INTRODUCTION TO LOGIC

4 The Syntax of Predicate Logic

Volker Halbach

I counsel you, dear friend, in sum,
That first you take collegium logicum.
Your spirit's then well broken in for you,
In Spanish boots laced tightly to,
That you henceforth may more deliberately keep
The path of thought and straight along it creep,
And not perchance criss-cross may go,
A- will-o'-wisping to and fro.
Then you'll be taught full many a day
What at one stroke you've done alway,
Like eating and like drinking free,
It now must go like: One! Two! Three!

Goethe, Faust I

The argument

> *Zeno is a tortoise. All tortoises are toothless. Therefore Zeno is toothless.*

is logically valid but not propositionally valid: replacing ‘Zeno is tortoise’, ‘All tortoises are toothless’, and ‘Zeno is toothless’ (uniformly) with other sentences doesn’t always yield another valid argument.

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In the language $\mathcal{L}_2$ of predicate logic such arguments can be analysed.

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The upper index is called the 'arity index'. It indicates how many designators the predicate takes. In the above examples

'is tall' takes one

'is bigger than' takes two

'opens … with' takes three

So the predicate expression 'is tall' can be translated as P^1, 'is bigger than' as Q^2, and 'opens … with' as R^3.

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Especially proper names are formalised in $\mathcal{L}_2$ as 'constants'. Constants are a, b, c, a_1, b_1, c_1, and so on.

Example

Tom hates Mary.

formalisation: P^2ab

dictionary:

$$\begin{aligned} P^2&: \quad \ldots \text{ hates } \ldots \\ a&: \quad \text{Tom} \\ b&: \quad \text{Mary} \end{aligned}$$

The arity index 2 is important: P^2 takes two constants (as 'hates' takes two designators).

The order of a and b matters.

Using the same dictionary

Example

Tom hates Mary or Mary hates Tom.

is formalised as $(P^2ab \lor P^2ba)$.
Sentences of $\mathcal{L}_2$ can be combined using connectives in the same way as $\mathcal{L}_1$-sentences.

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Example

A person is morally responsible if and only if she acts freely.

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the pronoun ‘she’ is used to express generalisation.

There are other ways to express generalisation, but pronouns offer a very flexible and efficient way of generalising.

Example

If an object is part of another object and it is part of still another object , then it is a part of it .

Example

If an object_1 is part of another object_2 and it_2 is part of still another object_3, then it_1 is a part of it_3.

Using numerical subscripts one can make the reference of the pronouns clear and unambiguous. In the language $\mathcal{L}_2$ the variables $x, y, z, x_1, y_1, z_1, x_2, \dots$ play the role of pronouns that are used for quantification.

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If an object x_1 is part of another object x_2 and x_2 is part of still another object x_3, then x_1 is a part of x_3.

To save on indices I'll use $x, y, z, x_1, y_1, z_1, x_2, \dots$

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Example

All tortoises are reptiles.

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Example

All tortoises are reptiles.

This generalisation can be reexpressed using 'something':

If something is a tortoise then it is a reptile.

Here is how to express a generalisation using only pronouns and generalisations over *all* objects.

Example

All tortoises are reptiles.

… or as a generalisation over everything:

For everything: if it is a tortoise then it is a reptile.

Here is how to express a generalisation using only pronouns and generalisations over *all* objects.

Example

All tortoises are reptiles.

Replacing pronouns with variables gives the logical form:

For all x: if x is a tortoise then x is a reptile

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Example

All tortoises are reptiles.

Replacing pronouns with variables gives the logical form:

For all x: if x is a tortoise then x is a reptile

So I need an expression in $\mathcal{L}_2$ that corresponds to 'for all'. The symbol $\forall$ is used for that purpose.

I didn't specify the syntax of $\mathcal{L}_2$, and I didn't say anything about the semantics of $\mathcal{L}_2$; thus we cannot really discuss translations from English into $\mathcal{L}_2$. But I'll sketch how we'll carry out formalisations in $\mathcal{L}_2$.

For formalisations one can again first give the logical form and then replace the English expressions by the correponding $\mathcal{L}_2$-symbols.

Example

All epistemologists are philosophers.

This is the original sentence.

Example

For everything: if it is an epistemologist then it is a philosopher.

I reexpress the general claim using a pronoun.

Example

For all x: if x is an epistemologist then x is a philosopher.

I replace the pronoun with a variable.

Example

For all x: (if x is an epistemologist then x is a philosopher)

'For all x' is in logical form. I turn to the remaining sentence 'if x is an epistemologist then x is a philosopher' and apply the methods from propositional logic. As 'if … then' is a standard connective I put the expression in brackets and turn to the subsentences.

Example

For all x: (if (x is an epistemologist) then x is a philosopher)

'x is an epistemologist' is a designator and a predicate: it cannot be sensibly be reformulated with a connective or a generalising expression such as 'for all'. so I enclose it in brackets and leave it alone.

