Estimating Value-at-Risk and Expected Shortfall Using the Intraday Low and Range Data

Xiaochun Meng
University of Sussex Business School, University of Sussex
and
James W. Taylor
Saïd Business School, University Of Oxford

European Journal of Operational Research, 2020, 280(1), 191-202.

Address for Correspondence:

Xiaochun Meng
Jubilee Building
University of Sussex Business School
University of Sussex
BN1 9SL, UK
Email: xiaochun.meng@sussex.ac.uk

Abstract

Value-at-Risk (VaR) is a popular measure of market risk. To convey information regarding potential exceedances beyond the VaR, Expected Shortfall (ES) has become the risk measure for trading book bank regulation. However, the estimation of VaR and ES is challenging, as it requires the estimation of the tail behaviour of daily returns. In this paper, we take advantage of recent research that develops joint scoring functions for VaR and ES. Using these functions, we present a novel approach to estimating the two risk measures based on intraday data. We focus on the intraday range, which is the difference between the highest and lowest intraday log prices. In contrast to intraday observations, the intraday low and high are widely available for many financial assets. To alleviate the challenge of modelling extreme risk measures, we propose the use of the intraday low series. We draw on a theoretical result for Brownian motion to show that a quantile of the daily returns can be estimated as the product of a constant term and a less extreme quantile of the intraday low returns, which we define as the difference between the lowest log price of the day and the log closing price of the previous day. In view of this, we use estimates of the VaR and ES of the intraday low returns to estimate the VaR and ES of the daily returns. We provide empirical support for the new proposals using data for five stock indices and five individual stocks.

KEY WORDS: Finance; Value-at-Risk; Expected Shortfall; Intraday Low; Joint Scoring Functions.

1 Introduction

Value-at-risk (VaR) has become a standard tool for risk management in financial and insurance institutions. It involves measuring the amount a certain portfolio can lose for a given probability level. Formally, VaR is defined as a tail quantile of the distribution of a financial return. The accurate forecasting of VaR is fundamental for internal risk control and financial regulation. Despite its importance and popularity, VaR has some undesirable features. It does not take into account the magnitude of the potential loss beyond the VaR, and it is not a sub-additive risk measure (Gordy and Juneja 2010; Du and Escanciano 2016), which means the measure for a portfolio can be less than the sum of the measures of the components of the portfolio. In view of these concerns, Expected Shortfall (ES) has been proposed as a risk measure for future regulatory frameworks (Embrechts et al. 2014; Du and Escanciano 2016). ES is defined as the expected value of exceedances beyond the VaR. In addition to their use as risk measures, the estimation of VaR and ES can be used as the basis for portfolio optimisation (Huang et al. 2010; Lwin et al. 2017). However, ES is not elicitable, which means there does not exist a scoring function for which the expectation is minimized by the true ES (Gneiting 2011). The ES is not elicitable because the level set for ES is not convex (Acerbi and Szekely 2014). This has posed difficulties for both the estimation and evaluation of ES in practice (Rockafellar et al. 2014). In recent work, Fissler and Ziegel (2016) find that ES and VaR are jointly elicitable, and provide a family of scoring functions that are minimized by the true VaR and ES. Such scoring functions enable the joint estimation and evaluation of VaR and ES. For example, Patton et al. (2017) and Taylor (2018) estimate dynamic models for VaR and ES based on joint scoring functions of this type.

Classical approaches to forecasting daily VaR and ES use only historical daily returns, such as generalised autoregressive conditional heteroskedasticity (GARCH) models, historical simulation, and conditional autoregressive value-at-risk (CAViaR) models (Engle and Manganelli 2004). The studies of Patton et al. (2017) and Taylor (2018) are also solely based on historical daily returns. With intraday data becoming increasingly available, efforts have been made to use it in the forecasting of VaR and ES for daily returns (see, for example, Clements et al. 2008). However, the use of intraday data in VaR and ES estimation has tended to involve high-frequency data, which is generally expensive, with only limited availability (Kumar and Maheswaran 2015). In this paper, we consider the use of the intraday low and high series to improve VaR and ES forecast accuracy. These data are readily available for most tradable assets for the past 30 years.

Our aim in this paper is to improve CAViaR modelling in order to generate more accurate VaR and ES forecasts. CAViaR models are appealing because they estimate the quantiles directly, and have been shown to perform competitively in comparison with other VaR models (Manganelli and Engle 2004; Chen et al. 2012). Using a joint scoring function, Taylor (2018) estimates the VaR and ES using a CAViaR model for the VaR, and ES modelled as a constant multiple of the VaR, or assumed to be equal to the VaR plus a dynamic model for the exceedances.

In this paper, we make two contributions. Firstly, to enrich the models proposed by Taylor

(2018), we incorporate the intraday range, which is defined as the absolute difference between the lowest and highest log prices of a day. By contrast, Taylor (2018) considered just dynamic models based on daily returns data. Secondly, we draw on a theoretical result for Brownian motion to show that a quantile of the daily returns can be estimated as the product of a constant term and a less extreme quantile of the *intraday low returns*, which we define as the difference between the lowest log price of the day and the log closing price of the previous day. With observations being scarce in the tails of the distribution, the estimation of a less extreme quantile is intuitively appealing. In view of this, we propose a novel method, in which the VaR and ES of the intraday low returns are used as the basis for estimating the VaR and ES of the daily returns.

In Section 2, we review established VaR and ES methods, and introduce a new model, which is a simple extension of existing approaches. Section 3 describes how VaR and ES estimated for the intraday low returns can be used to estimate VaR and ES for the daily returns. Section 4 uses data for five stock indices and five individual stocks to evaluate the performance of the proposed new methods, and to compare their estimation accuracy with the established benchmark methods. Section 5 provides a summary and some concluding remarks.

2 VaR and ES Methods

In this section, we review time series methods for VaR and ES, and the recently proposed joint scoring functions of Fissler and Ziegel (2016), which enable joint modelling. We then describe VaR models that incorporate intraday information. We end the section by proposing a new model that uses the intraday range, and can be estimated using a joint scoring function to enable both VaR and ES prediction.

2.1 Time Series Methods for VaR and ES

The methods available for VaR and ES forecasting can be classified as parametric, nonparametric or semiparametric. Nonparametric methods involve no parameterisation for the conditional volatility and no assumption for the return's distribution. Examples are historical simulation (see, for example, Hull 2012) and kernel density estimation (see, for example, Butler and Schachter 1997), which are applied to moving windows of observations. Although these methods are easy to implement, they implicitly assume that the returns distribution remains reasonably constant within the specified window length, which may be inappropriate (Manganelli and Engle 2004). Moreover, the choice of the window length is not clear.

Parametric models involve a parameterisation of the conditional volatility and an assumption for the distribution of the return. The VaR and ES estimates can then be obtained from the estimated distribution (see, for example, Gerlach and Wang 2016). A common example is a GARCH model with a Gaussian or Student-t distribution (Bollerslev 1987). Although parametric models provide an estimate of the complete conditional distribution of the return series, they usually suffer from some degree of misspecification, either from the model structure or from the distributional assumption.

Semiparametric methods involve either the use of extreme value theory (EVT), filtered historical simulation (FHS), or a direct modelling of VaR and ES in a regression framework. A popular EVT approach fits a generalised Pareto distribution to exceedances over some threshold of the standardised residuals from a parametric model (McNeil and Frey 2000). Filtered historical simulation avoids a distributional assumption by using the empirical distribution of standardised residuals (see, for example, Escanciano and Pei 2012). In a semiparametric regression approach, a model is estimated by minimizing a score function summed over the in-sample period. For example, the CAViaR models of Engle and Manganelli (2004) are estimated using quantile regression, which amounts to minimizing the quantile score. By directly modelling individual quantiles, CAViaR models have the advantage of making no distributional assumption, and allowing different quantiles to have distinct dynamics. An apparent limitation of these models is that they do not deliver ES predictions. However, this concern has been alleviated by the recent development of joint scoring functions for VaR and ES.

2.2 Joint Scoring Functions for VaR and ES

A fundamental problem for ES estimation has been that it is not elicitable, meaning that scoring functions do not exist for its estimation and evaluation. This has recently been addressed by Fissler and Ziegel (2016), who show that VaR and ES are jointly elicitable, and present the following set of consistent joint scoring functions for these two risk measures:

$$\begin{aligned} h_{FZ}(y_t, q_t, e_t; \theta, G_1, G_2, a) = (\mathbb{1}\{y_t \le q_t\} - \theta)\left(G_1(q_t) - G_1(y_t) + \frac{1}{\theta}G_2(e_t)q_t\right) \\ - G_2(e_t)\left(\frac{1}{\theta}\mathbb{1}\{y_t \le q_t\}y_t - e_t\right) - \mathscr{G}_2(e_t) + a(y_t) \end{aligned} \tag{1}$$

where y_t is the daily return; θ is the probability level; q_t and e_t are VaR and ES at the same probability level θ; G_1, G_2, $\mathscr{G}_2$ and a are real valued functions; G_1 is weakly increasing; G_2 is strictly increasing and positive; and $\mathscr{G}_2^{'} = G_2$. We refer to scores of this type as FZ scores.

The FZ scores enable the direct estimation of VaR and ES jointly. Taylor (2018) studies the joint scoring function obtained by selecting $G_1(x) = 0$, $G_2(x) = -\frac{1}{x}$, $\mathscr{G}_2(x) = -\log(-x)$ and $a(x) = 1 - \ln(1-\theta)$ in expression (1) to give:

$$h_{FZ}(y_t, q_t, e_t; \theta) = -\ln\left(\frac{\theta - 1}{e_t}\right) - \frac{(y_t - q_t)(\theta - \mathbb{1}\{y_t \le q_t\})}{\theta e_t} + \frac{y_t}{e_t} \tag{2}$$

Taylor (2018) notes that minimizing this score is equivalent to maximising a likelihood function based on the asymmetric Laplace distribution, which links the optimisation to quantile regression. We refer to this as the *AL score*. In independent work, Patton et al. (2017) use a score that differs only by a constant term from expression (2), and so their analysis applies equally to the AL score.

