# INTRODUCTION TO LOGIC

## 6 Natural Deduction

Volker Halbach

There's nothing you can't prove if
your outlook is only sufficiently limited.
*Dorothy L. Sayers*

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One way of showing that an argument is valid is to break it down into several steps and to show that one can arrive at the conclusion through some more obvious arguments.

It's not clear one can break down *every* valid argument into a sequence of steps from a predefined finite set of rules.
This is possible in the case of $\mathcal{L}_2$. There is a finite set of rules that allows one to *derive* the conclusion from the premisses of any valid argument.

The existence of such a set of rules is remarkable for various reasons:

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- An alternative definition of the validity of arguments becomes available: An argument is valid iff the conclusion can be derived from the premisses using the specified rules.
- The notion of proof can be precisely defined. In cases of disagreement, one can always break down an argument into elementary steps that are covered by these rules. The point is that all proofs could *in principle* be broken down into these elementary steps.

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- In order to prove that an argument in $\mathcal{L}_2$ is valid, one can use the proof system.
- An alternative definition of the validity of arguments becomes available: An argument is valid iff the conclusion can be derived from the premisses using the specified rules.
- The notion of proof can be precisely defined. In cases of disagreement, one can always break down an argument into elementary steps that are covered by these rules. The point is that all proofs could *in principle* be broken down into these elementary steps.
- The notion of proof becomes tractable, so one can obtain general results about provability.

The proof system is defined in purely syntactic terms. In a proof one can't appeal to semantic notions (such as 'this means the same as').

The rules describe how to manipulate symbols without referring to the 'meaning' (semantics) of the symbols.

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I write $\Gamma \vdash \phi$ iff there is a proof of $\phi$ from sentences in $\Gamma$ (this will be made precise below).

(cf. $\vDash$)

**Example**

$(P \land Q) \land R \vdash P$

Here is a proof…

$$(P \land Q) \land R$$

I write down the premiss as an *assumption*. This is covered by the

**ASSUMPTION RULE**

*The occurrence of a sentence $\phi$ with no sentence above it is an assumption. An assumption of $\phi$ is a proof of $\phi$.*

Any sentence can be assumed.

There is a rule that allows one to go from $\phi \wedge \psi$ to $\phi$:

**$\wedge$Elim1**

*The result of appending $\phi$ to a proof of $\phi \wedge \psi$ is a proof of $\phi$.*

$$\frac{(P \wedge Q) \wedge R}{P \wedge Q}$$

The rule is applied again.

$$\dfrac{\dfrac{(P \land Q) \land R}{P \land Q}}{P}$$

The result is a proof of the conclusion $P$ from the premiss $(P \wedge Q) \wedge R$.

$$\dfrac{\dfrac{(P \wedge Q) \wedge R}{P \wedge Q}}{P}$$

## Example

$Qb \wedge Pa, Ra \vdash Pa \wedge Ra$

$Qb \land Pa$

$Ra$

I write down the two premisses as *assumptions*. This is covered by the

**ASSUMPTION RULE**

*The occurrence of a sentence $\phi$ with no sentence above it is an assumption. An assumption of $\phi$ is a proof of $\phi$.*

Any sentence can be assumed.

$$\frac{Qb \wedge Pa}{Pa} \qquad Ra$$

There is a rule that allows one to go from $\phi \wedge \psi$ to $\psi$:

**$\wedge$Elim2**

*The result of appending $\psi$ to a proof of $\phi \wedge \psi$ is a proof of $\psi$.*

And there is a rule that allows one to go from $\phi$ and $\psi$ to the sentence $\phi \wedge \psi$:

### $\wedge$Intro

*The result of appending $\phi \wedge \psi$ to a proof of $\phi$ and a proof of $\psi$ is a proof of $\phi \wedge \psi$.*

$$\dfrac{\dfrac{Qb \wedge Pa}{Pa} \qquad Ra}{Pa \wedge Ra}$$

The result is a proof of $Pa \wedge Ra$ from the two premisses.

$$\dfrac{\dfrac{Qb \wedge Pa}{Pa} \qquad Ra}{Pa \wedge Ra}$$

In the proof I have used the rule for assumptions and introduction and elimination rules for $\wedge$.
The introduction rule for $\wedge$ is:

