Estimating Value at Risk and Expected Shortfall Using Expectiles

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Abstract

Expectile models are derived using asymmetric least squares. A simple formula relates the expectile to the expectation of exceedances beyond the expectile. We use this as the basis for estimating expected shortfall. It has been proposed that the θ quantile be estimated by the expectile for which the proportion of observations below the expectile is θ. In this way, an expectile can be used to estimate value at risk. Using expectiles has the appeal of avoiding distributional assumptions. For univariate modelling, we introduce conditional autoregressive expectiles (CARE). Empirical results for the new approach are competitive with established benchmarks methods.

Key words: Financial Risk; Asymmetric Least Squares; Expectiles; Conditional Autoregressive Expectiles.

Value at risk (VaR) measures the maximum potential loss of a given portfolio over a prescribed holding period at a given confidence level, which is typically chosen to be 1% or 5%. Therefore, assessing VaR amounts to estimating tail quantiles of the conditional distribution of a series of financial returns. Although VaR has become the standard measure of financial market risk, it has been criticised for reporting only a quantile, and thus disregarding outcomes beyond the quantile. In addition, VaR is not a subadditive risk measure. This property concerns the idea that the total risk on a portfolio should not be greater than the sum of the risks of the constituent parts of the portfolio (see Artzner, Delbain, Eber, and Heath, 1999; Acerbi and Tasche, 2002). Expected shortfall (ES) is a risk measure that overcomes these weaknesses, and that is becoming increasingly widely used. ES is defined as the conditional expectation of the return given that it exceeds the VaR (see Yamai and Yoshiba, 2002).

A recent development in the VaR literature is the conditional autoregressive value at risk (CAViaR) class of models (see Engle and Manganelli, 2004). This approach to VaR estimation has strong appeal in that it provides a modelling framework and does not rely on distributional assumptions. However, the focus is solely on VaR estimation, and it is not clear how to estimate the corresponding ES.

In this paper, we present a new modelling approach, which also avoids distributional assumptions, and which delivers estimates for both VaR and ES. The approach involves the use of asymmetric least squares (ALS) regression, which is the least squares analogue of quantile regression. The solution of an ALS regression is known as an expectile. This name was given by Newey and Powell (1987) who note that the ALS solution is determined by the properties of the expectation of exceedances beyond the solution. We use this as the basis for estimating ES. It has been shown that there exists a one-to-one mapping from expectiles to quantiles. In view of this, Efron (1991) proposes that the θ quantile be estimated by the expectile for which the proportion of in-sample observations lying below the expectile is θ. This idea can be used to enable VaR estimation from expectiles.

As the basis for conditional VaR and ES modelling, we introduce a new class of univariate expectile models: conditional autoregressive expectiles (CARE). Therefore, this paper can be viewed as developing, for conditional ES modelling, the analogue of the conditional VaR models provided by CAViaR.

In Section 1, we briefly review the literature on VaR and ES estimation. Section 2 describes how expectiles can be used to estimate VaR and ES, and Section 3 introduces the new class of expectile models. Section 4 uses stock indices to illustrate implementation of the new approach, and to evaluate its accuracy. Section 5 provides a summary and concluding comments.

1. A Review of Methods for Estimating VaR and Expected Shortfall

Manganelli and Engle (2004) divide VaR methods into three categories: parametric, semiparametric and nonparametric. *Parametric* approaches involve a parameterisation of the behaviour of prices. Conditional quantiles are estimated using a conditional volatility forecast with an assumption for the shape of the distribution. GARCH models are very often used to forecast the volatility (see Poon and Granger 2003). The distribution is typically chosen to be Gaussian or the Student-*t* distribution, and for these choices it is straightforward to calculate the ES.

Included in the *semiparametric* VaR category are methods based on extreme value theory (EVT) or quantile regression. The straightforward application of EVT is hampered by the heteroskedasticity present in series of financial returns. To overcome this, Diebold, Schuermann, and Stroughair (2000) and McNeil and Frey (2000) propose that the peaks over threshold EVT method be applied to residuals standardised by GARCH conditional volatility estimates. Using the derived exceedance distribution, the approach delivers an analytical formula for the ES (see McNeil, Frey and Embrechts, 2005, p. 283).

A recent proposal using quantile regression is the class of conditional autoregressive value at risk (CAViaR) models introduced by Engle and Manganelli (2004). Three of Engle and Manganelli's CAViaR models are presented in expressions (1) to (3). CAViaR models have similar structures to GARCH models, with the Asymmetric Slope model designed specifically to model the asymmetric leverage effect, which is the tendency for volatility to be greater following a negative return than a positive return of equal size.

Symmetric Absolute Value CAViaR: $$Q_t(\theta) = \beta_0 + \beta_1 Q_{t-1}(\theta) + \beta_2 \left| y_{t-1} \right| . \quad (1)$$

Asymmetric Slope CAViaR: $$Q_t(\theta) = \beta_0 + \beta_1 Q_{t-1}(\theta) + \beta_2 (y_{t-1})^+ + \beta_3 (y_{t-1})^- . \quad (2)$$

Indirect GARCH CAViaR: $$Q_t(\theta) = \left(1 - 2I\left(\theta < 0.5\right)\right)\left(\beta_0 + \beta_1 Q_{t-1}(\theta)^2 + \beta_2\, y_{t-1}^2\right)^{\frac{1}{2}}, \quad \beta_i > 0. \quad (3)$$

$Q_t(\theta)$ is the θ quantile conditional upon Ψ_{t-1}, the information set up to time t-1; the β_i are parameters; $I(x)$ is the indicator function; and $(x)^+ = \max(x,0)$ and $(x)^- = -\min(x,0)$. Note that we are modelling here a residual term, y_t, defined as $y_t = r_t - E(r_t | \Psi_{t-1})$, where r_t is the return and $E(r_t | \Psi_{t-1})$ is the conditional expectation, which is often assumed to be zero or a constant. Expression (4) presents a CAViaR model proposed by Kuester, Mittnik and Paolella (2006), which aims to model autocorrelation in the conditional mean of the returns series. The autocorrelation is captured by the parameter α_1.

Indirect ARGARCH CAViaR:

$$Q_t(\theta) = \alpha_1 r_{t-1} + \left(1 - 2I(\theta < 0.5)\right)\left(\beta_0 + \beta_1 \left(Q_{t-1}(\theta) - \alpha_1 r_{t-2}\right)^2 + \beta_2 \left(r_{t-1} - \alpha_1 r_{t-2}\right)^2\right)^{\frac{1}{2}}, \quad \beta_i > 0. \tag{4}$$

CAViaR model parameters are estimated using the quantile regression minimisation in the following expression, which was introduced by Koenker and Bassett (1978):

$$\min_{\boldsymbol{\beta}} \sum_t \left(\theta - I\left(y_t < Q_t(\theta)\right)\right)\left(y_t - Q_t(\theta)\right), \tag{5}$$

where y_t is the target variable; $\boldsymbol{\beta}$ is a vector of parameters in the model for $Q_t(\theta)$. Although CAViaR models provide an attractive means of estimating the conditional quantile, by their very nature they provide only a model for the quantile, and it is not clear how to calculate the corresponding ES.

