The analysis of the beginning would thus yield the notion of the unity of being and not-being – or, in a more reflected form, the unity of differentiatedness and non-differentiatedness, or the identity of identity and non-identity.  

*Hegel, The Science of Logic*
Assume Keith and Volker don’t share a car; they only have the same model of the same year (same colour etc).

**Example**

Keith and Volker have the same car.
Keith and Volker have identical cars.

This an example of (approximate) **qualitative identity**.
Assume Keith and Volker don’t share a car; they only have the same model of the same year (same colour etc).

Example

Keith and Volker have the same car.
Keith and Volker have identical cars.

This an example of (approximate) **qualitative identity**.

Qualitative identity can be formalised as a binary predicate letter expressing close similarity or sameness in all relevant aspects.
Example

This is the same car as the car that was seen at the crime scene.

This means probably that the *very same* car and not just a car of the same brand, the same colour etc was seen at the scene.
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This is an example of **numerical identity**.
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This means probably that the *very same* car and not just a car of
the same brand, the same colour etc was seen at the scene.

This is an example of **numerical identity**.

Occasionally it's ambiguous whether numerical or qualitative
identity is meant.

In what follows I talk about numerical identity.
The language $L_\equiv$ is $L_2$ plus an additional binary predicate letter $=$ that is always interpreted as identity.
The language $\mathcal{L}_= \text{ is } \mathcal{L}_2$ plus an additional binary predicate letter $=$ that is always interpreted as identity.

In $\mathcal{L}_2$ we can formalise ‘is identical to’ as a binary predicate letter, but this predicate letter can receive arbitrary relations as extension (semantic value).
The language $\mathcal{L}_=$ is $\mathcal{L}_2$ plus an additional binary predicate letter $=$ that is always interpreted as identity.

In $\mathcal{L}_2$ we can formalise ‘is identical to’ as a binary predicate letter, but this predicate letter can receive arbitrary relations as extension (semantic value).

In $\mathcal{L}_=$ the new binary predicate letter is always taken to express identity.
Definition (atomic formulae of $\mathcal{L}_=$)

All atomic formulae of $\mathcal{L}_2$ are atomic formulae of $\mathcal{L}_=$. Furthermore, if $s$ and $t$ are variables or constants, then $s = t$ is an atomic formula of $\mathcal{L}_=$.
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Example
$c = a$, $x = y_3$, $x_7 = x_7$, and $x = a$ are all atomic formulae of $\mathcal{L}_=$.
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All atomic formulae of $\mathcal{L}_2$ are atomic formulae of $\mathcal{L}_=$. Furthermore, if $s$ and $t$ are variables or constants, then $s = t$ is an atomic formula of $\mathcal{L}_=$.

Example

$c = a$, $x = y_3$, $x_7 = x_7$, and $x = a$ are all atomic formulae of $\mathcal{L}_=$.

The symbol ‘=’ now plays two roles: as symbol of $\mathcal{L}_=$ and as a symbol in the metalanguage.
One can use connectives and quantifiers to build formulae of $\mathcal{L}_\leq$ in the same ways as in $\mathcal{L}_2$. 
One can use connectives and quantifiers to build formulae of $\mathcal{L}_=$ in the same ways as in $\mathcal{L}_2$.

**Example**

$\neg x = y$ and $\forall x (Rx y_2 \rightarrow y_2 = x)$ are formulae of $\mathcal{L}_=$.
One can use connectives and quantifiers to build formulae of $\mathcal{L}_\leq$ in the same ways as in $\mathcal{L}_2$.

**Example**

$\neg x = y$ and $\forall x (Rx y_2 \rightarrow y_2 = x)$ are formulae of $\mathcal{L}_\leq$.

The notion of an $\mathcal{L}_\leq$-sentence is defined in analogy to the notion of an $\mathcal{L}_2$-sentence.
Everything is as for $\mathcal{L}_2$, except that an additional clause needs to be added to the definition of satisfaction, where $\mathcal{A}$ is an $\mathcal{L}_2$-structure, $s$ is a variable or constant, and $t$ is a variable or constant:

(ix) $|s = t|_\mathcal{A}^\alpha = T$ if and only if $|s|_\mathcal{A}^\alpha = |t|_\mathcal{A}^\alpha$.  

40
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All other definitions of Chapter 5 carry over to $\mathcal{L}_\equiv$, just with ‘$\mathcal{L}_2$’ replaced by ‘$\mathcal{L}_\equiv$’.
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All other definitions of Chapter 5 carry over to $\mathcal{L}_\equiv$, just with ‘$\mathcal{L}_2$’ replaced by ‘$\mathcal{L}_\equiv$’.