**$\wedge$Intro**

*The result of appending $\phi \wedge \psi$ to a proof of $\phi$ and a proof of $\psi$ is a proof of $\phi \wedge \psi$.*

So an application of the rule looks like this:

$$\dfrac{\begin{matrix}\vdots \\ \phi\end{matrix} \qquad \begin{matrix}\vdots \\ \psi\end{matrix}}{\phi \wedge \psi}\ \wedge\text{Intro}$$

The elimination rules are:

**$\wedge$Elim1**

*The result of appending $\phi$ to a proof of $\phi \wedge \psi$ is a proof of $\phi$.*

**$\wedge$Elim2**

*The result of appending $\psi$ to a proof of $\phi \wedge \psi$ is a proof of $\psi$.*

$$\dfrac{\begin{matrix}\vdots \\ \phi \wedge \psi\end{matrix}}{\phi}\ \wedge\mathit{Elim1} \qquad\qquad \dfrac{\begin{matrix}\vdots \\ \phi \wedge \psi\end{matrix}}{\psi}\ \wedge\mathit{Elim2}$$

## Example

$\exists y\, Py \to Qa, \exists y\, Py \vdash Qa$

I assume both premisses

$\exists y\, Py \qquad \exists y\, Py \to Qa$

## Example

$\exists y\, Py \to Qa, \exists y\, Py \vdash Qa$

$$\frac{\exists y\, Py \qquad \exists y\, Py \to Qa}{Qa}$$

I use the elimination rule for $\to$:

### $\to$Elim

*The result of appending $\psi$ to a proof of $\phi$ and a proof of $\phi \to \psi$ is a proof of $\psi$.*

This rule is graphically represented as follows:

$$\frac{\begin{matrix}\vdots \\ \phi\end{matrix} \qquad \begin{matrix}\vdots \\ \phi \to \psi\end{matrix}}{\psi}\ \to\text{Elim}$$

## Example

$\exists y\, Py \to Qa, \exists y\, Py \vdash Qa$

This is the completed proof.

$$\dfrac{\exists y\, Py \qquad \exists y\, Py \to Qa}{Qa}$$

## Example

$P, (P \wedge Q) \to R \vdash Q \to R$

I write down the first premiss as assumption.

$P$

## Example

$P, (P \land Q) \to R \vdash Q \to R$

To get $P \land Q$ I assume $Q$ although $Q$ isn't a premiss.

$P \qquad Q$

## Example

$P, (P \land Q) \to R \vdash Q \to R$

By applying $\land$Intro I obtain $P \land Q$.

$$\frac{P \qquad Q}{P \land Q}$$

## Example

$P, (P \wedge Q) \rightarrow R \vdash Q \rightarrow R$

I write down the second premiss as an assumption…

$$\dfrac{P \qquad Q}{P \wedge Q} \qquad (P \wedge Q) \rightarrow R$$

## Example

$P, (P \wedge Q) \to R \vdash Q \to R$

… and apply →Elim.

$$\dfrac{\dfrac{P \qquad Q}{P \wedge Q} \qquad (P \wedge Q) \to R}{R}$$

## Example

$P, (P \wedge Q) \to R \vdash Q \to R$

$$\dfrac{\dfrac{\dfrac{P \qquad [Q]}{P \wedge Q} \qquad (P \wedge Q) \to R}{R}}{Q \to R}$$

Finally I apply →Intro. $Q$ has only be assumed ‘for the sake of the argument’. The final sentence $Q \to R$ doesn’t depend on the assumption $Q$. Thus I ‘discharge’ the assumption $Q$ by enclosing it in square brackets.

## →Intro

*The result of appending $\phi \to \psi$ to a proof of $\psi$ and discharging all assumptions of $\phi$ in the proof of $\psi$ is a proof of $\phi \to \psi$.*

## Example

$P, (P \wedge Q) \to R \vdash Q \to R$

$$\dfrac{\dfrac{\dfrac{P \qquad [Q]}{P \wedge Q} \qquad (P \wedge Q) \to R}{R}}{Q \to R}$$

Discharged assumptions are not listed as premisses. So I have proved $P, (P \wedge Q) \to R \vdash Q \to R$.