Example

For all x: (if (x is an epistemologist) then (x is a philosopher))

The same applies to 'x is a philosopher'.

From the logical form the $\mathcal{L}_2$-sentence can be obtained by the following substitutions

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$$\forall x \ (\quad (x \text{ is an epistemologist}) \quad \rightarrow (x \text{ is a philosopher}))$$

The standard connectives are replaced with the respective symbols. 'for all' is replaced with $\forall$.

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Example

$$\forall x \; (\qquad Q^1 x \qquad \to \qquad P^1 x)$$

'x is an epistemologist' is formalised as the atomic formula Q^1x, and 'x is a philosopher is formalised as P^1x.' as in the case of propositional logic the brackets around sentence that are not further analysable are dropped.

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Example

$$\forall x \quad (\qquad Q^1x \quad \to \qquad P^1x)$$

So the sentence is formalised as $\forall x\,(Q^1x \to P^1x)$ with the following dictionary:

P^1: … is a philosopher
Q^1: … is an epistemologist

Example

Some philosophers are logicians.

This is the original sentence.

Example

At least one philosopher is a logician.

I understand the sentence as saying this.

Example

There is at least one thing such that it is a philosopher and it is a logician.

I reexpress the claim using a pronoun. Now this isn't a generalisation but rather an existential claim. So I put 'at least one thing' rather than 'everything'.

Example

There is at least one x: x is a philosopher and x is a logician.

'There is at least one x' is in logical form. I turn to the remaining sentence.

Example

There is at least one x: (x is a philosopher and x is a logician)

As ‘and’ is a standard connective I put the expression in brackets and turn to the subsentences.

Example

There is at least one x: ((x is a philosopher) and (x is a logician))

‘x is a philosopher’ and ‘x is a logician’ cannot be further analysed with a connective or an expression such as ‘for all’ ‘there is at least one’. So I enclose them in brackets and leave them alone.

Example

There is at least one x: ((x is a philosopher) and (x is a logician))

This is formalised as:

$$\exists x\,(P^1 x \wedge R^1 x)$$

P^1: … is a philosopher
R^1: … is a logician

Example

All persons have a soul.

This is my example.

Example

For all x: if x is a person then x has a soul

I reformulate the sentence using variables (I skip the step with pronouns.)

Example

For all x: (if x is a person then x has a soul)

'if … then' is a standard connective, so I enclose the expression with this connective in brackets…

Example

For all x: (if x is a person then x has a soul)

… and turn to the first subsentence, which cannot be further reformulated using connectives or quantifying expressions such as ‘for all’ or ‘there is at least one’.

Example

For all x: (if (x is a person) then x has a soul)

So I enclose it in brackets…

Example

For all x: (if (x is a person) then x has a soul)

… and turn to the other sentence. ‘x has a soul’ contains an existential claim. It means that x has at least one soul, that is, there is at least one y such that x has y and y is a soul.

Example

For all x: (if ((x is a person) then there is at least one y: x has y and y is a soul)

So I replace 'x has a soul' with this reformulation. Here he must use a *new* variable (ie, a variable different from x) because x should still refer back to 'forall x' and not get caught (bound) by 'there is at least one'.

Example

For all x: (if (x is a person) then there is at least one y: (x has y and y is a soul))

I introduce brackets for the standard connective ‘and’.

Example

For all x: (if (x is a person) then there is at least one y: ((x has y) and (y is a soul)))

'x has y' and 'y is a soul' cannot be further analysed and I am done.

Starting from the logical I go on to the formalisation:

Example

For all x: (if (x is a person) then there is at least one y: ((x has y) and (y is a soul)))

This is the logical form.

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Example

$$\forall x \, ((x \text{ is a person}) \rightarrow \quad \exists y \, ((x \text{ has } y) \wedge (y \text{ is a soul})))$$

I introduce the symbols for connectives and quantifiers.

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Example

$\forall x\,(P^1x \rightarrow \exists y\,(Q^2xy \wedge R^1y))$

Predicate expressions are replaced with predicate letters of an appropriate arity…

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Example

$\forall x\,(P^1x \to \exists y\,(Q^2xy \wedge R^1y))$

… using the following dictionary:

P^1: … is a person
Q^2: … has …
R^1: … is a soul

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Definition (predicate letters)

All expressions of the form P^k_n, Q^k_n, or R^k_n are predicate letters, where k and n are either missing (no symbol) or a numeral '1', '2', '3', ...

So the letter P with or without numerals '1', '2', and so on as upper and/or lower indices is a predicate letter, and similarly for Q and R. The sentence letters P, Q, R, P_1, Q_1, ... are also predicate letters, according to this definition. Furthermore, P^1, Q^1, R^1, P^1_1, Q^1_1, R^1_1, P^1_2, Q^1_2, R^1_2, ..., P^2_1, Q^2_1, R^2_1, P^2_2, Q^2_2, R^2_2, and so on, are predicate letters.