The most widely used *nonparametric* VaR method is historical simulation, which requires no distributional assumptions and estimates the VaR as the quantile of the empirical distribution of historical returns from a moving window of the most recent periods. For this approach, the ES can be estimated as the mean of the returns, in the moving window, that exceed the VaR estimate. A difficulty with the historical simulation method is the choice of how many past periods to include in the moving window. Including too few observations will lead to large sampling error, while using too many will result in estimates that are slow to react to changes in the true distribution. This issue and the strong appeal in giving more weight to more recent observations prompted Taylor (2006) to propose exponentially weighted quantile regression, which amounts to simple exponential smoothing of the cumulative distribution function (cdf). For this method, ES estimation can be performed using the cost function of the exponentially weighted quantile regression.

2. Expectiles

2.1. Expectiles and ALS Regression

Before introducing expectiles, it is useful first to appreciate that the population θ quantile of a random variable y is the parameter m that minimises the function $E[(\theta - I(y < m))(y - m)]$, where the expectation is taken with respect to the random variable y. In view of this, quantile regression is the natural means by which to estimate parameters in a conditional quantile model. Turning to expectiles, the population τ expectile of y is the parameter m that minimises the function $E[|\tau - I(y < m)|(y - m)^2]$. It seems natural to estimate the parameters of a conditional model for expectile $\mu_t(\tau)$ using asymmetric least squares (ALS) regression, which is the least squares analogue of quantile regression. The ALS minimisation is presented in expression (6). It was originally proposed by Aigner, Amemiya and Poirier (1976), and is considered further by Newey and Powell (1987). Note that for τ=0.5 expression (6) becomes the widely used (symmetric) LS regression. Comparing quantiles and expectiles, Koenker (2005) observes that expectiles have a more "global dependence on the form of the distribution". For example, altering the shape of the upper tail of a distribution does not change the quantiles of the lower tail, but it does impact all of the expectiles.

$$\min_{\boldsymbol{\beta}} \sum_t |\tau - I(y_t < \mu_t(\tau))| (y_t - \mu_t(\tau))^2 . \qquad (6)$$

2.2. Using Expectiles to Estimate VaR

In this paper, we use expectiles as estimators of quantiles. This was first proposed by Efron (1991) who was attracted by the computational simplicity of ALS relative to quantile regression. The proposal involves using, as an estimator of the θ quantile, the expectile for which the proportion of in-sample observations lying below the expectile is θ. This is based on the fact that, for each τ expectile, there is a corresponding θ quantile, though τ is typically not equal to θ. The existence of a one-to-one mapping from expectiles to quantiles is supported by the theoretical work of Jones (1994), Abdous and Remillard (1995) and Yao and Tong (1996). Empirical support for Efron's proposal is provided by Sin and Granger (1999) and Granger and Sin (2000), using macroeconomic data and absolute financial returns, respectively.

2.3. Using Expectiles to Estimate ES

Newey and Powell (1987) provide insight into the result of the ALS minimisation in expression (6) by considering the case where the expectile is a scalar parameter. They consider the minimisation of the function $E[|\tau - I(y<m)|(y-m)^2]$ over *m*. It is straightforward to show that the solution $\mu(\tau)$ of this minimisation satisfies expression (7):

$$\left(\frac{1-2\tau}{\tau}\right) E[(y-\mu(\tau))I(y<\mu(\tau))] = \mu(\tau) - E(y). \tag{7}$$

This is a rearrangement of Newey and Powell's expression (2.7). They explain that the expression indicates that the solution $\mu(\tau)$ is determined by the properties of the expectation of the random variable *y* conditional on *y* exceeding $\mu(\tau)$. This suggests a link between expectiles and ES. Expression (7) can be rewritten as

$$E(y|y<\mu(\tau)) = \left(1+\frac{\tau}{(1-2\tau)F(\mu(\tau))}\right)\mu(\tau) - \frac{\tau}{(1-2\tau)F(\mu(\tau))}E(y),$$

where *F* is the cdf of *y*. This expression provides a formula for the ES of the quantile that coincides with the τ expectile. Referring to this as the θ quantile, we can write $F(\mu(\tau))=\theta$. and rewrite the expression as

$$ES(\theta) = \left(1+\frac{\tau}{(1-2\tau)\theta}\right)\mu(\tau) - \frac{\tau}{(1-2\tau)\theta}E(y). \tag{8}$$

With y_t defined, as in Section 1, to be a zero mean residual term, this simplifies to the following:

$$ES(\theta) = \left(1+\frac{\tau}{(1-2\tau)\theta}\right)\mu(\tau). \tag{9}$$

This expression relates the ES associated with the θ quantile of a zero-mean distribution and the τ expectile that coincides with that quantile. The expression is for ES in the lower tail of the distribution. The expression for the upper tail of the distribution is produced by replacing τ and θ with $(1-\tau)$ and $(1-\theta)$, respectively.

Although expressions (8) and (9) are for the case where the expectile is a scalar parameter, similar expressions are satisfied by an expectile that is conditional on an information set up to period *t*-1. This conditional expectile, $\mu_t(\tau)$, satisfies the following expression for the conditional ES:

$$ES_t(\theta) = \left(1+\frac{\tau}{(1-2\tau)\theta}\right)\mu_t(\tau) - \frac{\tau}{(1-2\tau)\theta}E(y_t). \tag{10}$$

If y_t is defined to be a zero mean residual term, this becomes:

$$ES_t(\theta) = \left(1 + \frac{\tau}{(1-2\tau)\theta}\right)\mu_t(\tau). \quad (11)$$

As we explained in Section 2.2, we follow Efron's proposal of using a conditional model for the τ expectile to estimate the θ quantile. Expressions (10) and (11) serve as a simple way to calculate the ES associated with this estimate. Considering expression (11), note that it is intuitively reasonable that, over time, for a given value of θ, the conditional ES is proportional to the conditional quantile model $\mu_t(\tau)$. Indeed, this is also the case with some other widely used models for financial returns, such as when a conditional volatility model is used with a Gaussian or Student-*t* distribution. Although regressors could be included in the expectile models, VaR and ES are usually estimated using univariate models. This implies a need for univariate expectile models. In the next section, we present a new class of such models.

3. Conditional Autoregressive Expectiles (CARE)

The structure of the CAViaR models in expressions (1)-(4) can be used for conditional autoregressive expectile (CARE) models. For example, the Symmetric Absolute Value CARE model is shown in expression (12).

$$\mu_t(\tau) = \beta_0 + \beta_1\,\mu_{t-1}(\tau) + \beta_2\,|y_{t-1}|. \quad (12)$$

The model parameters can be estimated using ALS with a similar non-linear optimisation routine to that used by Engle and Manganelli (2004) for CAViaR models. We describe this optimisation in more detail in Section 4.1. Using expression (10) or (11), it is straightforward to convert the CARE models into conditional autoregressive ES models. For example, substituting $\mu_t(\tau)$ from expression (11) into expression (12) delivers the following conditional ES model:

$$ES_t(\theta) = \gamma_0 + \gamma_1\,ES_{t-1}(\theta) + \gamma_2\,|y_{t-1}|, \quad (13)$$

where $\gamma_1=\beta_1$, and, for i = 0 and 2, $\gamma_i = \left(1 + \frac{\tau}{(1-2\tau)\theta}\right)\beta_i$.