**Caution:** $\mathcal{L}_\equiv$-structures don’t assign semantic values to the symbol $\equiv$. There is no difference between $\mathcal{L}_\equiv$ and $\mathcal{L}_2$-structures!
Example

∀x ∀y x = y isn’t logically true.
Example

\[ \forall x \forall y \ x = y \] isn’t logically true.

Counterexample: Let $A$ be any $L_2$-structure with $\{1, 2\}$ as its domain.
Natural Deduction for $\mathcal{L}_-$ has the same rules as Natural Deduction for $\mathcal{L}_2$ except for rules for $=$:
Natural Deduction for $\mathcal{L}_-$ has the same rules as Natural Deduction for $\mathcal{L}_2$ except for rules for $=$:

$=$\textsc{Intro}

*Any assumption of the form $t = t$ where $t$ is a constant can and must be discharged.*

A proof with an application of $=$\textsc{Intro} looks like this:

$$
\begin{array}{c}
[t = t] \\
\vdots \\
\end{array}
$$
\[ \begin{align*}
\vdots & \quad \vdots \\
\phi[s/v] & \quad s = t \\
\phi[t/v] & \quad \text{=Elim}
\end{align*} \]

\[ \begin{align*}
\vdots & \quad \vdots \\
\phi[s/v] & \quad t = s \\
\phi[t/v] & \quad \text{=Elim}
\end{align*} \]

Strictly speaking, only one of the versions is needed, as from \( s = t \) one can always obtain \( t = s \) using only one of the rules.
Example

\[ \vdash \forall x \forall y \ (Rx \ y \to (x = y \to Ryx)) \]
Example

\[ \vdash \forall x \forall y (Rx \ y \rightarrow (x = y \rightarrow Ry \ x)) \]

Here is the proof:
Example

\[ \vdash \forall x \forall y (Rxy \rightarrow (x = y \rightarrow Ryx)) \]

Here is the proof:

\[ Rab \quad a = b \]
**Example**

\[ \vdash \forall x \forall y (Rxy \rightarrow (x = y \rightarrow Ryx)) \]

Here is the proof:

\[
\begin{array}{c}
= \text{Elim} \\
Rab & a = b \\
\hline
Raa
\end{array}
\]
Example

\[ \vdash \forall x \forall y (Rx y \to (x = y \to Ryx)) \]

Here is the proof:

\[
\begin{align*}
=\text{Elim} & \quad \frac{Rab \quad a = b}{Raa} \\
& \quad a = b
\end{align*}
\]
Example

⊢ ∀x ∀y (Rx y → (x = y → Ryx))

Here is the proof:

\[
\begin{align*}
= &\text{Elim} \quad \frac{Rab \quad a = b}{Raa} \\
= &\text{Elim} \quad \frac{Raa \quad a = b}{Rba}
\end{align*}
\]
Example

\[ \vdash \forall x \forall y (Rxy \rightarrow (x = y \rightarrow Ryx)) \]

Here is the proof:

\[
=\text{Elim} \quad \begin{array}{c} \text{Rab} \\ [a = b] \end{array} \quad \begin{array}{c} \text{Raa} \\ [a = b] \end{array} =\text{Elim} \\
\quad \begin{array}{c} \text{Rba} \\ a = b \rightarrow Rba \end{array}
\]
Example

⊢ ∀x ∀y (Rx y → (x = y → Ryx))

Here is the proof:

\[
\begin{align*}
&= \text{Elim} \\
&\quad \frac{[Rab]}{\text{[a = b]}} \\
&\quad \frac{\frac{Raa}{\text{[a = b]}}}{\frac{Rba}{a = b \to Rba}} \\
&\quad \frac{Rab \to (a = b \to Rba)}
\end{align*}
\]
Example

\[ \vdash \forall x \forall y (Rx \, y \rightarrow (x = y \rightarrow Ry \, x)) \]

Here is the proof:

\[
\begin{align*}
= \text{Elim} & \quad [Rab] & [a = b] \\
\quad & \rightarrow & \quad Raa \\
\text{=Elim} & \quad [a = b] \\
\quad & \rightarrow & \quad Rba \\
\quad & \rightarrow & \quad a = b \rightarrow Rba \\
\quad & \rightarrow & \quad Rab \rightarrow (a = b \rightarrow Rba) \\
\quad & \rightarrow & \quad \forall y (Ray \rightarrow (a = y \rightarrow Rya))
\end{align*}
\]
Example

\[ \vdash \forall x \forall y \left( Rxy \rightarrow (x = y \rightarrow Ryx) \right) \]

Here is the proof:

\[
\begin{align*}
\text{=Elim} & \quad \frac{\left[ Rab \right]}{\text{Raa}} \\
\text{=Elim} & \quad \frac{\left[ a = b \right]}{\text{Rba}} \\
\text{=Elim} & \quad \frac{a = b \rightarrow Rba}{Rab \rightarrow (a = b \rightarrow Rba)} \\
\text{=Elim} & \quad \frac{\forall y \left( Ray \rightarrow (a = y \rightarrow Rya) \right)}{\forall x \forall y \left( Rxy \rightarrow (x = y \rightarrow Ryx) \right)}
\end{align*}
\]
Theorem (ADEQUACY)

Assume that $\phi$ and all elements of $\Gamma$ are $L_=-$-sentences. Then $\Gamma \vdash \phi$ if and only if $\Gamma \models \phi$. 
Using $=$ one can formalise overt identity claims:

**Example**

William II is Wilhelm II.
Using = one can formalise overt identity claims:

**Example**

William II is Wilhelm II.