Taylor (2018) finds that estimation based on minimizing the AL score is particularly competitive in terms of forecasting accuracy when used for models composed of a CAViaR expression for the VaR, and ES expressed as a constant multiple of the VaR. We refer to such models, estimated by minimizing an FZ score, as CAViaR-FZ models, and present two in the following expressions:

CAViaR Symmetric Absolute Value estimated using an FZ score (CAViaR-FZ-SAV):

$$\begin{aligned} q_t(\boldsymbol{\beta}) &= \beta_1 + \beta_2 q_{t-1}(\boldsymbol{\beta}) + \beta_3 |y_{t-1}| \\ e_t(\boldsymbol{\beta}) &= \beta_4 q_t(\boldsymbol{\beta}) \end{aligned} \tag{3}$$

CAViaR Asymmetric Slope estimated using an FZ score (CAViaR-FZ-AS):

$$\begin{aligned} q_t(\boldsymbol{\beta}) &= \beta_1 + \beta_2 q_{t-1}(\boldsymbol{\beta}) + \beta_3 (y_{t-1})^+ + \beta_4 (y_{t-1})^- \\ e_t(\boldsymbol{\beta}) &= \beta_5 q_t(\boldsymbol{\beta}) \end{aligned} \tag{4}$$

where, in each model, $\boldsymbol{\beta}$ is a vector of parameters. The CAViaR-FZ-AS model accommodates the asymmetric leverage effect that is typically found in stock index returns. This effect refers to the empirical finding that volatility tends to be greater following a negative value of the return than a positive value of equal size (see, for example, Wang et al. 2015). It was first discussed by Black and Cox (1976), and has been widely applied in volatility, VaR and ES modelling (see, for example, Glosten et al. 1993; Chen et al. 2012).

2.3 Incorporating Intraday Information in VaR and ES Modelling

Intraday data provides information regarding the distribution of the price process within a day, and this can be useful in modelling the daily volatility (see, for example, Corsi 2009; Hansen et al. 2012; Sévi 2014). As VaR and ES are closely related to the daily volatility, intraday data should contain useful information for modelling VaR and ES. However, intraday data, recorded at a relatively high frequency, such as minute-by-minute, is generally expensive and not available for a long period. In contrast, the daily opening, closing, intraday high and intraday low prices are readily available for most tradable assets for the past 30 years. Although an abundance of literature can be found on volatility estimation based on the intraday range (see, for example, Alizadeh et al. 2002; Brandt and Jones 2006; Corsi 2009), very few studies have used the intraday range in the daily VaR and ES context. Chen et al. (2012) find that including range information can be beneficial for some daily CAViaR models, but they do not consider ES estimation. Gerlach and Wang (2016) show that the intraday range is competitive in terms of VaR and ES forecasting, when compared to other volatility measures based on high-frequency data for the realized GARCH models, which are proposed by Hansen et al. (2012) for volatility modelling.

2.4 A New Model Using the Intraday Range and an FZ Score

The CAViaR models of Chen et al. (2012) provide a synthesis of autoregressive quantile modelling and the intraday range. However, as they are simply quantile models, they do not provide ES predictions. To rectify this, we propose the model of expression (5), which incorporates the intraday range, and is estimated by minimizing an FZ score. Quite apart from the appeal of capturing intraday information, the intraday range has the advantage of being a more efficient and less noisy estimator of the daily volatility than the magnitude of the return, which is used in the more standard CAViaR formulations of expressions (3) and (4). Intuitively, when the market is very volatile, the daily return can still be small, while the intraday range will certainly be large. Parkinson (1980) establish theoretically that, as a daily volatility estimator, the intraday range is five times more efficient than the daily return.

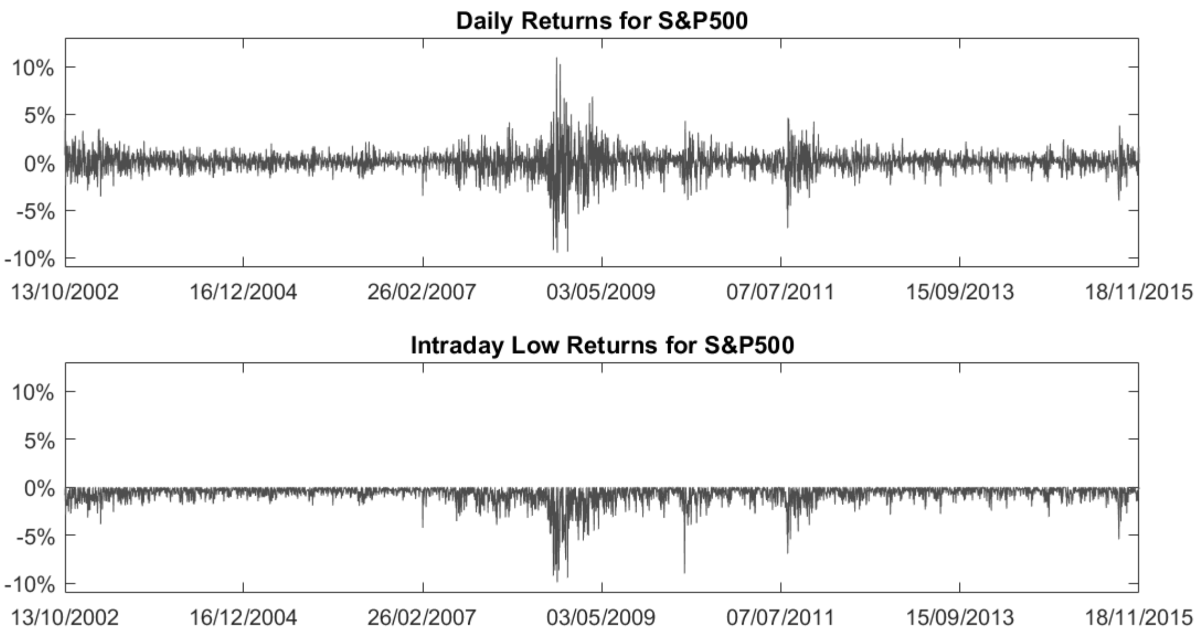
CAViaR with intraday range estimated using an FZ score (CAViaR-FZ-Range):

$$\begin{aligned} q_t(\boldsymbol{\beta}) &= \beta_1 + \beta_2 q_{t-1}(\boldsymbol{\beta}) + \beta_3 Range_{t-1} \\ e_t(\boldsymbol{\beta}) &= \beta_4 q_t(\boldsymbol{\beta}) \end{aligned} \tag{5}$$

3 Model Estimation Based on the Intraday Low

The accuracy of VaR and ES forecasting is influenced by the number of observations in the tail of the returns distribution. When the data is scarce in the tail, estimation can be challenging. The essence of the new approach that we propose is that extreme VaR and ES of the daily returns can be approximated by less extreme VaR and ES of the intraday low returns, which, as we stated in Section 1, we define as the difference between the lowest log price of the day and the log closing price of the previous day. (We define the daily return, in the usual way, as the difference between the log closing price on successive days.) Figure 1 plots a sample of daily returns and intraday low returns for the S&P 500 stock index. We note the similarity between the dynamics of the daily returns and the intraday low returns, with periods of relatively high and low volatility essentially coinciding for the two series.

Figure 1: Daily returns and intraday low returns for the S&P 500.



In Section 3.1, we describe how, for standard Brownian motion, there is an equivalence of a quantile of the daily return to a less extreme quantile of the intraday low return. Building on this, we propose that the intraday low return can be used as the basis for estimating VaR and ES for the daily return. We theoretically establish the estimation efficiency of this proposal. In Section 3.2, we describe approaches for implementing our proposal in practice.

3.1 Brownian Motion and the Intraday Low

Brownian motion has been widely used to model the log price in the context of stochastic volatility and option price modelling (see, for example, Black and Scholes 1973; Merton 1974; Back 1993; Kou 2002; Andersen and Piterbarg 2007). It is well-established that the distribution of the infimum L_t of standard Brownian motion y_s, starting at 0 over a closed interval $[0, t]$, and the distribution of y_t are related via the following expression (see, for example, Mörters and Peres 2010):

$$P(L_t < x) = 2P(y_t < x) \qquad if\ x < 0 \tag{6}$$

For Brownian motion with nonzero starting point, we can simply subtract the starting value from the Brownian motion. Suppose the intraday log price follows a Brownian motion starting from the log closing price of the previous day. In expression (6), y_t can be considered the daily return, and L_t can be considered the intraday low return. Expression (6) implies that the VaR and ES of the daily return, corresponding to probability level θ, are equal to the VaR and ES of the intraday low return, corresponding to probability level 2θ.