In Section 2.2, we explained that in this paper, we use expectiles as estimators of quantiles. More specifically, we use CARE to model a conditional θ quantile. The value of τ that we select for the CARE model is the value for which the proportion of in-sample observations lying below the conditional expectile model is θ. The success of the use of CARE models for VaR and ES estimation relies on θ being a close approximation of the conditional coverage of the model. We test this in the empirical study of Section 4.

The quantile or volatility forecast from another approach could be included as a regressor within the CARE models. This would enable a form of forecast combining. If confidence intervals are required for parameter estimates, or for VaR or ES predictions, we would suggest the use of a bootstrap resampling approach. In addition to CAViaR models, the quantile regression literature contains several other forms of autoregressive quantile models (e.g. Koenker and Zhao, 1996). The structure of these models could also be considered for autoregressive expectile modelling.

4. Empirical Illustration and Evaluation of VaR and ES Estimation

In this section, we describe a study that compared the accuracy of the VaR and ES estimates from our new methods with those from established methods. The study considered day-ahead estimation of the 1%, 5%, 95% and 99% conditional VaR and ES. We chose these quantiles because they are widely considered in practice. Our focus on day-ahead estimation is consistent with the holding period considered for internal risk control by most financial firms.

We used the following stock indices: the French CAC40, the German DAX30, British FTSE100, Japanese Nikkei225 and the US S&P500. The sample period used in our study consisted of daily data, from 1 September 1997 to 2 May 2005. This period delivered 2000 log returns. As in the study of Kuester, Mittnik and Paolella (2006), we used a moving window of 1000 observations to re-estimate repeatedly parameters for the various methods. Day-ahead post-sample VaR and ES estimates were produced from each method for the final 1000 days of each stock index series. We use these post-sample predictions as the basis of our comparison of methods. With the exception of the Indirect ARGARCH CAViaR and Indirect ARGARCH CARE models, we followed common practice by not estimating models for the conditional mean of each series (see Poon and Granger, 2003). For each moving window, we subtracted from each return, r_t, the mean, μ, of the 1000 in-sample returns. The quantile estimation methods were applied to the resultant residuals, $y_t = r_t - \mu$.

In the next section, we present the methods considered in our study. Although there are many VaR estimation methods that we could have implemented as benchmark methods (see, for example, Kuester, Mittnik and Paolella, 2006), we restricted ourselves to commonly used methods for which ES estimation is straightforward. All methods have been implemented in Gauss code, which is available on request.

4.1. Methods Used for Estimating VaR and ES

Benchmark Methods

We implemented the historical simulation method using moving windows of lengths 250, 500 and 1000 days. The VaR and ES estimates were produced from the method as described in Section 1.

We included the GARCH(1,1) model in our study. Our use of the (1,1) specification was based on the general popularity of this order for GARCH models. We also implemented the asymmetric GJRGARCH(1,1) model of Glosten, Jagannathan and Runkle (1993), but we found that it was outperformed by the standard symmetric GARCH model, and so in the remainder of this paper, we do not refer to GJRGARCH. We considered two versions of the GARCH(1,1) model. The first version, which we refer to as GARCH-*t*, used the Student-*t* distribution and the second version, which we term GARCH-SKEW*t*, used the generalized asymmetric *t*-distribution of Mittnik and Paolella (2000) that has density of the following form:

$$f(z;d,v,\psi) = I(z<0)C\left(1+\frac{(-z\psi)^d}{v}\right)^{-\left(v+\frac{1}{d}\right)} + I(z\geq 0)C\left(1+\frac{(z/\psi)^d}{v}\right)^{-\left(v+\frac{1}{d}\right)}, \qquad (14)$$

where d, v and ψ are positive parameters, $C = \left[\left(\psi+\psi^{-1}\right)d^{-1}v^{1/d}B\left(d^{-1},v\right)\right]^{-1}$, and $B(.,.)$ is the beta function. Using the respective distributions, we optimised model parameters using maximum likelihood. We then used the distributions to produce parametric estimates of the conditional quantile and ES. For the Student-*t* distribution, expressions for the VaR and ES are presented by McNeil, Frey and Embrechts (Section 2.2.4, 2005). For the generalised asymmetric distribution, Kuester, Mittnik and Paolella (2006) provide the expression for the cdf, which can be used to derive VaR estimates. They do not consider ES, but using the density function in expression (14), we get the following expression for the ES corresponding to the θ quantile, $Q(\theta)$:

$$ES(\theta) = \begin{cases} \dfrac{-C\,v^{2/d}}{\theta\psi^2 d} B(v-1/d\,,2/d) F_{beta}(L, v-1/d\,,2/d) & \text{if } \theta \le 0.50 \\ \dfrac{C\,v^{2/d}\psi}{(1-\theta)d} B(v-1/d\,,2/d)\left(1-F_{beta}(U,2/d\,,v-1/d)\right) & \text{if } \theta > 0.50 \end{cases}$$

where $L = v/\left(v+(-Q(\theta)\psi)^d\right)$, $U = (Q(\theta)/\psi)^d/\left(v+(Q(\theta)/\psi)^d\right)$ and $F_{beta}(.,.,)$ is the cdf of a beta distribution, which is also known as the incomplete beta. In addition to the parametric estimates, for both GARCH versions, we also produced conditional quantile and ES estimates by applying the peaks over threshold EVT method to the standardised residuals (see Section 7.2.3, McNeil, Frey and Embrechts, 2005). In this approach, we set the threshold as the 10% unconditional quantile for the lower tail, and as the 90% unconditional quantile for the upper tail. The GARCH-SKEW*t* model, in conjunction with this EVT approach, performed very well in terms of VaR estimation in the study of Kuester, Mittnik and Paolella (2006).

CAViaR Models

We estimated the CAViaR models presented in Section 1 using a procedure similar to that described by Engle and Manganelli (2004). For each model, we first generated 10^5 vectors of parameters from a uniform random number generator between 0 and 1, or between -1 and 0, depending on the appropriate sign of the parameter. For each of the vectors, we then evaluated the QR Sum, which we define as the summation in the quantile regression objective function presented in expression (5). The 10 vectors that produced the lowest values of the QR Sum were used as initial values in a quasi-Newton algorithm. The QR Sum was then calculated for each of the 10 resulting vectors, and the vector producing the lowest value of the QR Sum was chosen as the final parameter vector. Due to computational running times, it was impractical to implement this estimation procedure for each moving window in our study. However, this was not a significant issue because much of the procedure is aimed at deriving suitable initial values for the quasi-Newton algorithm. In our study, we implemented the full estimation procedure for the first moving window, and for each subsequent moving window we simply performed the quasi-Newton algorithm using as initial parameter vector the optimal vector from the previous moving window.

