$ .

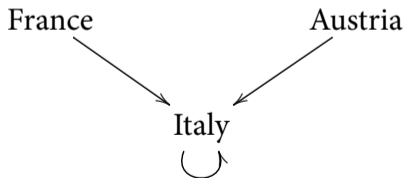


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The relation with the following diagram is a function:



There is at most one arrow leaving from every point in the diagram of a function.

## Example

The set of all ordered pairs  $\langle d, e \rangle$  such that  $e$  is the (biological) mother of  $d$  is a function.

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You might know examples of the following kind from school:

### Example

The set of all pairs  $\langle d, d^2 \rangle$  where  $d$  is some real number is a function.

One can't write down all the pairs, but the function would look like this:

$\{\langle 2, 4 \rangle, \langle 1, 1 \rangle, \langle 5, 25 \rangle, \langle \frac{1}{2}, \frac{1}{4} \rangle \dots\}$

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One also think of a function as something that yields an 'output', e.g. 25 when given an input, e.g. 5, or that 'assigns the value 25 to the argument 5'.