In the VaR and ES context, this is a potentially interesting result, because it implies that, instead of estimating the VaR and ES of the daily return, we can estimate a less extreme VaR and ES of the intraday low return, which is appealing in terms of estimation accuracy. For example, if we consider the θ VaR of the daily return (which is also the 2θ VaR of the intraday low return), then on average there will be θ observations of the daily return beyond it, while there will be 2θ observations of the intraday low return beyond it. In other words, there will be twice as many observations beyond the VaR if we use the intraday low return, than if we use the daily return. This point has not been recognised previously in the literature on VaR and ES estimation. Moreover, suppose we use some semiparametric model based on an FZ score to estimate the VaR and ES of the daily return for probability level θ, and use the same approach to estimate the VaR and ES of the intraday low return for probability level 2θ, it is more efficient to estimate the model parameters for the intraday low return, as will be shown in Theorem 1 below. In fact, we can loosen the assumption (in expression (6)) that the ratio $\frac{P(L_t<x)}{P(y_t<x)}$ is equal to 2. As we show in Theorem 1, for any stochastic process y_t, which is not necessarily Brownian motion, provided the ratio $\frac{P(L_t<x)}{P(y_t<x)}$ is equal to a constant (greater than 1) within the estimation window, improved efficiency will be achieved by estimating the quantile for the intraday low return. In the rest of the paper, we use the notation $\tilde{}$ to distinguish all the quantities related to the intraday low return from those of the daily return. For example, we use $\tilde{q}_t$ and $\tilde{e}_t$ to denote the VaR and ES of the intraday low return.

Assumption 1. *Let y_t and L_t be two time series. Assume the following:*

(A). There exist a scalar x_0 and a constant $\lambda > 1$, such that

$$\Pr(L_t < x) = \lambda \Pr(y_t < x) \qquad \textit{for all } x < x_0. \tag{7}$$

(B). y_t, L_t and their corresponding VaR and ES satisfy the regularity conditions in Assumptions 1 and 2 in Patton et al. (2017).

Assumption 1(A) immediately implies that the θ VaR and ES of the daily returns are exactly the same as the $\lambda\theta$ VaR and ES of the intraday low returns for θ sufficiently small. If we consider a semiparametric specification $\{q_t(\boldsymbol{\beta}), e_t(\boldsymbol{\beta})\}$ for the θ VaR and ES of the daily returns, then we can estimate the parameters by minimizing the AL score for y_t. Alternatively, we can estimate the parameters by minimizing the AL score for L_t by viewing $\{q_t(\boldsymbol{\beta}), e_t(\boldsymbol{\beta})\}$ as the $\lambda\theta$ VaR and ES of L_t. The next theorem compares the estimation efficiency of these two approaches.

Theorem 1. *Under Assumption 1, using a semiparametric specification $\{q_t(\boldsymbol{\beta}), e_t(\boldsymbol{\beta})\}$, estimating the VaR and ES of the intraday low return L_t for probability level $\lambda\theta$ is more efficient than estimating the VaR and ES of the daily return y_t for probability level θ.*

A proof of this theorem is presented in the appendix. Theorem 1 provides support for using the intraday low return to estimate VaR and ES of the daily return. However, we should emphasize that the theorem requires Assumption 1. There are two important practical aspects with Assumption 1 and Theorem 1. The first is the practical issue of obtaining the value of λ, or, more specifically, the value of the probability level $\lambda\theta$ to use with the intraday low returns, as the data generating process of the intraday log return is unknown. Secondly, the intraday log return does not necessarily satisfy Assumption 1(A), which could potentially violate the results of Theorem 1. We address these two issues in Section 3.2.

3.2 Estimating a Model for VaR and ES using the Intraday Low Returns

Our aim is to use the VaR and ES of the intraday low returns to improve the forecasting accuracy of the VaR and ES of the daily returns. We propose two alternative approaches. In the first approach, we estimate the value $\tilde{\theta} = \lambda\theta$, discussed in Section 3.1, using in-sample observations, and then model the $\tilde{\theta}$ VaR and ES of the intraday low returns as the estimated VaR and ES of the daily returns. In the second approach, acknowledging that Theorem 1 may not hold in practice, we adopt a practical approach, where we model the θ VaR and ES estimates of the returns by rescaling the $\tilde{\theta}$ VaR and ES estimates of the intraday low returns, which we obtain in the first approach. The two approaches are summerised as follows:

1. As forecasts of the VaR and ES of the daily returns with probability level θ, use forecasts of the VaR and ES of the intraday low returns with the following empirical probability level:

$$\tilde{\theta} = \frac{1}{T}\sum_{t=1}^{T} I(L_t < Q_\theta) \tag{8}$$

where Q_θ is the unconditional θ quantile of the daily returns, and T is the size of the estimation sample. To enable this, model the VaR and ES of the intraday low returns using one of the CAViaR-FZ models presented in expressions (3)-(5).

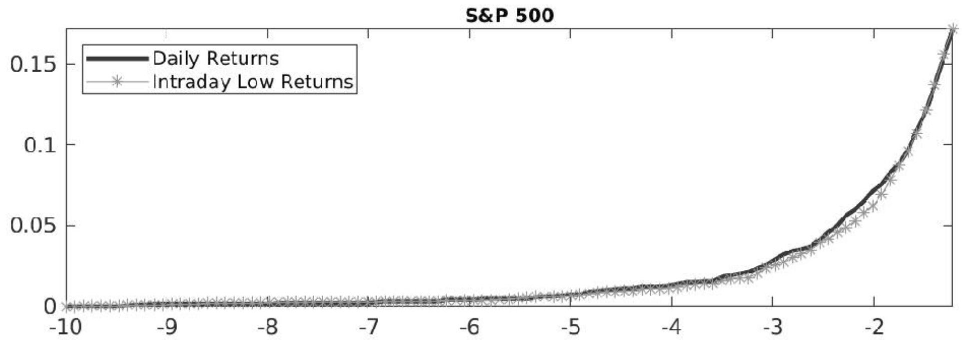
2. As stock indices are not perfectly described by Brownian motion, there may be benefit in rescaling the VaR and ES forecasts produced for the intraday low returns for probability level $\tilde{\theta}$. To do this, we propose the following model:

$$\begin{aligned} q_t(\boldsymbol{\gamma}) &= \gamma_1 + \gamma_2 \tilde{q}_t(\boldsymbol{\beta}) \\ e_t(\boldsymbol{\gamma}) &= \gamma_3 q_t(\boldsymbol{\gamma}) \end{aligned} \tag{9}$$

where $\tilde{q}_t(\boldsymbol{\beta})$ is the VaR of the intraday low returns for probability level $\tilde{\theta}$, estimated using one of the CAViaR-FZ models presented in expressions (3)-(5); and $q_t(\boldsymbol{\gamma})$ and $e_t(\boldsymbol{\gamma})$ are the VaR and ES of the daily returns for probability level θ. The parameter vector $\boldsymbol{\gamma}$ is estimated by minimizing an FZ score for the daily returns with probability level θ.

The performance of the proposed approach largely depends on the validity of Assumption 1(A). As the conditional distributions of y_t and L_t are unobservable, we use the empirical distributions of y_t and L_t to provide some empirical support for Assumption 1(A). For the S&P 500, Figure 2 plots $\lambda \Pr(y_t < x)$, which is the scaled empirical distribution of the daily return, and $\Pr(L_t < x)$, which is the empirical distribution of the intraday low. In this figure, we consider θ=0.01, so that Q_θ is the unconditional 0.01 quantile of the daily returns, and $x < Q_\theta$; and we compute $\lambda = \frac{\tilde{\theta}}{\theta}$, where $\tilde{\theta}$ is computed as in expression (8). The resulting value of λ was 1.72, which is a little lower than the value of 2, which would result for Brownian motion. The figure shows that $\lambda \Pr(y_t < x)$ and $\Pr(L_t < x)$ are indeed very close to each other, which gives some support for Assumption 1(A).

Figure 2: Plot of empirical $\lambda \Pr(y_t < x)$ and $\Pr(L_t < x)$ for the 3300 observations of S&P 500 between 10 October 2002 and 18 November 2015.



In the first approach, where we use the CAVaR-FZ models for L_t, we assume the ES is a multiple of the VaR. In practice, the data will not necessarily satisfy Assumption 1(A), which could potentially misspecify the ratio between the ES and VaR. The second approach aims to overcome this potential misspecification by rescaling the VaR and ES estimates for L_t.

4 Empirical Study of VaR and ES Forecasting

4.1 Stock Index Data

To evaluate VaR and ES forecast accuracy, we used the daily opening, daily closing, intraday low and intraday high price series of the following five stock indices: CAC 40, DAX 30, FTSE 100, Nikkei 225 and S&P 500. For each index, our sample period consisted of the series of 3300 observations ending on 18 November 2015. Due to different holiday periods in each country, the starting periods differed for the five indices. The starting periods were 27 December 2002, 22 November 2002, 28 October 2002, 6 June 2002, and 10 October 2002 for the CAC 40, DAX 30, FTSE 100, Nikkei 225 and S&P 500, respectively.

We subtracted the log closing price of the previous day from the log closing price, and from the lowest log price of the current day, to obtain the daily return y_t and the intraday low return

L_t, respectively. The intraday range $Range_t$ is the difference between the highest and lowest log price. For each index, descriptive statistics for the daily returns y_t are shown in Table 1.

We considered the 0.5%, 2.5%, 1% and 5% probability levels for the VaR and ES of the daily returns. 1% and 5% are two commonly studied probability levels for VaR estimation; 2.5% is the required probability level for ES estimation in the Basel III Accord (Basel Committee on Banking Supervision 2012); and 0.5% is included to evaluate more extreme tails. We used a rolling window of 1800 observations for repeated re-estimation of the parameters of each method. This window length is of a similar order of magnitude to sample sizes used for estimation in other studies in the literature on VaR and ES estimation (see, for example, Chen et al. 2012; Gerlach and Wang 2016). This produced post-sample forecasts from each method for the final 1500 days in each series.

Table 1: Descriptive statistics for the full sample of 3300 daily returns of the five stock indices.