**FORMALISATION**

\[ a = b \]

\[ a: \text{ William II} \]
\[ b: \text{ Wilhelm II} \]
Don’t confuse identity with predication.
Don’t confuse identity with predication.

Example
William is an emperor.
Don’t confuse identity with predication.

Example
William is an emperor.

FORMALISATION
Qa

a:  William
Q:  ... is an emperor

Here ‘is’ forms part of the predicate ‘is an emperor’.
Don’t confuse identity with predication.

Example
William is an emperor.

FORMALISATION

\( Qa \)

\[ \begin{align*}
    a & : \text{William} \\
    Q & : \ldots \text{is an emperor}
\end{align*} \]

Here ‘is’ forms part of the predicate ‘is an emperor’.

Example
William is the emperor.

Here ‘is’ expresses identity.
Identity can also be used in formalisations of sentences that do not involve identity explicitly.
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**Example**

There is exactly one perfect being.
Identity can also be used in formalisations of sentences that do not involve identity explicitly.

<table>
<thead>
<tr>
<th>Example</th>
</tr>
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<tbody>
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<td>There is exactly one perfect being.</td>
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<table>
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<tr>
<th>FORMALISATION</th>
</tr>
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<tbody>
<tr>
<td>$\exists x \ (Px \land \forall y \ (Py \rightarrow x = y))$</td>
</tr>
</tbody>
</table>

$P$: ... is a perfect being
Identity can also be used in formalisations of sentences that do not involve identity explicitly.

**Example**

There is exactly one perfect being.

**FORMALISATION**

$$\exists x (P x \land \forall y (P y \rightarrow x = y))$$

*P:* … is a perfect being

Similar tricks work for various other numerical quantifiers ‘at least three’, ‘at most 2’, and so on.

There is no reference to numbers.
Definite descriptions

The following expressions are definite descriptions:
Definite descriptions

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  - the present king of France
Definite descriptions

The following expressions are definite descriptions:

- the present king of France
- Tim’s car
Definite descriptions

The following expressions are definite descriptions:

- the present king of France
- Tim’s car
- the person who has stolen a book from the library and forgotten his or her bag in the library

Formalising definite descriptions as constants brings various problems as the semantics of definite descriptions doesn’t match the semantics of constants in $L_\equiv$. 
Russell’s trick

Example

Tim’s car is red.
### Russell's trick

<table>
<thead>
<tr>
<th>Example</th>
<th>Paraphrase</th>
</tr>
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<tbody>
<tr>
<td>Tim's car is red.</td>
<td>Tim owns exactly one car and it is red.</td>
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</table>
Russell’s trick

Example
Tim’s car is red.

Paraphrase
Tim owns exactly one car and it is red.

FORMALISATION
\[ \exists x (Qx \land Rbx \land \forall y (Qy \land Rby \to x = y) \land Px) \]

\[ b: \] Tim
\[ Q: \] … is a car
\[ R: \] … owns …
\[ P: \] … is red
This formalisation is much better than the formalisation of ‘Tim’s car’ as a constant.
For instance, the following argument comes out as valid if Russell’s trick is used (but not if a constant is used):

**Example**

Tim’s car is red. Therefore there is a red car.

**FORMALISATION**

\[ \exists x (Qx \land Rbx \land \forall y (Qy \land Rby \rightarrow x = y) \land Px) \vdash \exists x (Px \land Qx) \]

The proof is in the Manual.
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For instance, the following argument comes out as valid if Russell’s trick is used (but not if a constant is used):

**Example**

Tim’s car is red. Therefore there is a red car.

**FORMALISATION**

\[ \exists x (Qx \land Rbx \land \forall y (Qy \land Rby \rightarrow x = y) \land Px) \vdash \exists x (Px \land Qx) \]

The proof is in the Manual.

So the English argument is valid in predicate logic with identity.
By using Russell’s trick one can formalise definite descriptions in such way that the definite description may fail to refer to something. Constants, in contrast, are assigned objects in any $\mathcal{L}_2$-structure.
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Using Russell’s trick offers more ways to analyse sentences containing definite descriptions and negations.
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**Example**

- Volker’s private jet is red.
- Volker’s private jet isn’t red.