CARE Models

We implemented the Symmetric Absolute Value, Asymmetric Slope, Indirect GARCH and Indirect ARGARCH CARE models introduced in Section 3. As with the GARCH models, we found that the Asymmetric Slope CAViaR and CARE models were outperformed by the symmetric versions of the models, and so in the remainder of this paper, we do not consider further the asymmetric models. We estimated CARE model parameters using the same procedure as described in the previous section for the CAViaR models, except that the QR Sum was replaced by the ALS summation presented in expression (6).

As described in Section 2.2, we set, as estimator of the θ quantile, the τ expectile for which the proportion of in-sample observations lying below the expectile is θ. To find the optimal value of τ, we estimated models for different values of τ over a grid with step size of 0.0001. The final optimal value of τ was derived by linearly interpolating between grid values. We used just the first moving window of observations to optimise τ. The resulting τ values are reported in Table 1. For a given θ quantile, the table shows similar values of τ for the three different types of CARE model. The table shows that values of τ are more extreme than their corresponding values of θ, and this is consistent with Newey and Powell's (1987) results for the Gaussian distribution and Granger and Sin's (2000) results for the absolute value of financial returns. In Figure 1, we present a similar plot to Newey and Powell's Figure 1. Our figure plots the unconditional θ quantiles and unconditional τ expectiles against θ and τ, respectively. The figure shows that for the unconditional θ quantile and τ expectile to be identical, the value of τ has to be more extreme than the value of θ.

---------- Table 1 and Figure 1 ----------

Due to the need to estimate the value of τ, the estimation of CARE models would appear to be more computationally demanding than CAViaR models. However, it is interesting to note that, even with this extra task, the computational running times for one full implementation of the CARE models for each series was less than for the CAViaR models. This is because the ALS minimisation is somewhat less challenging than the quantile regression minimisation.

Let us now illustrate how we use a CARE model to produce predictive VaR and ES models. The Symmetric Absolute Value CARE model and inferred conditional ES model were presented in expressions (12) and (13), respectively. As indicated in Table 1, for the FTSE100 returns, the 5% VaR is estimated using the Symmetric Absolute Value CARE model with τ=0.0126. Expression (15) presents the model with parameters estimated from the first moving window of 1000 observations.

$$\mu_t(0.0126) = -0.00179 + 0.869\mu_{t-1}(0.0126) - 0.107|y_{t-1}| . \quad (15)$$

Replacing $\mu_t(0.0126)$ with $Q_t(0.05)$ in expression (15) gives the following autoregressive VaR model:

$$Q_t(0.05) = -0.00179 + 0.869Q_{t-1}(0.05) - 0.107|y_{t-1}| .$$

Substituting θ=0.05 and τ=0.0126 in expression (11) gives $ES_t(0.05) = 1.259\mu_t(0.0126)$. Using this, we can substitute for $\mu_t(0.0126)$ in expression (15) to give the following conditional ES model:

$$ES_t(0.05) = -0.00225 + 1.094ES_{t-1}(0.05) - 0.135|y_{t-1}| .$$

Similar steps are used with the other CARE models to produce conditional VaR and ES models. For the Indirect ARGARCH CARE model, note that the ES model is produced using expression (10), rather than expression (11), because this CARE model does not assume that the condtional mean of the returns is constant.

Tables 2 and 3 provide parameters for the Symmetric Absolute Value CARE model of expression (12) and the Indirect GARCH CARE model, which for clarity we present in expression (16).

$$\mu_t(\tau) = (1 - 2I(\tau < 0.5))\left(\beta_0 + \beta_1\ \mu_{t-1}(\tau)^2 + \beta_2\ y_{t-1}^2\right)^{\frac{1}{2}}, \qquad \beta_i>0. \quad (16)$$

---------- Tables 2 and 3 and Figure 2 ----------

Figure 2 presents the post-sample 5% and 95% VaR and ES estimates, produced for the 1000 post-sample days of the FTSE100 returns, using the Symmetric Absolute Value CARE model. The VaR and ES estimates can be seen to change with the volatility in the returns.

4.2. VaR Results

To evaluate the post-sample conditional quantile estimates, we use the hit percentage and dynamic quantile (DQ) test. The hit percentage assesses the percentage of observations falling below the estimator. Ideally, for estimation of the conditional θ quantile, the percentage should be θ. We examined significant difference from this ideal using a test based on the binomial distribution. Engle and Manganelli's (2004) DQ test is a development of the test proposed by Christoffersen (1998), which evaluates the dynamic properties of a conditional quantile estimator. The DQ test involves the joint test of whether the hit variable, defined as $Hit_t = I(y_t \leq \hat{Q}_t(\theta)) - \theta$, is distributed i.i.d. Bernoulli with probability θ, and is independent of the conditional quantile estimator. Ideally, Hit_t will have zero unconditional and conditional expectations, and this is the null hypothesis in the test. As in the empirical study of Engle and Manganelli, we included four lags of Hit_t in the test's regression to deliver a DQ test statistic, which, under the null hypothesis, is distributed as $\chi^2(6)$.

Tables 4 and 5 present the values of the hit percentage measure for each method applied to each of the five stock indices for estimation of the 95% and 99% quantiles, respectively. P-values are presented in parentheses for the significance test with perfect hit percentage as null hypothesis. The final column presents a count for the number of series for which the null is rejected at the 5% level. (In all our tables of results, smaller counts and larger p-values are desirable.) The results are poor for the historical simulation method based on a moving window of 1000 days, and for the GARCH model with Student-*t* distribution assumption. The results for the latter method are improved when EVT is applied to the standardised residuals. Of the CARE models, the Indirect GARCH model performed particularly well for both the 95% and 99% quantiles.

---------- Tables 4 to 7 ----------

Tables 6 and 7 report the DQ test p-values for the 95% and 99% quantiles, respectively. As in Tables 4 and 5, the final column presents a count for the number of series for which the null is rejected at the 5% level. The DQ test results are poor for the historical simulation approach when based on either 1000 or 500 days in the moving window. As with the hit percentage results, the DQ results for the GARCH model, estimated using the Student-*t* distribution, were improved when EVT was used to construct the quantiles rather than simply using the Student-*t* distribution. The DQ test results in Tables 6 and 7 are a little disappointing for the GARCH model estimated using the skewed-*t* distribution. The tables show that the

CAViaR and CARE models performed well for both the 95% and 99% quantiles. In Section 4.4, we summarise the VaR results for all four quantiles.