Index	Mean	Std. Dev.	Min	Max	Skewness	Kurtosis
Stock Indices						
CAC	0.01	1.42	-9.47	10.59	0.03	9.15
DAX	0.04	1.43	-7.43	10.80	0.00	8.38
FTSE	0.01	1.18	-9.27	9.38	-0.12	10.95
NIKKEI	0.02	1.53	-12.11	13.23	-0.50	10.17
S&P	0.03	1.22	-9.47	10.96	-0.29	13.71

4.2 VaR and ES Forecasting Methods

In this section, we describe the methods that we included in the empirical study. We implemented historical simulation with window lengths ranging from 25 to 300 days, but the methods performed poorly, and so we omit the results from this paper. We also implemented the standard GARCH(1,1) and GJR-GARCH(1,1) models, as well as the log-linear realized GARCH model based on the intraday range, as proposed by Gerlach and Wang (2016). For each GARCH model, we estimated the parameters using a Student-t distribution. To estimate VaR and ES, we used the following three different approaches: the Student-t distribution; filtered historical simulation applied to the 1800 in-sample observations; and the EVT approach of McNeil and Frey (2000), with threshold chosen as the 10% quantile (see, for example, Gerlach and Wang 2016). These three approaches essentially use the same volatility estimates, but assume different distributions for the standardised daily returns. The first approach simply uses the estimated Student-t distribution; the second approach uses the empirical distribution of the in-sample standardised daily returns; and the third approach fits a generalised Pareto distribution to the lowest 10% of the in-sample standardised daily returns. The VaR and ES of each of these distributions is then multiplied by the volatility forecast to deliver VaR and ES estimates for the daily returns.

We implemented Taylor's (2018) CAViaR-FZ-SAV and CAViaR-FZ-AS models of expressions (3) and (4), and our CAViaR-FZ-Range model of expression (5). These models are estimated using daily returns. All of the CAViaR-FZ models in this paper were estimated using the AL score in expression (2), as the asymptotic theories have been established in Patton et al. (2017). We use a numerical procedure similar to that described by Engle and Manganelli (2004) for CAViaR models. For each model, 10^d initial trial vectors of parameters were randomly generated, where d is the number of parameters. The first $d-1$ entries of each vector were sampled from a uniform distribution between 0 and 1; these entries correspond to the parameters of the VaR models. The final entry in each vector corresponds to the ES; it was sampled from a uniform distribution between 1 and 10. For each initial trial vector, we calculated the particular AL score. The six vectors producing the lowest values of the AL score were passed to the interior point algorithm as starting vectors. The resulting vector that produced the lowest score was selected as the optimal parameter vector.

We implemented the two new approaches of Section 3.2. Each involves fitting one of the CAViaR-FZ models of expressions (3)-(5) to the intraday low returns. In the first approach, we estimate the probability level $\tilde{\theta}$ to use with the intraday low returns. For each of the 1500 post-sample periods of the S&P 500 returns, Figure 3 plots the value of $\tilde{\theta}$ that we estimated using expression (8) for each moving window of 1800 days. In the figure, the $\tilde{\theta}$ values deviate quite clearly from 2θ, which would be appropriate for Brownian motion. Therefore, to use a probability level of 2θ with the S&P 500 intraday low returns would potentially deliver inaccurate VaR and ES forecasts. Table 2 presents the estimated parameters for four CAViaR-FZ-Range models derived using the first estimation window of 1800 days of the S&P 500 series of intraday low returns. The three models correspond to probability levels of $\tilde{\theta}$, where $\tilde{\theta} = 0.9\%$, 1.5%, 3.7% and 7.4%. In the second approach of Section 3.2, we rescale the VaR and ES forecasts produced for the intraday low returns for probability level $\tilde{\theta}$ in the first step by using expression (9). Table 3 presents the estimated parameters of expression (9), used to rescale each of the CAViaR-FZ-Range models in Table 2. The standard errors are obtained through the asymptotic theory of Patton et al. (2017).

Figure 3: For the 1500 post-sample periods of the S&P 500 intraday low returns, empirical probability levels $\tilde{\theta}$ estimated using expression (8). Dot-dashed horizontal lines correspond to $\tilde{\theta}=2\theta$, which is appropriate for Brownian motion.

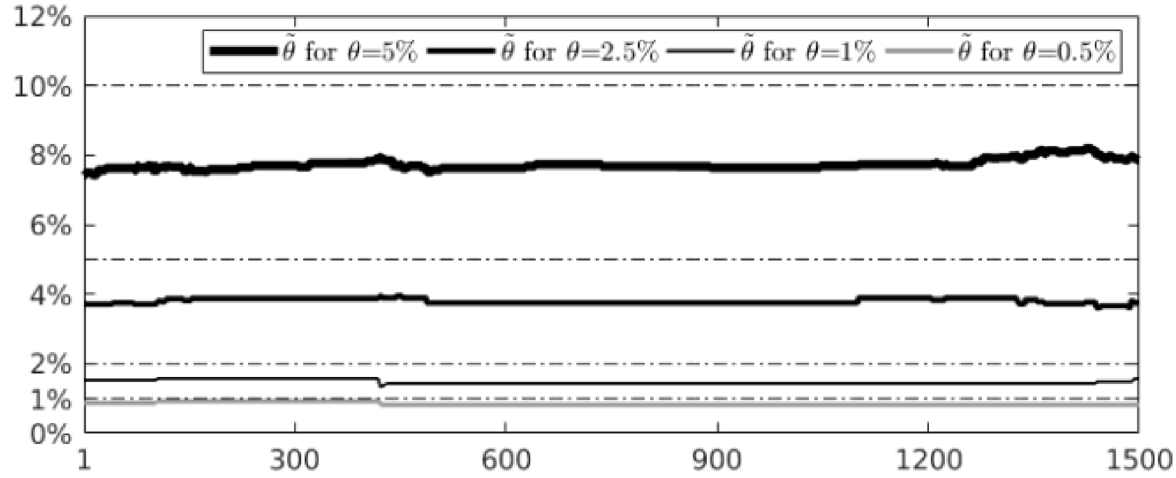


Table 2: Parameter estimates and their standard errors for the CAViaR-FZ-Range model with probability level $\tilde{\theta}$, fitted to the intraday low returns L_t, derived using the first moving window of 1800 days of the S&P 500.

$\tilde{\theta}\times 100$	0.9	1.5	3.7	7.4
β_1	0.104	0.050	0.015	0.014
	(0.251)	(0.138)	(0.097)	(0.087)
β_2	0.779	0.843	0.849	0.848
	(0.258)	(0.199)	(0.164)	(0.217)
β_3	0.394	0.266	0.229	0.191
	(0.403)	(0.308)	(0.199)	(0.247)
β_4	1.214	1.215	1.261	1.301
	(0.178)	(0.231)	(0.185)	(0.272)

Table 3: Parameter estimates and their standard errors obtained for rescaling the models of Table 2. Parameters estimated using probability level θ and the daily returns y_t for the first moving window of 1800 days of the S&P 500.

$\theta\times 100$	0.5	1	2.5	5
γ_1	0.089	0.096	-0.076	-0.002
	(1.485)	(1.268)	(0.773)	(0.497)
γ_2	0.969	0.976	1.058	0.995
	(0.564)	(0.508)	(0.379)	(0.280)
γ_3	1.235	1.193	1.254	1.320
	(0.084)	(0.064)	(0.053)	(0.033)

4.3 VaR and ES Evaluation Methods

VaR evaluation methods can be briefly classified into two categories, coverage tests and scoring functions (Taylor 2018). A review of the coverage tests can be found in Christoffersen (2010). We implemented most of the tests reviewed in Christoffersen (2010), including the unconditional coverage test and dynamic quantile (DQ) test for VaR evaluation, and the bootstrap test for ES evaluation.

Scoring functions provide a different approach to VaR and ES evaluation. Scoring functions enable honest assessment of VaR and ES estimation, as they are always minimized by the perfect VaR and ES estimates (Gneiting and Raftery 2007). We considered the quantile score for VaR evaluation, which is now standard in the VaR literature (see, for example, Gneiting and Raftery 2007; Ehm et al. 2016). To evaluate jointly the VaR and ES performance, we used the AL score, as well as two other FZ scores, because different FZ scores can lead to different ranking of methods (Nolde and Ziegel 2017). It could be suggested that the AL score may give unfair advantage to the CAViaR-FZ models that are estimated based on the AL score, and so it is important that we use additional FZ scores in our empirical study.

4.3.1 Coverage Tests

For VaR, coverage tests are essentially based on the fact that if the model is correctly specified, VaR exceedances should occur with probablility θ, and should occur randomly. Mathematically, we can define a variable

$$Hit_t = \begin{cases} 1-\theta & if \quad y_t < q_t \\ \theta & if \quad y_t \geq q_t \end{cases}$$

Unconditional coverage requires that the mean of Hit_t in the postsample period is zero, and conditional coverage has the additional requirement that the sequence of VaR exceedances is i.i.d.. For unconditional coverage, we used a standard binomial test to test whether the observed mean of Hit_t in the postsample period is significantly deviated from 0. For conditional coverage, we used the dynamic quantile (DQ) test of Engle and Manganelli (2004) with four lags in the test's regression.

To evaluate the unconditional coverage of the ES forecasts, we used the bootstrap test of McNeil and Frey (2000), which focuses on the discrepancies between the VaR exceedances and the ES forecasts. The test examines whether the standardised discrepancies have zero unconditional expectation. As volatility estimates are not available for semiparametric methods, we adopt the approach of Acerbi and Szekely (2014), which standardises the discrepancies by dividing each by the corresponding ES estimate. In view of the relatively small number of VaR exceedances, conditional coverage is typically not considered for ES forecasts.