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Definition (variables)

$x, y, z, x_1, y_1, z_1, x_2, \ldots$ are variables.

Definition (atomic formulae of $\mathcal{L}_2$)

If Z is a predicate letter of arity n and each of $t_1, \ldots, t_n$ is a variable or a constant, then $Zt_1 \ldots t_n$ is an atomic formula of $\mathcal{L}_2$.

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P^2cy

$P^3_5x_{31}c_4y$

R^2xx

Definition

A quantifier is an expression $\forall v$ or $\exists v$ where v is a variable.

Thus, $\forall x_{348}$ and $\exists z$ are quantifiers.

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(iii) If v is a variable and ϕ is a formula, then $\forall v\, \phi$ and $\exists v\, \phi$ are formulae of $\mathcal{L}_2$.

Example

The following expressions are formulae of $\mathcal{L}_2$:

$$\forall x\,(P^2 xa \to Q^1 x)$$

$$\forall z_{77}\, \neg\exists y_3\, \exists z_{45}(P^2 xy \to \exists x_2(R^4_3 z_{77} c_3 x z_{77} \wedge Q))$$

$$(\exists x\, P^1 x \leftrightarrow \neg\exists y\, \exists y\, Q^2 yy)$$

$$\forall x\, \exists z\, R^2 az$$

The formula $P^1 x$ isn't a sentence. Only once the variable x is used or *bound* by some quantifier is becomes a sentence.

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Example

$$(P^1 x \to Q^2 a x)$$

In this formula both occurrences of the variable x are free.

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Example

$\forall x\,(P^1 x \to Q^2 a x)$

Now both occurrences of the variable x refer back to the quantifier $\forall x$, so they are both bound.

Example

$(\forall x\, P^1 x \rightarrow Q^2 a x)$

In this formula only the first red occurrence of x refers back to $\forall x$; it's bound by this quantifier; the second (i.e. green) occurrence is free.

Definition

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An occurrence of a variable is bound in a formula if and only if it is not free.

I look at my example again to illustrate the definition:

Example

P^1x

P^1x is an atomic formula…

I look at my example again to illustrate the definition:

Example

$\forall x\, P^1 x$

Writing $\forall\, x$ in front of $P^1 x$ binds the green occurrence of x.

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Example

$\forall x\, P^1 x \quad Q^2 a x$

$Q^2 a x$ is an atomic formula, so the red occurrence of x is free.
$Q^2 a x$ is still not related to $P^1 x$.

I look at my example again to illustrate the definition:

Example

$(\forall x\, P^1 x \to Q^2 a x)$

Now I combine the two formulae using $\to$ but that doesn't make the red occurrence of x a bound occurrence.

Definition

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Definition (sentence of $\mathcal{L}_2$)

A formula of $\mathcal{L}_2$ is a sentence of $\mathcal{L}_2$ if and only if no variable occurs freely in the formula.

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The bracketing conventions of $\mathcal{L}_1$ apply also to $\mathcal{L}_2$ formulae.

Convention

An $\mathcal{L}_2$-formula may be abbreviated by dropping the arity indices.

So instead of P^2xy one may write Pxy.

Example

$$\forall x\,((P^1x \land R^2_5xa) \to \exists y_2((R^2_5xy_2 \land Q^1x) \land P^1y_2))$$

This is the sentence. Now I apply the rules for abbreviating formulae of $\mathcal{L}_2$.

Example

$\forall x\,((P^1x \wedge R^2_5 xa) \rightarrow \exists y_2((R^2_5 x y_2 \wedge Q^1x) \wedge P^1y_2))$

The connective $\wedge$ binds more strong than $\rightarrow$; so I drop the the red brackets

Example

$$\forall x\,(\;P^1x \wedge R^2_5xa\; \to \exists y_2((R^2_5xy_2 \wedge Q^1x) \wedge P^1y_2))$$

The pair of green brackets can be dropped because in chains of formulae with $\wedge$ 'left-bracketing' applies.

Example

$$\forall x\,(\ \ P^1 x \land R^2_5 x a \ \ \to \exists y_2(\ \ R^2_5 x y_2 \land Q^1 x \ \ \land P^1 y_2))$$

And then, according to the new rule, I can drop the arity indices from all predicate letters.

Example

$$\forall x\,(\;P\,x \wedge R_5 x a \;\rightarrow \exists y_2(\;R_5 x y_2 \wedge Q\,x \;\wedge P\,y_2))$$

Note that there is no pair of outer brackets that I could drop. The formula cannot be further abbreviated.

In the abbreviation

$$Pa \wedge Pab$$

the two occurrences of P stand for different predicate letters, which becomes obvious when arity indices are added:

$$(P^1 a \wedge P^2 ab)$$

This is only my first stab on formalisation in $\mathcal{L}_2$. In Chapter 7 I'll take up the topic again after specifying the semantics of $\mathcal{L}_2$.