4.3. ES Results

To evaluate conditional ES estimation, we follow the approach of McNeil and Frey (2000), which focuses on the discrepancy between an observation and the conditional ES estimate for only those periods for which the observation exceeds the conditional quantile estimate. McNeil and Frey note that these discrepancies, when standardised by the conditional volatility, should be i.i.d. with a mean of zero. In order to avoid distributional assumptions, they use a bootstrap test to test for zero mean (see page 224 of Efron and Tibshirani, 1993). We were forced to adapt this test for use in our study because the CARE models do not involve the estimation of the conditional volatility. In our version of the test, instead of standardising with the volatility, we standardise using the conditional quantile estimate for each method. Manganelli and Engle (2004) also standardise using this estimator in their application of extreme value theory to standardised quantile residuals. In Tables 8 and 9, we report p-values for the bootstrap test for the post-sample conditional 95% and 99% ES estimates. As with the earlier tables for VaR evaluation, the final columns in these tables present a count for the number of series for which the null is rejected at the 5% level. Interestingly, the ES results for the CARE models are very competitive with the other methods.

Unfortunately, testing whether the standardised discrepancies are i.i.d. is problematic due to the low number of discrepancies, and this is particularly so for 1% and 99% estimation. Indeed, McNeil and Frey do not perform a test for i.i.d. discrepancies. We tested for zero autocorrelation in each series of discrepancies corresponding to 5% and 95% estimation. For each of the methods, except historical simulation, we found that, when testing at the 5% level, the total number of rejections of the null hypothesis across these two quantiles and the five series was zero or one. For simplicity, we do not present these results in detail here.

---------- Tables 8 to 10 ----------

4.4. Summary of VaR and ES Results

Table 10 is a summary, for all four quantiles, of the results for the two VaR tests and the ES test described in the previous two sections. The table presents the number of test rejections at the 5% significance level, as presented for the 95% and 99% quantiles in the final columns of Tables 4 to 9. As we used five indices in our study, for a given quantile, the maximum number of test rejections for any single test is five. For simplicity, in Table 10, we shall focus on the three columns labelled "Total", which contain the total number of rejections across the four quantiles.

The DQ test results suggest that it is inadvisable to use historical simulation regardless of the number of days used in the moving window. For the GARCH model estimated using a Student-*t* distribution, the results indicate that it is far preferable to construct VaR estimates using EVT than simply to use the Student-*t* distribution. By contrast with the results of Kuester, Mittnik and Paolella (2006), in our study, VaR estimation was not improved by the use of EVT with the GARCH skewed-*t* distribution. Our results also differ from those of Kuester, Mittnik and Paolella in terms of the CAViaR models; the Indirect GARCH CAViaR model performs very well, and is not outperformed by the Indirect ARGARCH CAViaR model. The fact that the CAViaR models are so competitive in terms of VaR estimation motivates consideration of models with similar characteristics that produce not only VaR estimates but also ES estimates. This is the motivation for CARE models. The columns corresponding to the two VaR tests indicate that each CARE model is slightly outperformed by its corresponding CAViaR model. This is not surprising, as the CAViaR models are estimated using quantile regression, while the CARE models are expectile approximations of quantile models. However, it is reassuring to see from the VaR results that the approximation is not poor.

Turning to the ES estimation results, there is not a substantial difference between the methods. There is a suggestion that the use of EVT is beneficial for the GARCH models, and that the GARCH model estimated using the skewed-*t* distribution is a good candidate for ES estimation. This last point compliments the findings of Kuester, Mittnik and Paolella (2006) who promote the use of the method in the context of VaR estimation. Although the ES estimation results do not indicate superior performance for the CARE models, it is interesting to see from the final column in Table 10 that the methods are competitive in this respect.

In the analysis reported so far, for all the methods except historical simulation we used a moving window of 1000 periods. We also considered windows of lengths 500 and 250 periods. In terms of VaR estimation, the accuracy of the various methods did not weaken when using the smaller window sizes. This was also the case for ES estimation, with the one exception being a reduction in accuracy when using a smaller window size for the approach based on the CARE models. As previously described, this approach requires, as estimator of the θ quantile, the τ expectile for which the proportion of in-sample observations lying below the expectile is θ. It seems that the smaller window size causes difficulty for the derivation of the optimal value of τ for a given θ quantile. This point was confirmed by the substantial improvement in the ES estimation accuracy from the CARE models that resulted when we reran the analysis with window size of 250 periods using the values of τ derived from the 1000 periods immediately prior to the first forecast origin. (These values of τ were used in the initial analysis with moving window size of 1000 periods.)

5. Summary and Concluding Comments

In this paper, we have introduced a new approach to estimating conditional VaR and conditional ES using ALS regression. Following the suggestion of Efron (1991), we estimate the θ quantile by the expectile for which the proportion of in-sample observations lying below the expectile is θ. The main contribution of this paper is that we show that the corresponding ES estimator is a simple function of the expectile. Therefore, a conditional expectile estimator can be used as a conditional quantile estimator, and also, after a simple transformation, as a conditional ES estimator. A further contribution of the paper is the introduction of CARE, which is a new class of univariate expectile models inspired by Engle and Manganelli's CAViaR models. CARE models enable conditional autoregressive ES modelling. Our empirical study suggests that the CARE models are competitive in terms of both VaR and ES estimation.

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Figure Legends

Figure 1 Unconditional θ quantiles and τ expectiles plotted against θ and τ, respectively, for the first moving window of 1000 daily FTSE100 daily stock index returns.

Figure 2 FTSE100 daily stock index returns for the 1000 post-sample days with VaR and ES estimates from the Symmetric Absolute Value CARE model and the inferred conditional ES model, respectively.

References

Abdous, B., B. Remillard. (1995). "Relating Quantiles and Expectiles Under Weighted-Symmetry. *Annals of the Institute of Statistical Mathematicas* 47, 371-384.

Acerbi, C., D. Tasche. (2002). On the Coherence of Expected Shortfall. *Journal of Banking and Finance* 26, 1487-1503.

Aigner, D.J., T. Amemiya, D.J. Poirier. (1976). On the Estimation of Production Frontiers: Maximum Likelihood Estimation of the Parameters of a Discontinuous Density Function. *International Economic Review* 17, 377-396.

Artzner, P.F., F. Delbain, J.M. Eber, D. Heath. (1999). Coherent Measures of Risk. *Mathematical Finance* 9, 203-228.

Christoffersen, P.F. (1998). Evaluating Interval Forecasts. *International Economic Review* 39, 841-862.

Diebold, F.X., T. Schuermann, J.D. Stroughair. (2000). Pitfalls and Opportunities in the Use of Extreme Value Theory in Risk Management. *Journal of Risk Finance* 1 (Winter), 30-36.

Efron, B. (1991). Regression Percentiles Using Asymmetric Squared Error Loss. *Statistica Sinica* 1, 93-125.

Efron, B., J. Tibshirani. (1993). *An Introduction to the Bootstrap*. New York: Chapman and Hall.

Engle, R.F., S. Manganelli. (2004). CAViaR: Conditional Autoregressive Value at Risk by Regression Quantiles. *Journal of Business and Economic Statistics* 22, 367-381.

Glosten, L.R., R. Jagannathan, D.E. Runkle. (1993). On the Relation Between the Expected Value and the Volatility of the Nominal Excess Return on Stocks. *Journal of Finance* 48, 1779-1801.