4.3.2 Scoring Functions

In terms of scoring functions, we evaluated VaR forecasts using the quantile score, which is standard for VaR estimation (Gneiting and Raftery 2007; Ehm et al. 2016; Taylor 2018). The quantile score is expressed as:

$$QS_t = (\theta - \mathbb{1}(y_t < q_t))(y_t - q_t)$$

This is a consistent scoring function for a quantile, which means that it is minimized by the true quantile. Models with lower average quantile scores in the postsample period are considered better.

As discussed previously in this paper, ES is not elicitable by itself, but it is jointly elicitable with the VaR. We considered the following three different forms of the FZ joint score for VaR and ES: the AL score considered by Taylor (2018) and Patton et al. (2017) in expression (2); the score considered in Nolde and Ziegel (2017) with $G_1 = 0$, $G_2 = \frac{1}{2}(-x)^{-\frac{1}{2}}$, $\mathscr{G}_2(x) = -(-x)^{\frac{1}{2}}$ and $a(x) = 0$ in expression (1), which we term the *NZ score*; and the score considered by Fissler et al. (2016) with $G_1(x) = x$, $G_2 = \frac{\exp(x)}{1+\exp(x)}$, $\mathscr{G}_2(x) = \ln(1 + \exp(x))$ and $a(x) = 0$ in expression (1), which we term the *FZG score*. The FZG score can have negative values, therefore, in order to make the values more easily comparable, we set $a(x) = \ln(2)$, which led to positive values for the score. We summarise the AL, NZ and FZG scores in Table 4.

As the quantile and FZ scores are not easy to interpret, we calculated skill scores as the ratio of each method's score to the score of a benchmark method, then subtracted this ratio from

1, and multiplied the result by 100. We chose the benchmark method as the GARCH model with Student-t distribution. For all skill scores, higher values are preferable. To summarise performance across multiple stock indices, for each method, we calculated the geometric mean of the ratios of the method's score to the score of the benchmark method, then subtracted this from one, and multiplied the result by 100.

Table 4: A selection of FZ scoring functions for the joint estimation and evaluation of VaR and ES.

	$G_1(x)$	$G_2(x)$	$\mathscr{G}_2(x)$	$a(x)$
AL	0	$-\frac{1}{x}$	$-\ln(-x)$	$1-\ln(1-\theta)$
NZ	0	$\frac{1}{2}(-x)^{-\frac{1}{2}}$	$-(-x)^{\frac{1}{2}}$	0
FZG	x	$\frac{\exp(x)}{1+\exp(x)}$	$\ln(1+\exp(x))$	$\ln(2)$

4.4 Post-Sample Results

Table 5 summarises the coverage test results for the five stock indices. For each of the three CAViaR-FZ models, the results of three methods are presented. The first method fits the model to the daily returns y_t using probability level θ, while the other two fit the model to the intraday low returns L_t using the two different approaches described in Section 3.2. Table 5 reports the numbers of test rejections at the 5% significance level for each method across the five indices. Smaller numbers of rejections are preferred. All the methods perform well in terms of unconditional coverage. For the DQ test, it is interesting to see that the methods tend to perform better for the 5% probability level than the other probability levels, which reflects the tendency for the more extreme quantiles to be more difficult to estimate. The best results correspond to GJR-GARCH, and the CAViaR-FZ-AS models, with one of the CAViaR-FZ-Range models also performing well. For the ES bootstrap test, little insight can be observed in terms of model performance. This is partly due to the fact that there are few observations beyond the VaR. This motivates us to evaluate the model performance using FZ scores.

Next, we present the evaluation results for the quantile score and the three FZ scores. We present the results in Tables 6 and 7, where we have indicated in bold the best two performing methods for each probability level in each table. Recall that the AL, NZ and FZG scores are all types of FZ score, which enable the joint evaluation of the VaR and ES. In Table 6, for the 0.5% probability level, the best quantile and AL skill score results correspond to the CAViaR-FZ-Range model fitted to the intraday low returns using either of the approaches described in Section 3.2. This is also the case when using the AL skill score to evaluate the results for the 1% probability level. For the 2.5% and 5% probability levels, the best quantile and AL skill score results again correspond to the models fitted to the intraday low returns, but for these less extreme probability levels, the CAViaR-FZ-AS model also performs well. The ranking of the models, in terms of the NZ and FZG skill scores in Table 7, are similar to those for the AL score in Table 6. Overall, the proposed CAViaR-FZ-Range models and the CAViaR-FZ-AS models based on the intraday low returns perform well. These results are supportive of the new proposals put forward in this paper, namely the use of the CAViaR-FZ-Range model of expression (5), and the use of the intraday low returns to estimate CAViaR-FZ models.

Table 5: Summary of the coverage test results for each probability level for the stock indices. Number of test rejections for the five indices (at 5% significance level).

	VaR Hit				VaR DQ				ES Bootstrap			
$\theta \times 100$	0.5	1	2.5	5	0.5	1	2.5	5	0.5	1	2.5	5
GARCH												
Student-t	0	1	4	0	3	4	2	3	0	0	0	0
FHS	0	0	0	0	3	4	1	3	0	0	0	0
EVT	0	0	0	0	3	2	1	3	0	0	0	0
GJR-GARCH												
Student-t	0	0	3	2	2	1	2	1	1	0	0	0
FHS	0	0	0	0	2	1	1	0	1	1	0	0
EVT	0	0	0	0	1	0	0	0	1	0	0	0
Realized GARCH												
Student-t	0	1	4	0	3	3	4	3	1	0	0	2
FHS	0	0	0	0	3	2	2	0	1	0	0	0
EVT	0	0	4	0	3	3	5	0	0	0	3	0
CAViaR-FZ-SAV												
θ VaR and ES of y_t	0	0	0	0	3	2	2	2	0	0	0	0
$\tilde{\theta}$ VaR and ES of L_t	0	0	0	0	3	2	1	2	0	1	0	0
Rescaling the above	0	0	0	0	3	2	1	1	0	0	1	0
CAViaR-FZ-AS												
θ VaR and ES of y_t	0	0	0	0	2	0	1	0	0	0	0	0
$\tilde{\theta}$ VaR and ES of L_t	0	0	0	0	2	0	0	0	0	0	0	1
Rescaling the above	0	0	0	0	2	0	1	0	0	0	0	0
CAViaR-FZ-Range												
θ VaR and ES of y_t	0	0	0	0	3	2	2	1	1	0	0	0
$\tilde{\theta}$ VaR and ES of L_t	0	0	0	0	3	0	1	0	1	1	0	0
Rescaling the above	0	0	0	0	3	2	2	1	1	0	0	0

Note: Smaller values are preferred. y_t is daily return and L_t is intraday low return.

Table 6: Summary of the quantile and AL skill scores for each probability level for the stock indices. Skill scores averaged over the five indices.

	Quantile Score				AL Score			
$\theta \times 100$	0.5	1	2.5	5	0.5	1	2.5	5
GARCH								
Student-t	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
FHS	-0.4	0.1	0.6	-0.1	-0.2	0.2	0.5	0.2
EVT	-0.7	0.2	0.4	0.0	0.2	0.4	0.5	0.2
GJR-GARCH								
Student-t	4.4	4.6	3.6	2.5	1.0	1.7	1.7	1.6
FHS	3.8	4.2	4.0	2.5	1.0	1.8	2.2	1.9
EVT	3.2	3.9	3.9	2.6	1.2	1.9	2.2	1.9
Realized GARCH								
Student-t	3.6	3.8	2.3	1.9	1.7	2.0	1.4	1.4
FHS	2.5	4.0	3.4	2.0	1.2	2.3	2.3	1.7
EVT	3.1	4.1	1.5	2.1	1.8	2.6	0.4	1.8
CAViaR-FZ-SAV								
θ VaR and ES of y_t	2.2	0.6	0.6	-0.1	1.0	0.6	0.6	0.3
$\tilde{\theta}$ VaR and ES of L_t	1.0	0.8	1.2	0.2	0.6	0.8	0.9	0.5
Rescaling the above	2.5	1.5	1.0	0.2	1.3	1.2	0.8	0.4
CAViaR-FZ-AS								
θ VaR and ES of y_t	5.1	**4.8**	4.1	2.6	2.1	2.3	2.1	1.8
$\tilde{\theta}$ VaR and ES of L_t	3.7	4.1	**4.5**	**2.9**	1.1	1.9	**2.4**	**2.0**
Rescaling the above	5.3	**4.8**	**4.3**	**2.9**	2.2	2.3	2.2	**2.0**
CAViaR-FZ-Range								
θ VaR and ES of y_t	4.7	3.8	3.2	2.1	3.1	2.7	2.3	1.7
$\tilde{\theta}$ VaR and ES of L_t	**6.8**	4.5	3.2	2.1	**3.6**	**3.0**	**2.4**	1.9
Rescaling the above	**6.3**	4.6	3.2	2.0	**3.6**	**3.0**	2.3	1.7

Note: Large skill scores are better. The best two methods in each column are indicated in bold. y_t is daily return and L_t is intraday low return.

Table 7: Summary of the NZ and FZG skill scores for each probability level for the stock indices. Skill scores averaged over the five indices.