## Example

$P, (P \land Q) \to R \vdash Q \to R$

Graphical representation of →Intro:

$$\dfrac{\dfrac{\dfrac{P \qquad [Q]}{P \land Q} \qquad (P \land Q) \to R}{R}}{Q \to R}$$

$$\dfrac{\begin{array}{c}[\phi] \\ \vdots \\ \psi\end{array}}{\phi \to \psi} \to\text{Intro}$$

Now I have explained what it means for an assumption to be discharged in a proof. This allows me to give the official definition of $\vdash$.

## Definition

The sentence $\phi$ is provable from $\Gamma$ (where $\Gamma$ is a set of $\mathcal{L}_2$-sentences) if and only if there is a proof of $\phi$ with only sentences in $\Gamma$ as non-discharged assumptions.

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The phrase '$\phi$ is provable from $\Gamma$' is abbreviated as $\Gamma \vdash \phi$.

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## Definition

The sentence $\phi$ is provable from $\Gamma$ (where $\Gamma$ is a set of $\mathcal{L}_2$-sentences) if and only if there is a proof of $\phi$ with only sentences in $\Gamma$ as non-discharged assumptions.

The phrase '$\phi$ is provable from $\Gamma$' is abbreviated as $\Gamma \vdash \phi$.

If $\Gamma$ is empty, $\Gamma \vdash \phi$ is abbreviated as $\vdash \phi$. If $\Gamma$ contains exactly the sentences $\psi_1, \dots, \psi_n$, one may write $\psi_1, \dots, \psi_n \vdash \phi$ instead of $\{\psi_1, \dots, \psi_n\} \vdash \phi$.

There are two rules for introducing $\vee$. Applications of them look like this:

$$\dfrac{\begin{array}{c}\vdots\\ \phi\end{array}}{\phi \vee \psi}\ \vee\text{Intro1} \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ \psi\end{array}}{\phi \vee \psi}\ \vee\text{Intro2}$$

An application of the rule for eliminating $\vee$ looks like this:

$$\dfrac{\begin{matrix} \\ \vdots \\ \phi \vee \psi \end{matrix} \qquad \begin{matrix} [\phi] \\ \vdots \\ \chi \end{matrix} \qquad \begin{matrix} [\psi] \\ \vdots \\ \chi \end{matrix}}{\chi} \vee\text{Elim}$$

So one infers $\chi$ from $\phi \vee \psi$ by making a case distinction: one derives $\chi$ from $\phi$ and one derives $\chi$ from $\psi$ to show that $\chi$ follows in either case.

**Example**

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

$$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P)$$

I write down the premiss as an assumption.

## Example

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

$$\neg P \wedge Q \qquad\qquad \exists x\, Qx \wedge \neg P$$

$$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P)$$

To apply →Elim I write down the two ‘cases’ as assumptions.

**Example**

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

$$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \qquad \dfrac{\neg P \wedge Q}{\neg P} \qquad \exists x\, Qx \wedge \neg P$$

Using $\wedge$Elim1 I infer $\neg P$ from $\neg P \wedge Q$.

## Example

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

$$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \qquad \frac{\neg P \wedge Q}{\neg P} \qquad \frac{\exists x\, Qx \wedge \neg P}{\neg P}$$

Similarly, by applying $\wedge$Elim2 I infer $\neg P$ in the other case.

## Example

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

$$\dfrac{(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \qquad \dfrac{[\neg P \wedge Q]}{\neg P} \qquad \dfrac{[\exists x\, Qx \wedge \neg P]}{\neg P}}{\neg P}$$

By applying $\vee$Elim I infer $\neg P$ and discharge the two assumption that were only made for the sake of the argument to distinguish the two cases.

An application of ¬Intro looks like this:

$$\dfrac{\begin{array}{cc} [\phi] & [\phi] \\ \vdots & \vdots \\ \psi & \neg\psi \end{array}}{\neg\phi}\ \neg\text{Intro}$$

An application of ¬Intro looks like this:

$$\dfrac{\begin{array}{c}[\phi] \\ \vdots \\ \psi\end{array} \qquad \begin{array}{c}[\phi] \\ \vdots \\ \neg\psi\end{array}}{\neg\phi}\ \neg\text{Intro}$$

The proof technique is also called '*reductio ad absurdum*'.