Granger, C.W.J., C.-Y. Sin. (2000). Modelling the Absolute Returns of Different Stock Indices: Exploring the Forecastability of an Alternative Measure of Risk. *Journal of Forecasting* 19, 277-298.

Jones, M.C. (1994). Expectiles and M-quantiles are Quantiles. *Statistics and Probability Letters* 20, 149-153.

Koenker, R.W. (2005). *Quantile Regression*. Cambridge, UK: Cambridge University Press.

Koenker, R.W., G.W. Bassett. (1978). Regression Quantiles. *Econometrica* 46, 33-50.

Koenker, R.W., Q. Zhao. (1996). Conditional Quantile Estimation and Inference for ARCH Models. *Econometric Theory* 12, 793-813.

Kuester, K., S. Mittnik, M.S. Paolella. (2006). Value-at-Risk Prediction: A Comparison of Alternative Strategies. *Journal of Financial Econometrics* 4, 53-89.

Manganelli S., R.F. Engle. (2004). A Comparison of Value-at-Risk Models in Finance. Szegö G., ed., *Risk Measures for the 21st Century*, Chichester, UK: Wiley.

McNeil, A.J., R. Frey. (2000). Estimation of Tail-Related Risk Measures for Heteroscedastic Financial Time Series: An Extreme Value Approach. *Journal of Empirical Finance* 7, 271-300.

McNeil, A.J., R. Frey, P. Embrechts. (2005). *Quantitative Risk Management*. New Jersey: Princeton University Press.

Mittnik, S., M.S. Paolella. (2000). Conditional Density and Value-at-Risk Prediction of Asian Currency Exchange Rates. *Journal of Forecasting* 19, 313-333.

Newey, W. K., J.L. Powell. (1987). Asymmetric Least Squares Estimation and Testing. *Econometrica* 55, 819-847.

Poon, S., C.J.W. Granger. (2003). Forecasting Volatility in Financial Markets: A Review. *Journal of Economic Literature* 41, 478-639.

Sin, C.-Y., C.W.J. Granger. (1999). Estimating and Forecasting Quantiles with Asymmetric Least Squares. Working Paper, University of San Diego.

Taylor, J.W. (2006). Estimating Value at Risk Using Exponentially Weighted Quantile Regression. Working Paper, University of Oxford, UK.

Yamai, Y., T. Yoshiba. (2002). On the Validity of Value-at-Risk: Comparative Analyses with Expected Shortfall. *Monetary and Economic Studies* 20, 57-85.

Yao, Q., H. Tong. (1996). Asymmetric Least Squares Regression Estimation: A Nonparametric Approach. *Nonparametric Statistics* 6, 273-292.

Table 1 For a given θ quantile, optimal values of ALS parameter τ ($\times 100$) for the three CARE models, derived using the first moving window of 1000 days.

CARE model	CAC40	DAX30	FTSE100	NIKKEI225	S&P500	Mean for each model	Mean for each quantile
1% quantile							
Sym Abs Value	0.110	0.125	0.115	0.245	0.260	0.171	
Indirect GARCH	0.110	0.163	0.100	0.237	0.320	0.186	0.182
Indirect ARGARCH	0.100	0.170	0.102	0.240	0.330	0.188	
5% quantile							
Sym Abs Value	1.330	1.350	1.260	1.540	1.910	1.478	
Indirect GARCH	1.310	1.510	1.360	1.480	1.800	1.492	1.493
Indirect ARGARCH	1.410	1.370	1.348	1.540	1.870	1.508	
95% quantile							
Sym Abs Value	98.690	98.395	99.110	98.000	98.660	98.571	
Indirect GARCH	98.750	98.510	98.855	98.140	98.377	98.526	98.520
Indirect ARGARCH	98.745	98.420	98.690	98.110	98.348	98.463	
99% quantile							
Sym Abs Value	99.845	99.800	99.890	99.780	99.900	99.843	
Indirect GARCH	99.850	99.770	99.880	99.760	99.880	99.828	99.823
Indirect ARGARCH	99.865	99.710	99.890	99.670	99.850	99.797	

Table 2 **For the Symmetric Absolute Value CARE model presented in expression (12), parameter estimates derived using the first moving window of 1000 days.**

	CAC40	DAX30	FTSE100	NIKKEI225	S&P500
1% quantile					
$\tau \times 100$	0.110	0.125	0.115	0.245	0.260
β_0	-0.00162	-0.00513	-0.00748	-0.00232	-0.00863
β_1	0.900	0.691	0.716	0.870	0.505
β_2	-0.189	-0.631	-0.135	-0.208	-0.977
5% quantile					
$\tau \times 100$	1.330	1.350	1.260	1.540	1.910
β_0	-0.00055	-0.00227	-0.00179	-0.00141	-0.00773
β_1	0.911	0.763	0.869	0.856	0.508
β_2	-0.155	-0.354	-0.107	-0.198	-0.253
95% quantile					
$\tau \times 100$	98.690	98.395	99.110	98.000	98.660
β_0	0.00171	0.00144	0.00010	0.00057	-0.00013
β_1	0.819	0.857	0.943	0.922	0.893
β_2	0.214	0.149	0.110	0.115	0.245
99% quantile					
$\tau \times 100$	99.845	99.800	99.890	99.780	99.900
β_0	0.00957	0.00305	0.00064	0.00106	0.00021
β_1	0.585	0.863	0.935	0.925	0.906
β_2	0.372	0.150	0.113	0.199	0.283

Table 3 **For the Indirect GARCH CARE model presented in expression (16), parameter estimates derived using the first moving window of 1000 days.**

	CAC40	DAX30	FTSE100	NIKKEI225	S&P500
1% quantile					
$\tau \times 100$	0.110	0.163	0.100	0.237	0.320
β_0	0.000065	0.000052	0.000295	0.000115	0.000354
β_1	0.890	0.832	0.645	0.844	0.570
β_2	0.415	0.896	0.400	0.403	0.933
5% quantile					
$\tau \times 100$	1.310	1.510	1.360	1.480	1.800
β_0	0.000068	0.000065	0.000095	0.000069	0.000153
β_1	0.782	0.714	0.753	0.773	0.528
β_2	0.363	0.645	0.098	0.351	0.360
95% quantile					
$\tau \times 100$	98.750	98.510	98.855	98.140	98.377
β_0	0.000071	0.000067	0.000048	0.000068	0.000033
β_1	0.706	0.758	0.720	0.772	0.730
β_2	0.409	0.267	0.440	0.321	0.477
99% quantile					
$\tau \times 100$	99.850	99.770	99.880	99.760	99.880
β_0	0.000261	0.000057	0.000032	0.000024	0.000017
β_1	0.606	0.896	0.899	0.882	0.871
β_2	0.718	0.263	0.264	0.872	0.658

Table 4 Evaluation of estimators of 95% VaR. Hit percentage for 1000 post-sample estimates of 95% conditional quantile (p-values in parentheses).