	NZ Score				FZG Score			
$\theta \times 100$	0.5	1	2.5	5	0.5	1	2.5	5
GARCH								
Student-t	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
FHS	-0.2	0.1	0.4	0.1	-0.1	0.2	0.5	0.2
EVT	-0.1	0.3	0.3	0.1	0.3	0.4	0.5	0.2
GJR-GARCH								
Student-t	1.8	2.2	1.8	1.5	-0.6	0.3	1.0	1.4
FHS	1.6	2.1	2.2	1.6	-0.2	0.7	1.5	1.6
EVT	1.5	2.1	2.2	1.6	0.1	0.9	1.6	1.6
Realized GARCH								
Student-t	1.9	2.0	1.3	1.2	0.9	1.3	1.1	1.3
FHS	1.3	2.3	2.0	1.3	0.6	1.5	1.9	1.6
EVT	1.8	2.4	0.6	1.4	1.1	1.8	0.2	1.6
CAViaR-FZ-SAV								
θ VaR and ES of y_t	1.2	0.5	0.5	0.1	0.5	0.5	0.5	0.3
$\tilde{\theta}$ VaR and ES of L_t	0.6	0.7	0.7	0.3	0.2	0.6	0.8	0.5
Rescaling the above	1.5	1.1	0.7	0.3	0.6	0.8	0.7	0.5
CAViaR-FZ-AS								
θ VaR and ES of y_t	2.6	2.5	2.2	1.6	0.6	1.1	1.5	1.6
$\tilde{\theta}$ VaR and ES of L_t	1.7	2.1	**2.4**	**1.7**	-0.1	0.8	1.7	**1.8**
Rescaling the above	2.7	2.5	**2.3**	**1.8**	0.7	1.1	1.5	**1.8**
CAViaR-FZ-Range								
θ VaR and ES of y_t	3.0	2.4	2.0	1.4	**1.8**	1.9	**1.9**	1.6
$\tilde{\theta}$ VaR and ES of L_t	**3.8**	**2.8**	2.0	1.4	**1.7**	**2.1**	**2.0**	1.7
Rescaling the above	**3.7**	**2.8**	2.0	1.3	**1.8**	**2.0**	**1.9**	1.6

Note: Large skill scores are better. The best two methods in each column are indicated in bold. y_t is daily return and L_t is intraday low return.

4.5 Empirical Study of Individual Stocks

It could be suggested that the reliance of our proposed models on the intraday low returns will be particularly useful for returns with notably high kurtosis, which essentially implies particularly long distributional tails. With this in mind, we selected the five S&P 500 companies with highest market capitalisation that had kurtosis exceeding the kurtosis of the five stock indices that we have considered so far in this paper. The five individual stocks are AMZN, BAC, JPM, UHN and XOM. For these stocks, we used data from the same period that we considered for the S&P 500 data. Descriptive statistics for the returns of these indices are shown in Table 8.

The empirical results for the individual stocks are presented in Tables 9-11. We report the coverage test results in Table 9, the quantile and AL scores in Table 10, and the NZ and FZG scores in Table 11. Overall, the results show that the best performing methods were our new CAViaR-FZ-Range model applied to the intraday low, and also the realized GARCH model with intraday range. The CAViaR-FZ-AS model, which performed well for the stock indices, did not perform well for the individual stocks. One explanation is that the intraday range contains more predicative power for forecasting the VaR and ES of individual stock returns with heavy tails.

Table 8: Descriptive statistics for the full sample of 3300 daily returns of the five individual stocks.

Index	Mean	Std. Dev.	Min	Max	Skewness	Kurtosis
Stock Indices						
AMZN	0.11	2.63	-24.62	23.86	0.39	15.30
BAC	-0.02	3.23	-34.21	30.21	-0.35	28.79
JPM	0.04	2.51	-23.23	22.39	0.35	19.09
UNH	0.05	2.07	-20.62	29.83	0.30	26.67
XOM	0.03	1.52	-15.03	15.86	0.00	16.55

Table 9: Summary of the coverage test results for each probability level for the individual stocks. Number of test rejections for the five stocks (at 5% significance level).

	VaR Hit				VaR DQ				ES Bootstrap			
$\theta \times 100$	0.5	1	2.5	5	0.5	1	2.5	5	0.5	1	2.5	5
GARCH												
Student-t	0	0	1	2	4	2	3	2	0	0	0	0
FHS	1	0	0	0	4	1	3	3	1	1	1	1
EVT	1	0	1	0	4	2	3	3	0	1	1	1
GJR-GARCH												
Student-t	0	0	0	2	3	1	2	1	0	0	0	0
FHS	0	1	1	0	3	1	2	2	0	1	0	1
EVT	1	0	1	0	3	1	2	2	1	1	1	1
Realized GARCH												
Student-t	0	0	1	1	3	3	2	3	1	0	0	0
FHS	0	0	0	0	2	1	2	4	0	1	1	1
EVT	1	2	0	0	2	3	3	3	0	1	1	1
CAViaR-FZ-SAV												
θ VaR and ES of y_t	0	0	0	0	4	2	4	2	0	0	0	1
$\tilde{\theta}$ VaR and ES of L_t	1	2	2	1	3	1	3	3	0	1	1	1
Rescaling the above	0	1	0	0	4	2	3	2	0	1	0	1
CAViaR-FZ-AS												
θ VaR and ES of y_t	0	0	0	0	5	2	2	2	0	1	1	1
$\tilde{\theta}$ VaR and ES of L_t	1	2	2	1	2	2	2	1	0	0	1	1
Rescaling the above	0	1	1	0	2	1	2	1	0	1	1	0
CAViaR-FZ-Range												
θ VaR and ES of y_t	0	0	0	0	3	0	2	3	0	1	1	1
$\tilde{\theta}$ VaR and ES of L_t	0	2	0	0	2	1	1	2	1	0	1	1
Rescaling the above	1	0	0	0	2	0	2	2	0	1	1	1

Note: Smaller values are preferred. y_t is daily return and L_t is intraday low return.

Table 10: Summary of the quantile and AL skill scores for each probability level for the individual stocks. Skill scores averaged over the five stocks.

	Quantile Score				AL Score			
$\theta \times 100$	0.5	1	2.5	5	0.5	1	2.5	5
GARCH								
Student-t	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
FHS	-1.4	-0.2	-0.4	0.1	-1.0	-0.4	-0.3	0.0
EVT	-0.6	-0.7	-0.3	0.2	-0.4	-0.4	-0.3	0.0
GJR-GARCH								
Student-t	2.1	1.3	1.0	0.5	0.8	0.6	0.6	0.4
FHS	0.6	1.0	0.8	0.7	-0.1	0.3	0.4	0.4
EVT	1.4	0.9	0.7	0.8	0.4	0.3	0.3	0.5
Realized GARCH								
Student-t	6.4	5.0	2.8	2.0	**2.8**	**2.5**	1.9	1.7
FHS	5.9	**5.3**	3.4	2.3	2.1	**2.3**	2.0	1.8
EVT	5.7	**5.2**	3.4	2.4	2.2	**2.3**	2.0	1.8
CAViaR-FZ-SAV								
θ VaR and ES of y_t	0.1	1.4	-0.2	0.0	-0.9	0.2	0.0	0.1
$\tilde{\theta}$ VaR and ES of L_t	-0.4	0.1	-0.2	0.2	-0.6	-0.1	-0.1	0.3
Rescaling the above	-0.1	1.3	0.0	0.2	-0.6	0.3	0.1	0.2
CAViaR-FZ-AS								
θ VaR and ES of y_t	-1.2	1.5	1.3	0.9	-1.4	0.3	0.8	0.8
$\tilde{\theta}$ VaR and ES of L_t	1.0	1.6	1.5	1.3	-0.2	0.6	0.8	1.0
Rescaling the above	-0.3	2.1	2.0	1.2	-0.8	0.5	1.0	0.9
CAViaR-FZ-Range								
θ VaR and ES of y_t	5.2	5.0	3.2	**2.5**	1.5	2.2	2.0	**1.9**
$\tilde{\theta}$ VaR and ES of L_t	**7.9**	5.1	**3.7**	**2.5**	**2.6**	**2.3**	**2.1**	**1.9**
Rescaling the above	**7.9**	**5.2**	**3.5**	2.3	2.1	2.2	**2.1**	1.8

Note: Large skill scores are better. The best two methods in each column are indicated in bold. y_t is daily return and L_t is intraday low return.

Table 11: Summary of the NZ and FZG skill scores for each probability level for the individual stocks. Skill scores averaged over the five stocks.

	NZ Score				FZG Score			
$\theta \times 100$	0.5	1	2.5	5	0.5	1	2.5	5
GARCH								
Student-t	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
FHS	-1.1	-0.4	-0.3	0.0	-0.1	-0.1	-0.2	0.0
EVT	-0.5	-0.5	-0.3	0.0	0.0	-0.1	-0.2	0.0
GJR-GARCH								
Student-t	1.1	0.8	0.6	0.4	0.1	0.1	0.3	0.3
FHS	0.1	0.4	0.5	0.5	0.0	0.0	0.1	0.3
EVT	0.7	0.5	0.4	0.5	0.1	0.1	0.1	0.3
Realized GARCH								
Student-t	**3.6**	**3.0**	1.9	1.5	**0.8**	**1.1**	**1.3**	1.4
FHS	2.9	**2.9**	2.1	1.6	0.7	**1.0**	**1.3**	1.4
EVT	3.0	**2.9**	2.1	**1.7**	0.7	**1.0**	**1.3**	1.4
CAViaR-FZ-SAV								
θ VaR and ES of y_t	-0.4	0.6	-0.1	0.1	-0.7	-0.2	0.0	0.1
$\tilde{\theta}$ VaR and ES of L_t	-0.4	0.0	-0.1	0.2	-0.2	-0.1	-0.1	0.2
Rescaling the above	-0.4	0.5	0.0	0.2	-0.4	0.0	0.0	0.2
CAViaR-FZ-AS								
θ VaR and ES of y_t	-1.1	0.7	0.9	0.7	-1.1	-0.1	0.4	0.6
$\tilde{\theta}$ VaR and ES of L_t	0.2	0.8	0.9	0.9	-0.2	0.1	0.4	0.7
Rescaling the above	-0.5	1.0	1.2	0.9	-0.9	-0.1	0.6	0.7
CAViaR-FZ-Range								
θ VaR and ES of y_t	2.5	2.8	2.0	**1.7**	0.4	0.9	1.2	**1.5**
$\tilde{\theta}$ VaR and ES of L_t	**3.9**	**2.9**	**2.3**	**1.8**	**0.9**	**1.0**	**1.3**	**1.5**
Rescaling the above	3.5	2.8	**2.2**	**1.7**	0.6	0.7	1.2	1.4

Note: Large skill scores are better. The best two methods in each column are indicated in bold. y_t is daily return and L_t is intraday low return.