## Example

$\neg(P \to Q) \vdash \neg Q$

## Example

$\neg(P \to Q) \vdash \neg Q$

$$Q \qquad\qquad \neg(P \to Q)$$

I write down the premiss as an assumption and I assume $Q$.

### Example

$\neg(P \to Q) \vdash \neg Q$

$$\dfrac{Q}{P \to Q} \qquad \neg(P \to Q)$$

From $Q$ I infer $P \to Q$ (although I have never assumed $P$). So I have a contradiction.

## Example

$\neg(P \to Q) \vdash \neg Q$

$$\dfrac{\dfrac{[Q]}{P \to Q} \qquad \neg(P \to Q)}{\neg Q}$$

By applying ¬Intro, ie,

$$\dfrac{\begin{matrix}[\phi] \\ \vdots \\ \psi\end{matrix} \qquad \begin{matrix}[\phi] \\ \vdots \\ \neg\psi\end{matrix}}{\neg\phi}\ \neg\text{Intro}$$

I discharge the assumption of $Q$ and infer $\neg Q$.

The rule for eliminating ¬ looks like this:

$$\dfrac{\begin{matrix}[\neg\phi] \\ \vdots \\ \psi\end{matrix} \qquad \begin{matrix}[\neg\phi] \\ \vdots \\ \neg\psi\end{matrix}}{\phi}\ \neg\text{Elim}$$

For $\leftrightarrow$ I use the following rules:

$$\dfrac{\begin{matrix}[\phi] & & [\psi] \\ \vdots & & \vdots \\ \psi & & \phi\end{matrix}}{\phi \leftrightarrow \psi}\;\leftrightarrow\text{Intro}$$

$$\dfrac{\begin{matrix}\vdots & & \vdots \\ \phi \leftrightarrow \psi & & \phi\end{matrix}}{\psi}\;\leftrightarrow\text{Elim1} \qquad \dfrac{\begin{matrix}\vdots & & \vdots \\ \phi \leftrightarrow \psi & & \psi\end{matrix}}{\phi}\;\leftrightarrow\text{Elim2}$$

Here is an example for an application of the rule for $\forall$-elimination:

## Example

$\forall x\,(Px \to Qx), Pa \vdash Qa$

I assume the first premiss.

$$\forall x\,(Px \to Qx)$$

Here is an example for an application of the rule for $\forall$-elimination:

### Example

$\forall x\,(Px \to Qx), Pa \vdash Qa$

$$\frac{\forall x\,(Px \to Qx)}{Pa \to Qa}$$

I apply the rule for eliminating $\forall$ by deleting $\forall x$ and by replacing all free occurrences of $x$ in the formula by the constant $a$.

Here is an example for an application of the rule for $\forall$-elimination:

## Example

$\forall x\,(Px \to Qx), Pa \vdash Qa$

I assume the other premiss…

$$Pa \qquad \dfrac{\forall x\,(Px \to Qx)}{Pa \to Qa}$$

Here is an example for an application of the rule for $\forall$-elimination:

**Example**

$\forall x\,(Px \to Qx), Pa \vdash Qa$

… and apply $\to$Elim to get the conclusion.

$$\dfrac{Pa \qquad \dfrac{\forall x\,(Px \to Qx)}{Pa \to Qa}}{Qa}$$

Here is another example of an application of $\forall$Elim:

**Example**

$\forall z\,(Pz \lor \exists z\, Qzz) \vdash Pc \lor \exists z\, Qzz$

I assume the premiss.

$\forall z\,(Pz \lor \exists z\, Qzz)$

Here is another example of an application of $\forall$Elim:

**Example**

$\forall z\,(Pz \lor \exists z\, Qzz) \vdash Pc \lor \exists z\, Qzz$

$$\frac{\forall z\,(Pz \lor \exists z\, Qzz)}{Pc \lor \exists z\, Qzz}$$

I apply the rule for eliminating $\forall$ by deleting $\forall z$ and by replacing all *free* occurrences of $z$ in the formula by the constant $c$.