	CAC40	DAX30	FTSE100	NIKKEI225	S&P500	Number significant at 5% level
Benchmark methods						
Hist Sim 1000	94.8 (0.773)	94.5 (0.471)	95.7 (0.312)	96.4 (0.042)	96.9 (0.006)	2
Hist Sim 500	95.2 (0.773)	94.9 (0.886)	96.2 (0.082)	96.2 (0.082)	96.7 (0.013)	1
Hist Sim 250	95.7 (0.312)	95.4 (0.564)	96.3 (0.059)	96.6 (0.020)	95.7 (0.312)	1
GARCH Student-*t*	97.3 (0.001)	96.4 (0.042)	97.3 (0.001)	97.6 (0.000)	97.1 (0.002)	5
GARCH Student-*t* EVT	94.9 (0.886)	94.9 (0.886)	95.6 (0.387)	96.1 (0.111)	94.9 (0.886)	0
GARCH Skew-*t*	96.3 (0.059)	95.8 (0.248)	96.0 (0.148)	96.3 (0.059)	95.8 (0.248)	0
GARCH Skew-*t* EVT	96.2 (0.082)	95.3 (0.666)	96.0 (0.148)	96.3 (0.059)	95.4 (0.564)	0
CAViaR models						
Sym Abs Value CAViaR	95.2 (0.773)	94.6 (0.564)	95.9 (0.193)	95.5 (0.471)	95.0 (1.000)	0
Indirect GARCH CAViaR	95.3 (0.666)	94.5 (0.471)	95.8 (0.248)	95.5 (0.471)	95.0 (1.000)	0
Indirect ARGARCH CAViaR	95.2 (0.773)	94.6 (0.564)	95.2 (0.773)	95.7 (0.312)	95.1 (0.886)	0
CARE models						
Sym Abs Value CARE	94.8 (0.773)	94.0 (0.148)	96.3 (0.059)	94.3 (0.312)	95.7 (0.312)	0
Indirect GARCH CARE	95.2 (0.773)	94.4 (0.387)	95.8 (0.248)	94.7 (0.666)	95.2 (0.773)	0
Indirect ARGARCH CARE	95.0 (1.000)	93.8 (0.082)	94.9 (0.886)	94.5 (0.471)	94.8 (0.773)	0

Table 5 Evaluation of estimators of 99% VaR. Hit percentages for 1000 post-sample estimates of 99% conditional quantile (p-values in parentheses).

	CAC40	DAX30	FTSE100	NIKKEI225	S&P500	Number significant at 5% level
Benchmark methods						
Hist Sim 1000	98.2 (0.010)	98.1 (0.004)	98.3 (0.024)	99.8 (0.010)	99.1 (0.760)	4
Hist Sim 500	98.5 (0.114)	98.6 (0.211)	98.3 (0.024)	99.4 (0.221)	99.2 (0.539)	1
Hist Sim 250	98.7 (0.353)	98.7 (0.353)	98.7 (0.353)	99.6 (0.055)	98.8 (0.539)	0
GARCH Student-*t*	99.5 (0.114)	99.5 (0.114)	99.8 (0.010)	99.8 (0.010)	99.8 (0.010)	3
GARCH Student-*t* EVT	98.7 (0.353)	99.2 (0.539)	99.1 (0.760)	99.8 (0.010)	99.5 (0.114)	1
GARCH Skew-*t*	98.9 (0.760)	99.1 (0.760)	99.5 (0.114)	99.8 (0.010)	99.5 (0.114)	1
GARCH Skew-*t* EVT	98.8 (0.539)	99.1 (0.760)	99.2 (0.539)	99.8 (0.010)	99.5 (0.114)	1
CAViaR models						
Sym Abs Value CAViaR	98.3 (0.024)	98.9 (0.760)	98.9 (0.760)	99.5 (0.114)	99.3 (0.353)	1
Indirect GARCH CAViaR	98.4 (0.055)	98.9 (0.760)	99.0 (1.000)	99.5 (0.114)	99.0 (1.000)	0
Indirect ARGARCH CAViaR	98.3 (0.024)	98.8 (0.539)	98.8 (0.539)	99.5 (0.114)	99.2 (0.539)	1
CARE models						
Sym Abs Value CARE	98.4 (0.055)	98.9 (0.760)	98.9 (0.760)	99.7 (0.024)	99.7 (0.024)	2
Indirect GARCH CARE	98.5 (0.114)	98.7 (0.353)	99.1 (0.760)	99.4 (0.211)	99.6 (0.055)	0
Indirect ARGARCH CARE	98.3 (0.024)	98.3 (0.024)	99.0 (1.000)	99.3 (0.353)	99.5 (0.114)	2

Table 6 Evaluation of estimators of 95% VaR. DQ test p-values for 1000 post-sample estimates of 95% conditional quantile.

	CAC40	DAX30	FTSE100	NIKKEI225	S&P500	Number significant at 5% level
Benchmark methods						
Hist Sim 1000	0.000	0.000	0.000	0.116	0.003	4
Hist Sim 500	0.002	0.000	0.000	0.723	0.012	4
Hist Sim 250	0.687	0.062	0.000	0.178	0.261	1
GARCH Student-*t*	0.024	0.222	0.011	0.008	0.022	4
GARCH Student-*t* EVT	0.562	0.850	0.099	0.253	0.192	0
GARCH Skew-*t*	0.044	0.084	0.002	0.235	0.121	2
GARCH Skew-*t* EVT	0.020	0.161	0.002	0.081	0.065	2
CAViaR models						
Sym Abs Value CAViaR	0.494	0.839	0.442	0.961	0.501	0
Indirect GARCH CAViaR	0.946	0.796	0.478	0.962	0.308	0
Indirect ARGARCH CAViaR	0.845	0.970	0.672	0.432	0.456	0
CARE models						
Sym Abs Value CARE	0.521	0.525	0.178	0.829	0.684	0
Indirect GARCH CARE	0.831	0.746	0.036	0.891	0.417	1
Indirect ARGARCH CARE	0.758	0.586	0.257	0.849	0.539	0

Table 7 Evaluation of estimators of 99% VaR. DQ test p-values for 1000 post-sample estimates of 99% conditional quantile.

	CAC40	DAX30	FTSE100	NIKKEI225	S&P500	Number significant at 5% level
Benchmark methods						
Hist Sim 1000	0.000	0.000	0.000	0.321	0.001	4
Hist Sim 500	0.000	0.000	0.000	0.924	0.070	3
Hist Sim 250	0.051	0.000	0.000	0.690	0.304	2
GARCH Student-*t*	0.001	0.863	0.368	0.350	0.376	1
GARCH Student-*t* EVT	0.087	0.978	0.094	0.353	0.627	0
GARCH Skew-*t*	0.022	0.724	0.000	0.365	0.684	2
GARCH Skew-*t* EVT	0.006	0.857	0.021	0.351	0.637	2
CAViaR models						
Sym Abs Value CAViaR	0.018	0.981	0.226	0.861	0.951	1
Indirect GARCH CAViaR	0.043	0.678	0.181	0.847	0.999	1
Indirect ARGARCH CAViaR	0.021	0.979	0.284	0.863	0.992	1
CARE models						
Sym Abs Value CARE	0.003	0.743	0.019	0.553	0.554	2
Indirect GARCH CARE	0.011	0.692	0.066	0.945	0.727	1
Indirect ARGARCH CARE	0.003	0.057	0.168	0.984	0.856	1

Table 8 Evaluation of estimators of 95% ES. Bootstrap test p-values for zero mean standardised discrepancies based on 1000 post-sample estimates of conditional 95% ES.