5 Conclusion

In this paper, we have introduced a new approach to estimating conditional VaR and ES. In this approach, we not only consider the use of intraday information in model specification, but also use it in parameter estimation. Based on theory for Brownian motion, we argue that the dynamics of the VaR and ES of the intraday low return contains useful information for forecasting the dynamics of the VaR and ES of the daily returns. For a chosen probability level, we estimate the VaR and ES of the daily returns using the VaR and ES of the intraday low returns corresponding to a less extreme probability level. We also consider rescaling the proposed models for the intraday returns in a second step. Our empirical results suggest that the proposed approach is particularly useful for more extreme probability levels. A further contribution of the paper is that we support previous research that has shown that the intraday range is a useful explanatory variable for VaR and ES modelling, and that minimizing a joint VaR and ES scoring function is a useful way to estimate models with CAViaR structure for VaR and ES prediction.

Acknowledgements

We are very grateful to two referees for providing comments that helped greatly to improve the paper.

Appendix

In this appendix, we provide a proof for Theorem 1 under Assumption 1. Before we prove Theorem 1, we state the results in Theorem 2 of Patton et al. (2017): Under suitable regularity conditions, which are given in Assumption 2 of Patton et al. (2017), the parameter vector estimator $\hat{\boldsymbol{\beta}}$ for y_t satisfies the following asymptotic normality:

$$\sqrt{T} A_T^{-\frac{1}{2}} D_T(\hat{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_0) \xrightarrow{d} N(0, I) \ as \ T \to \infty$$

where

$$D_T = \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} \frac{f_t(q_t(\boldsymbol{\beta}_0)|\mathcal{F}_{t-1})}{-e_t(\boldsymbol{\beta}_0)\theta} \nabla q_t(\boldsymbol{\beta}_0)' \nabla q_t(\boldsymbol{\beta}_0) + \frac{1}{e_t(\boldsymbol{\beta}_0)^2} \nabla e_t(\boldsymbol{\beta}_0)' \nabla e_t(\boldsymbol{\beta}_0)\right]$$

$$A_T = \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} g_t(\boldsymbol{\beta}_0) g_t(\boldsymbol{\beta}_0)'\right]$$

$$g_t(\boldsymbol{\beta}_0) = \nabla q_t(\boldsymbol{\beta}_0)' \frac{1}{-e_t(\boldsymbol{\beta}_0)} \left(\frac{1}{\theta}\mathbb{1}\{y_t \le q_t(\boldsymbol{\beta}_0)\} - 1\right) + \nabla e_t(\boldsymbol{\beta}_0)' \frac{1}{e_t(\boldsymbol{\beta}_0)^2} \left(\frac{1}{\theta}\mathbb{1}\{y_t \le q_t(\boldsymbol{\beta}_0)\}(q_t(\boldsymbol{\beta}_0) - y_t) - q_t(\boldsymbol{\beta}_0) + e_t(\boldsymbol{\beta}_0)\right)$$

$f_t(q_t(\boldsymbol{\beta}_0)|\mathcal{F}_{t-1})$ denotes the density of y_t conditional on past information $\mathcal{F}_{t-1}$ evaluated at the θ conditional quantile $q_t(\boldsymbol{\beta}_0)$, $\boldsymbol{\beta}_0$ is the true parameter, and $\hat{\boldsymbol{\beta}}_T$ is the parameter estimator using the in-sample observations of length T.

Efficiency is shown by proving positive semi-definiteness of the difference between two asymptotic parameter covariance matrices (see, for example, Komunjer and Vuong 2010). To prove our Theorem 1 in Section 3.1, we need to prove the positive semi-definiteness of $(D_T)^{-1}A_T(D_T)^{-1} - (\tilde{D}_T)^{-1}\tilde{A}_T(\tilde{D}_T)^{-1}$, which is the difference between the asymptotic parameter covariance matrices of the θ VaR and ES for the daily returns and the $\lambda\theta$ VaR and ES for the intraday low returns. Now we begin our proof:

Proof. Under Assumption 1, we have that the $\lambda\theta$ VaR and ES of L_t are equal to the θ VaR and ES of y_t, that is, $\tilde{q}_t(\tilde{\boldsymbol{\beta}}^0) = q_t(\boldsymbol{\beta}^0)$ and $\tilde{e}_t(\tilde{\boldsymbol{\beta}}^0) = e_t(\boldsymbol{\beta}^0)$. Therefore, $\tilde{\boldsymbol{\beta}}^0 = \boldsymbol{\beta}^0$ by Assumption 1(A) in Patton et al. (2017). As a consequence, $\nabla q_t(\tilde{\boldsymbol{\beta}}^0) = \nabla q_t(\boldsymbol{\beta}^0)$. For convenience, we use q_t to denote $\tilde{q}_t(\tilde{\boldsymbol{\beta}}^0)$ and $q_t(\boldsymbol{\beta}^0)$, and we use e_t to denote $\tilde{e}_t(\tilde{\boldsymbol{\beta}}^0)$ and $e_t(\boldsymbol{\beta}^0)$, in the rest of the proof. First observe that:

$$\tilde{D}_T = \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} \frac{\tilde{f}_t(q_t|\mathcal{F}_{t-1})}{-e_t\lambda\theta} \nabla q_t' \nabla q_t + \frac{1}{e_t^2} \nabla e_t' \nabla e_t\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\frac{f_t(q_t|\mathcal{F}_{t-1})}{-e_t\theta}\nabla q_t'\nabla q_t + \frac{1}{e_t^2}\nabla e_t'\nabla e_t\right] \\
&= D_T
\end{aligned}
$$

where the second equality uses the fact that $\tilde{f}_t(q_t|\mathcal{F}_{t-1}) = \lambda f_t(q_t|\mathcal{F}_{t-1})$, which is a direct consequence of Assumption 1(A). Next we look at $\mathbb{E}\left[\tilde{g}_t(\tilde{g}_t)'\right]$:

$$
\begin{aligned}
\mathbb{E}\left[\tilde{g}_t(\tilde{g}_t)'\right] &= \mathbb{E}\Bigg[\left(\nabla q_t'\frac{1}{-e_t}\left(\frac{1}{\lambda\theta}\mathbb{1}\{L_t \le q_t\} - 1\right) + \nabla e_t'\frac{1}{e_t^2}\left(\frac{1}{\lambda\theta}\mathbb{1}\{L_t \le q_t\}(q_t - L_t) - q_t + e_t\right)\right) \\
&\quad \left(\nabla q_t'\frac{1}{-e_t}\left(\frac{1}{\lambda\theta}\mathbb{1}\{L_t \le q_t\} - 1\right) + \nabla e_t'\frac{1}{e_t^2}\left(\frac{1}{\lambda\theta}\mathbb{1}\{L_t \le q_t\}(q_t - L_t) - q_t + e_t\right)\right)'\Bigg] \\
&= \mathbb{E}\Bigg[\frac{1}{e_t^2}\left(\frac{\mathbb{1}\{L_t \le q_t\}}{\lambda\theta}\left(-\nabla q_t + \frac{\nabla e_t}{e_t}q_t - \frac{\nabla e_t}{e_t}L_t\right) + \left(\frac{\nabla e_t}{e_t}(e_t - q_t) + \nabla q_t\right)\right)' \\
&\quad \left(\frac{\mathbb{1}\{L_t \le q_t\}}{\lambda\theta}\left(-\nabla q_t + \frac{\nabla e_t}{e_t}q_t - \frac{\nabla e_t}{e_t}L_t\right) + \left(\frac{\nabla e_t}{e_t}(e_t - q_t) + \nabla q_t\right)\right)\Bigg] \\
&= \mathbb{E}\Bigg[\frac{1}{e_t^2}\left(\frac{\mathbb{1}\{L_t \le q_t\}}{\lambda^2\theta^2}\left(-\nabla q_t + \frac{\nabla e_t}{e_t}q_t - \frac{\nabla e_t}{e_t}L_t\right)'\left(-\nabla q_t + \frac{\nabla e_t}{e_t}q_t - \frac{\nabla e_t}{e_t}L_t\right)\right) \\
&\quad + \frac{1}{e_t^2}\left(\frac{\nabla e_t}{e_t}(e_t - q_t) + \nabla q_t\right)'\left(\frac{\nabla e_t}{e_t}(e_t - q_t) + \nabla q_t\right) + \frac{\mathbb{1}\{L_t \le q_t\}}{\lambda\theta e_t^2} \\
&\quad \Bigg(\left(-\nabla q_t + \frac{\nabla e_t}{e_t}q_t - \frac{\nabla e_t}{e_t}L_t\right)'\left(\frac{\nabla e_t}{e_t}(e_t - q_t) + \nabla q_t\right) \\
&\quad + \left(\frac{\nabla e_t}{e_t}(e_t - q_t) + \nabla q_t\right)'\left(-\nabla q_t + \frac{\nabla e_t}{e_t}q_t - \frac{\nabla e_t}{e_t}L_t\right)\Bigg)\Bigg]
\end{aligned}
$$