An application of the rule for eliminating $\forall$ looks like this where $\phi$ is an $\mathcal{L}_2$-formula in which only the variable $v$ occurs freely; $t$ is a constant, $\phi[t/v]$ is the sentence obtained by replacing all free occurrences of $v$ in $\phi$ by $t$.

$$\dfrac{\begin{array}{c}\vdots\\ \forall v\, \phi\end{array}}{\phi[t/v]}\ \forall\text{Elim}$$

Another example for $\phi[t/v]$.

**Example**

$((Pz \vee R^2 az) \rightarrow \exists z\,(Pz \wedge \forall y\, Rzy))$

Consider this formula.

Another example for $\phi[t/v]$.

**Example**

$$((Pz \vee R^2az) \to \exists z\,(Pz \wedge \forall y\, Rzy))$$

The free occurrences of $z$ are shown in green, the bound occurrences in red. No other variable occurs freely.

Another example for $\phi[t/v]$.

## Example

$((Pz \lor R^2az) \to \exists z\,(Pz \land \forall y\, Rzy))\,[c/z]$

Now I replace all free (green) occurrence of $z$ with $c$.

Another example for $\phi[t/v]$.

## Example

$((Pc \vee R^2ac) \to \exists z\,(Pz \wedge \forall y\, Rzy))$

So $((Pz \vee R^2az) \to \exists z\,(Pz \wedge \forall y\, Rzy))[c/z]$ is the sentence shown above.

Here is an example for an application of the rule for $\forall$-introduction:

## Example

$\vdash \forall z\,(Pz \to Qz \lor Pz)$

Here is an example for an application of the rule for $\forall$-introduction:

**Example**

$\vdash \forall z\,(Pz \to Qz \lor Pz)$

I assume $Pa$.

$Pa$

Here is an example for an application of the rule for $\forall$-introduction:

**Example**

$\vdash \forall z\,(Pz \to Qz \lor Pz)$

I apply $\lor$Intro2.

$$\frac{Pa}{Qa \lor Pa}$$

Here is an example for an application of the rule for $\forall$-introduction:

## Example

$\vdash \forall z\,(Pz \to Qz \vee Pz)$

I apply $\to$Intro by inferring $Pa \to (Qa \vee Pa)$ and discharging the assumption $Pa$

$$\dfrac{\dfrac{[Pa]}{Qa \vee Pa}}{Pa \to (Qa \vee Pa)}$$

Here is an example for an application of the rule for $\forall$-introduction:

## Example

$\vdash \forall z\,(Pz \to Qz \lor Pz)$

Finally I apply the rule for introducing $\forall$.

$$\dfrac{\dfrac{\dfrac{[Pa]}{Qa \lor Pa}}{Pa \to (Qa \lor Pa)}}{\forall z\,(Pz \to (Qz \lor Pz))}$$

Here is an example for an application of the rule for $\forall$-introduction:

**Example**

$\vdash \forall z\,(Pz \to Qz \lor Pz)$

$$\dfrac{\dfrac{\dfrac{[Pa]}{Qa \lor Pa}}{Pa \to (Qa \lor Pa)}}{\forall z\,(Pz \to (Qz \lor Pz))}$$

One must make sure that the constant $a$ doesn't occur in any undischarged assumption above $Pa \to (Qa \lor Pa)$ when applying $\forall$Intro. Also $a$ must not occur in the inferred sentence. Moreover, when replacing $a$ with $z$ I must make sure that the variable $z$ isn't bound by another occurrence of a quantifier.

An application of the rule for introducing $\forall$ looks like this. All the restrictions on the previous slide are contained in the following formulation:

$$\dfrac{\begin{array}{c}\vdots\\ \phi[t/v]\end{array}}{\forall v\,\phi}\;\forall\text{Intro}$$

provided the constant $t$ does not occur in $\phi$ or in any undischarged assumption in the proof of $\phi[t/v]$.

In the *Manual* I have explained why one is imposing the restrictions on $\phi$ and $t$.