The first sentence is false, but is the second sentence true?
By using Russell’s trick one can formalise definite descriptions in such way that the definite description may fail to refer to something. Constants, in contrast, are assigned objects in any $L_2$-structure.

Using Russell’s trick offers more ways to analyse sentences containing definite descriptions and negations.

**Example**

- Volker’s private jet is red.
- Volker’s private jet isn’t red.

The first sentence is false, but is the second sentence true?

There is a reading under which both sentences are false. This reading can be made explicit in $L_\neg$ using Russell’s analysis of definite descriptions.
Example
Volker’s private jet isn’t red.

FORMALISATION
\[ \exists x \left( (Qx \land Rax) \land \forall y (Qy \land Ray \rightarrow x = y) \land \neg Px \right) \]

\( a \): Volker
\( Q \): … is a private jet
\( R \): … owns …
\( P \): … is red

This formalisation expresses that Volker has exactly one private jet and that it isn’t red.
Example

Volker’s private jet isn’t red.

FORMALISATION

\[ \exists x ((Qx \land Rax) \land \forall y (Qy \land Ray \rightarrow x = y)) \land \neg Px \]

\( a \): Volker
\( Q \): … is a private jet
\( R \): … owns …
\( P \): … is red

This formalisation expresses that Volker has exactly one private jet and that it isn’t red.

Under this analysis ‘Volker’s private jet is red’ and ‘Volker’s private jet isn’t red’ are both false.
Example

It’s not the case (for whatever reason) that Volker’s private jet is red.

I tend to understand this sentence in the following way:

FORMALISATION

$$\neg \exists x \left( (Qx \land Rax) \land \forall y (Qy \land Ray \rightarrow x = y) \land Px \right)$$
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Perhaps the sentence ‘Volker’s private jet isn’t red’ can be understood as saying the same; so it is ambiguous (scope ambiguity concerning \( \neg \)).

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$$\exists x \left( (Qx \land Rx) \land \forall y (Qy \land Ray \rightarrow x = y) \land \neg Px \right)$$
Logical constants

I have treated identity, the connectives and expressions like ‘all’ etc. as subject-independent vocabulary. Perhaps there are more such expressions:

- many, few, infinitely many
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- it’s obligatory that

At any rate the logical vocabulary of $\mathcal{L}_\equiv$ is sufficient for analysing the validity of arguments in (large parts of) the sciences and mathematics.

You can learn more about extensions of $\mathcal{L}_\equiv$ in the *Philosophical Logic* paper.

Perhaps the above expressions can be analysed in $\mathcal{L}_\equiv$ in the framework of specific theories.
The dark side

So far you have seen the logician mainly as a kind of philosophical hygienist, who makes sure that philosophers don’t blunder by using logically invalid arguments, e.g., by messing up the scopes of quantifiers.

Logic seems to be an auxiliary discipline for sticklers who secure the foundations of other disciplines.
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But there is also a dark side.

Here is an example.
Russell’s paradox

If there are any safe foundations in any discipline, then the foundations of mathematics and logic should be unshakable.

Large parts of various disciplines (mathematics, sciences, various parts of philosophy) are founded on set theory. I have used sets for the foundations of logic. Functions and relations are sets; $L_2$-structures are defined in terms of sets.

But the theory of sets is threatened by paradox.
Example (Exercise 7.6)

There is no set \( \{ d : d \notin d \} \) that contains exactly those things that do not have themselves as elements.

Thus, very simple assumptions about sets are inconsistent. You cannot define sets by \( \{ d : \ldots d \ldots \} \) without some restrictions. Presumably the assumptions about sets you used at school form an inconsistent set of assumptions: anything can be proved from them.
Russell’s paradox shattered Frege’s foundations of mathematics.
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But there remained doubts in the hearts of some mathematicians and philosophers: they still didn’t know that the theory of sets (and therefore the foundations of mathematics) is consistent.

The hope: one day a white knight would come and prove, using the instruments of logic, that the revised theory of sets is (syntactically or semantically) consistent and thereby secure the foundations of mathematics, philosophy, etc.

Many logicians tried to find such a proof…
In the end, a black knight came and, using the methods of logic, proved roughly the following:

*If there is a proof of the consistency of set theory, using the tools of logic and set theory, then set theory is inconsistent.*

We can *never* prove, perhaps never know, that the foundations are safe (consistent). Not only did the white knights fail, they failed by necessity.
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Gödel’s proof is so devastatingly general that replacing set theory with a tamer theory will not help against Gödel’s result. One can prove the consistency of one’s standpoint only if that standpoint is inconsistent.
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Gödel’s proof is so devastatingly general that replacing set theory with a tamer theory will not help against Gödel’s result. One can prove the consistency of one’s standpoint only if that standpoint is inconsistent.

What remains is, perhaps, faith…