	CAC40	DAX30	FTSE100	NIKKEI225	S&P500	Number significant at 5% level
Benchmark methods						
Hist Sim 1000	0.004	0.015	0.002	0.077	0.520	3
Hist Sim 500	0.056	0.153	0.008	0.081	0.916	1
Hist Sim 250	0.126	0.455	0.097	0.158	0.733	0
GARCH Student-*t*	0.997	0.076	0.075	0.014	0.006	2
GARCH Student-*t* EVT	0.467	0.498	0.765	0.102	0.042	1
GARCH Skew-*t*	0.994	0.470	0.012	0.067	0.048	2
GARCH Skew-*t* EVT	0.356	0.756	0.837	0.139	0.075	0
CARE models						
Sym Abs Value CARE	0.776	0.395	0.076	0.000	0.659	1
Indirect GARCH CARE	0.207	0.698	0.422	0.006	0.299	1
Indirect ARGARCH CARE	0.358	0.269	0.601	0.009	0.140	1

Table 9 **Evaluation of estimators of 99% ES. Bootstrap test p-values for zero mean standardised discrepancies based on 1000 post-sample estimates of conditional 99% ES.**

	CAC40	DAX30	FTSE100	NIKKEI225	S&P500	Number significant at 5% level
Benchmark methods						
Hist Sim 1000	0.392	0.572	0.398	0.488	0.983	0
Hist Sim 500	0.047	0.247	0.271	0.227	0.787	1
Hist Sim 250	0.112	0.623	0.046	0.252	0.762	1
GARCH Student-*t*	0.019	0.306	0.511	0.507	0.511	1
GARCH Student-*t* EVT	0.022	0.454	0.413	0.511	0.288	1
GARCH Skew-*t*	0.029	0.065	0.114	0.511	0.152	1
GARCH Skew-*t* EVT	0.048	0.347	0.559	0.507	0.288	1
CARE models						
Sym Abs Value CARE	0.383	0.536	0.093	0.751	0.609	0
Indirect GARCH CARE	0.070	0.911	0.032	0.209	0.703	1
Indirect ARGARCH CARE	0.292	0.128	0.063	0.169	0.764	0

Table 10 Summary of VaR and ES results. Number of test rejections at 5% significance level for each of the four θ quantiles. Note that CAViaR models produce only VaR estimates.

	VaR Hit % Test					VaR DQ Test					ES Bootstrap Test				
	θ (×100)					θ (×100)					θ (×100)				
	1	5	95	99	Total	1	5	95	99	Total	1	5	95	99	Total
Benchmark methods															
Hist Sim 1000	1	0	2	4	7	4	5	4	4	17	0	0	3	0	3
Hist Sim 500	0	1	1	1	3	3	5	4	3	15	0	0	1	1	2
Hist Sim 250	0	0	1	0	1	4	4	1	2	11	0	0	0	1	1
GARCH Student-*t*	1	2	5	3	11	1	0	4	1	6	0	0	2	1	3
GARCH Student-*t* EVT	0	0	0	1	1	0	0	0	0	0	0	0	1	1	2
GARCH Skew-*t*	1	0	0	1	2	0	0	2	2	4	0	0	2	1	3
GARCH Skew-*t* EVT	1	0	0	1	2	0	0	2	2	4	0	0	0	1	1
CAViaR models															
Sym Abs Value CAViaR	0	0	0	1	1	0	1	0	1	2	-	-	-	-	-
Indirect GARCH CAViaR	0	0	0	0	0	0	0	0	1	1	-	-	-	-	-
Indirect ARGARCH CAViaR	0	1	0	1	2	1	0	0	1	2	-	-	-	-	-
CARE models															
Sym Abs Value CARE	0	1	0	2	3	1	2	0	2	5	0	0	1	0	1
Indirect GARCH CARE	0	1	0	0	1	0	2	1	1	4	0	0	1	1	2
Indirect ARGARCH CARE	1	0	0	2	3	1	1	0	1	3	0	1	1	0	2

Figure 1 Unconditional quantiles, $Q(\theta)$, and expectiles, $\mu(\tau)$, plotted against θ and τ, respectively, for the first moving window of 1000 daily FTSE100 daily stock index returns.

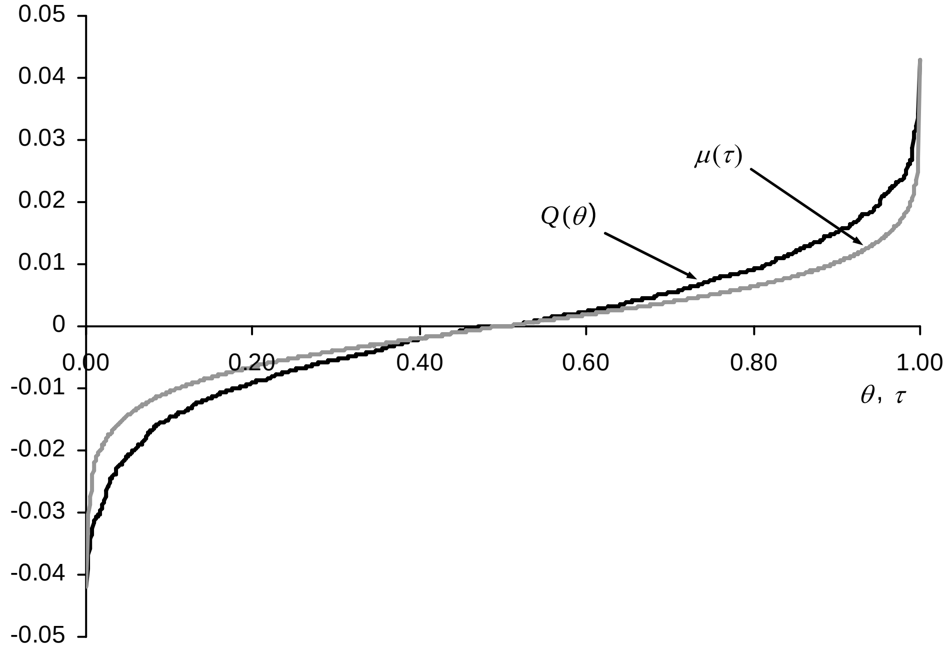

Figure 2 **FTSE100 daily stock index returns for the 1000 post-sample days with VaR and ES estimates from the Symmetric Absolute Value CARE model and the inferred conditional ES model, respectively.**