### Example

$\forall y\,(Py \to Qy), \forall z\,(Qz \to Rz) \vdash \forall y\,(Py \to Ry)$

$$\forall y\,(Py \to Qy)$$

I write down the first premiss as an assumption…

## Example

$\forall y\,(Py \to Qy), \forall z\,(Qz \to Rz) \vdash \forall y\,(Py \to Ry)$

$$\frac{\forall y\,(Py \to Qy)}{Pa \to Qa}$$

… and apply $\forall$Elim.

## Example

$\forall y\,(Py \to Qy), \forall z\,(Qz \to Rz) \vdash \forall y\,(Py \to Ry)$

$$Pa \qquad \frac{\forall y\,(Py \to Qy)}{Pa \to Qa}$$

Hoping to be able to infer $Ra$, I assume $Pa$.

## Example

$\forall y\,(Py \to Qy), \forall z\,(Qz \to Rz) \vdash \forall y\,(Py \to Ry)$

$$\dfrac{Pa \qquad \dfrac{\forall y\,(Py \to Qy)}{Pa \to Qa}}{Qa}$$

An application of $\to$Elim gives $Qa$.

## Example

$\forall y\,(Py \to Qy), \forall z\,(Qz \to Rz) \vdash \forall y\,(Py \to Ry)$

$$\dfrac{Pa \qquad \dfrac{\forall y\,(Py \to Qy)}{Pa \to Qa}}{Qa} \qquad \forall z\,(Qz \to Rz)$$

Next I write down the second premiss as an assumption…

## Example

$\forall y\,(Py \to Qy), \forall z\,(Qz \to Rz) \vdash \forall y\,(Py \to Ry)$

$$\dfrac{Pa \qquad \dfrac{\forall y\,(Py \to Qy)}{Pa \to Qa}}{Qa} \qquad \dfrac{\forall z\,(Qz \to Rz)}{Qa \to Ra}$$

… and apply $\forall$Elim with the constant $a$ again. Note that nothing prevents the use of $a$ again.

## Example

$\forall y\,(Py \to Qy), \forall z\,(Qz \to Rz) \vdash \forall y\,(Py \to Ry)$

$$\dfrac{\dfrac{Pa \qquad \dfrac{\forall y\,(Py \to Qy)}{Pa \to Qa}}{Qa} \qquad \dfrac{\forall z\,(Qz \to Rz)}{Qa \to Ra}}{Ra}$$

I apply $\to$Elim.

## Example

$\forall y\,(Py \to Qy), \forall z\,(Qz \to Rz) \vdash \forall y\,(Py \to Ry)$

$$\dfrac{\dfrac{[Pa] \qquad \dfrac{\forall y\,(Py \to Qy)}{Pa \to Qa}}{Qa} \qquad \dfrac{\forall z\,(Qz \to Rz)}{Qa \to Ra}}{\dfrac{Ra}{Pa \to Ra}}$$

Applying →Intro I infer $Pa \to Ra$ and discharge the assumption $Pa$.

## Example

$\forall y\,(Py \to Qy), \forall z\,(Qz \to Rz) \vdash \forall y\,(Py \to Ry)$

$$\dfrac{\dfrac{\dfrac{\dfrac{[Pa] \qquad \dfrac{\forall y\,(Py \to Qy)}{Pa \to Qa}}{Qa} \qquad \dfrac{\forall z\,(Qz \to Rz)}{Qa \to Ra}}{Ra}}{Pa \to Ra}}{\forall y\,(Py \to Ry)}$$

Finally I apply $\forall$Intro. I need to check that

- provided the constant $a$ does not occur in $(Py \to Ry)$, and
- $a$ does not occur in any undischarged assumption in the proof of $Pa \to Ra$

Here is an example for ∃Intro.

**Example**

$Rcc \vdash \exists y\, Rcy$

I assume the premiss.

$Rcc$

Here is an example for ∃Intro.

**Example**

$Rcc \vdash \exists y\, Rcy$

$$\frac{Rcc}{\exists y\, Rcy}$$

I infer the conclusion by replacing one (or more or all or none) occurrence(s) of a constant with the variable $y$ and prefixing the resulting formula with $\exists y$.

An application of ∃Intro looks like this:

$$\frac{\phi[t/v]}{\exists v\,\phi}\ \exists\text{Intro}$$

Here is an example of an application of the rule for eliminating $\exists$:

**Example**

$\exists x\, Px, \forall x\,(Px \to Qx) \vdash \exists x\, Qx$

$$\forall x\,(Px \to Qx)$$

$\exists x\, Px$

I write down the two premisses as assumptions.

Here is an example of an application of the rule for eliminating ∃:

**Example**

$\exists x\, Px, \forall x\,(Px \to Qx) \vdash \exists x\, Qx$

$$\forall x\,(Px \to Qx)$$

$$Pc$$

$$\exists x\, Px$$

For the sake of the argument, I assume that $Pc$. If I can prove the conclusion, which doesn't say anything specific about $c$, I can discharge the assumption by ∃Elim.

Here is an example of an application of the rule for eliminating $\exists$:

**Example**

$\exists x\, Px, \forall x\,(Px \to Qx) \vdash \exists x\, Qx$

$$Pc \qquad \frac{\forall x\,(Px \to Qx)}{Pc \to Qc}$$

$\exists x\, Px$

I apply $\forall$Elim.

Here is an example of an application of the rule for eliminating $\exists$:

**Example**

$\exists x\, Px, \forall x\, (Px \to Qx) \vdash \exists x\, Qx$

$$\dfrac{Pc \qquad \dfrac{\forall x\,(Px \to Qx)}{Pc \to Qc}}{Qc}$$

$\exists x\, Px$

An application of $\to$Elim gives $Qc$.

Here is an example of an application of the rule for eliminating $\exists$:

## Example

$\exists x\, Px, \forall x\,(Px \to Qx) \vdash \exists x\, Qx$

$$\exists x\, Px \qquad \dfrac{\dfrac{Pc \qquad \dfrac{\forall x\,(Px \to Qx)}{Pc \to Qc}}{Qc}}{\exists x\, Qx}$$

From $Qc$ I obtain $\exists x\, Qx$ using $\exists$Intro.

Here is an example of an application of the rule for eliminating $\exists$:

## Example

$\exists x\, Px, \forall x\,(Px \to Qx) \vdash \exists x\, Qx$

$$\dfrac{\exists x\, Px \qquad \dfrac{\dfrac{[Pc] \qquad \dfrac{\forall x\,(Px \to Qx)}{Pc \to Qc}}{Qc}}{\exists x\, Qx}}{\exists x\, Qx}$$

In this step, an application of $\exists$Elim I repeat the conclusion. The point of this step is that I can discharged the assumption of $Pc$.

Here is an example of an application of the rule for eliminating $\exists$:

## Example

$\exists x\, Px, \forall x\,(Px \to Qx) \vdash \exists x\, Qx$

$$\dfrac{\exists x\, Px \qquad \dfrac{\dfrac{[Pc] \qquad \dfrac{\forall x\,(Px \to Qx)}{Pc \to Qc}}{Qc}}{\color{red}{\exists x\, Qx}}}{\exists x\, Qx}$$

In deriving the red occurrence of $\exists x\, Qx$ I did not make use of any undischarged assumption involving $c$ – except of course for $Pc$ itself. Also one must apply $\exists$Elim only if the sentence corresponding to the red sentence here doesn't contain the crucial constant $c$.

$$\dfrac{\begin{array}{c} \\ \vdots \\ \exists v\, \phi \end{array} \qquad \begin{array}{c} [\phi[t/v]] \\ \vdots \\ \psi \end{array}}{\psi}\ \exists\text{Elim}$$

provided the constant $t$ does not occur in $\exists v\, \phi$, or in $\psi$, or in any undischarged assumption other than $\phi[t/v]$ in the proof of $\psi$.

For more examples of Natural Deduction proofs as pdf slides see
http://logicmanual.philosophy.ox.ac.uk/

Can we prove everything we want to prove?

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## Theorem (adequacy)

Assume that $\phi$ and all elements of $\Gamma$ are $\mathcal{L}_2$-sentences. Then $\Gamma \vdash \phi$ if and only if $\Gamma \vDash \phi$.