One may note that

$$
\mathbb{E}\left[\frac{1}{e_t^2}\left(\frac{\nabla e_t}{e_t}(e_t - q_t) + \nabla q_t\right)'\left(\frac{\nabla e_t}{e_t}(e_t - q_t) + \nabla q_t\right)\right]
$$

is independent of λ, and one may also notice the following relationship:

$$
\begin{aligned}
\mathbb{E}\left[\mathbb{1}\{L_t \le q_t\}h(L_t, q_t, e_t)\right] &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}\{L_t \le q_t\}h(L_t, q_t, e_t)|\mathcal{F}_{t-1}\right]\right] && (10) \\
&= \mathbb{E}\left[\lambda\mathbb{E}\left[\mathbb{1}\{y_t \le q_t\}h(y_t, q_t, e_t)|\mathcal{F}_{t-1}\right]\right] && (11) \\
&= \lambda\mathbb{E}\left[\mathbb{1}\{y_t \le q_t\}h(y_t, q_t, e_t)\right] && (12)
\end{aligned}
$$

for any function h, which can be easily derived from Assumption 1(A) using the fact that q_t and e_t are completely determined by $\mathcal{F}_{t-1}$. This implies that

$$
\begin{aligned}
\mathbb{E}\Bigg[\frac{\mathbb{1}\{L_t \le q_t\}}{\lambda\theta e_t^2}\Bigg(&\left(-\nabla q_t + \frac{\nabla e_t}{e_t}q_t - \frac{\nabla e_t}{e_t}L_t\right)'\left(\frac{\nabla e_t}{e_t}(e_t - q_t) + \nabla q_t\right) \\
&+ \left(\frac{\nabla e_t}{e_t}(e_t - q_t) + \nabla q_t\right)'\left(-\nabla q_t + \frac{\nabla e_t}{e_t}q_t - \frac{\nabla e_t}{e_t}L_t\right)\Bigg)\Bigg]
\end{aligned}
$$

is independent of λ. Thus all the other terms apart from

$$\mathbb{E}\left[\frac{\mathbb{1}\{L_t \leq q_t\}}{\lambda^2\theta^2 e_t^2}\left(-\nabla q_t + \frac{\nabla e_t}{e_t}q_t - \frac{\nabla e_t}{e_t}y_t\right)'\left(-\nabla q_t + \frac{\nabla e_t}{e_t}q_t - \frac{\nabla e_t}{e_t}y_t\right)\right]$$

are independent of λ. In $\mathbb{E}[g_t g_t'] - \mathbb{E}[\tilde{g}_t\tilde{g}_t']$, these terms are cancelled out. One can further obtain the following by the relationship in expressions (10)-(12),

$$\begin{aligned}&\mathbb{E}\left[\frac{\mathbb{1}\{L_t \leq q_t\}}{\lambda^2\theta^2 e_t^2}\left(-\nabla q_t + \frac{\nabla e_t}{e_t}q_t - \frac{\nabla e_t}{e_t}L_t\right)'\left(-\nabla q_t + \frac{\nabla e_t}{e_t}q_t - \frac{\nabla e_t}{e_t}L_t\right)\right] \\ &= \frac{1}{\lambda}\mathbb{E}\left[\frac{\mathbb{1}\{y_t \leq q_t\}}{\theta^2 e_t^2}\left(-\nabla q_t + \frac{\nabla e_t}{e_t}q_t - \frac{\nabla e_t}{e_t}y_t\right)'\left(-\nabla q_t + \frac{\nabla e_t}{e_t}q_t - \frac{\nabla e_t}{e_t}y_t\right)\right]\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathbb{E}[g_t g_t'] - \mathbb{E}[\tilde{g}_t\tilde{g}_t'] = \frac{\lambda-1}{\lambda}\mathbb{E}\Bigg[&\frac{\mathbb{1}\{y_t \leq q_t\}}{\theta^2 e_t^2}\left(-\nabla q_t + \frac{\nabla e_t}{e_t}q_t - \frac{\nabla e_t}{e_t}y_t\right)' \\ &\left(-\nabla q_t + \frac{\nabla e_t}{e_t}q_t - \frac{\nabla e_t}{e_t}y_t\right)\Bigg]\end{aligned}$$

which is positive semi-definite as $\lambda > 1$ by Assumption 1(A). Consequently, $A_T - \tilde{A}_T$ is positive semi-definite, since

$$\begin{aligned}A_T - \tilde{A}_T &= \mathbb{E}\left[\sum_{t=1}^T g_t g_t'\right] - \mathbb{E}\left[\sum_{t=1}^T \tilde{g}_t\tilde{g}_t'\right] \\ &= \sum_{t=1}^T \left(\mathbb{E}[g_t g_t'] - \mathbb{E}[\tilde{g}_t\tilde{g}_t']\right)\end{aligned}$$

We conclude the proof by observing that

$$(D_T)^{-1}A_T(D_T)^{-1} - (\tilde{D}_T)^{-1}\tilde{A}_T(\tilde{D}_T)^{-1} = (D_T)^{-1}\left(A_T - \tilde{A}_T\right)(D_T)^{-1}$$

is also positive semi-definite, since $D_T = \tilde{D}_T$ and is symmetric by its definition. □

References

Acerbi, C. and Szekely, B. (2014). Back-testing expected shortfall. *Risk*, 27:76–81.

Alizadeh, S., Brandt, M. W., and Diebold, F. X. (2002). Range-based estimation of stochastic volatility models. *Journal of Finance*, 57(3):1047–1091.

Andersen, L. B. and Piterbarg, V. V. (2007). Moment explosions in stochastic volatility models. *Finance and Stochastics*, 11(1):29–50.

Back, K. (1993). Asymmetric information and options. *The Review of Financial Studies*, 6(3):435–472.

Basel Committee on Banking Supervision (2012). Fundamental review of the trading book, May

2012. Available online at:www.bis.org/publ/bcbs219.htm.

Black, F. and Cox, J. C. (1976). Valuing corporate securities: Some effects of bond indenture provisions. *The Journal of Finance*, 31(2):351–367.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654.

Bollerslev, T. (1987). A conditionally heteroskedastic time series model for speculative prices and rates of return. *Review of Economics and Statistics*, 69(3):542–547.

Brandt, M. W. and Jones, C. S. (2006). Volatility forecasting with range-based EGARCH models. *Journal of Business & Economic Statistics*, 24(4):470–486.

Butler, J. and Schachter, B. (1997). Estimating value-at-risk with a precision measure by combining kernel estimation with historical simulation. *Review of Derivatives Research*, 1(4):371–390.

Chen, C. W., Gerlach, R., Hwang, B. B., and McAleer, M. (2012). Forecasting value-at-risk using nonlinear regression quantiles and the intra-day range. *International Journal of Forecasting*, 28(3):557–574.

Christoffersen, P. (2010). Backtesting. *Encyclopedia of Quantitative Finance*.

Clements, M. P., Galvão, A. B., and Kim, J. H. (2008). Quantile forecasts of daily exchange rate returns from forecasts of realized volatility. *Journal of Empirical Finance*, 15(4):729–750.

Corsi, F. (2009). A simple approximate long-memory model of realized volatility. *Journal of Financial Econometrics*, 7(2):174–196.

Du, Z. and Escanciano, J. C. (2016). Backtesting expected shortfall: accounting for tail risk. *Management Science*, 63(4):940–958.

Ehm, W., Gneiting, T., Jordan, A., and Krüger, F. (2016). Of quantiles and expectiles: consistent scoring functions, Choquet representations and forecast rankings. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 78(3):505–562.

Embrechts, P., Puccetti, G., Rüschendorf, L., Wang, R., and Beleraj, A. (2014). An academic response to Basel 3.5. *Risks*, 2(1):25–48.

Engle, R. F. and Manganelli, S. (2004). CAViaR: Conditional autoregressive value at risk by regression quantiles. *Journal of Business & Economic Statistics*, 22(4):367–381.

Escanciano, J. C. and Pei, P. (2012). Pitfalls in backtesting historical simulation var models. *Journal of Banking & Finance*, 36(8):2233–2244.

Fissler, T., Ziegel, J., and Gneiting, T. (2016). Expected shortfall is jointly elicitable with value at risk - implications for backtesting. *Risk*, Janurary:58–61.

Fissler, T. and Ziegel, J. F. (2016). Higher order elicitability and Osband's principle. *The Annals of Statistics*, 44(4):1680–1707.

Gerlach, R. and Wang, C. (2016). Forecasting risk via realized GARCH, incorporating the realized range. *Quantitative Finance*, 16(4):501–511.

Glosten, L. R., Jagannathan, R., and Runkle, D. E. (1993). On the relation between the expected value and the volatility of the nominal excess return on stocks. *Journal of Finance*, 48(5):1779–1801.

Gneiting, T. (2011). Making and evaluating point forecasts. *Journal of the American Statistical Association*, 106(494):746–762.

Gneiting, T. and Raftery, A. E. (2007). Strictly proper scoring rules, prediction, and estimation. *Journal of the American Statistical Association*, 102(477):359–378.

Gordy, M. B. and Juneja, S. (2010). Nested simulation in portfolio risk measurement. *Management Science*, 56(10):1833–1848.

Hansen, P. R., Huang, Z., and Shek, H. H. (2012). Realized GARCH: A joint model for returns and realized measures of volatility. *Journal of Applied Econometrics*, 27(6):877–906.

Huang, D., Zhu, S., Fabozzi, F. J., and Fukushima, M. (2010). Portfolio selection under distributional uncertainty: A relative robust CVaR approach. *European Journal of Operational Research*, 203(1):185–194.

Hull, J. (2012). *Risk management and financial institution*, volume 733. John Wiley & Sons.

Komunjer, I. and Vuong, Q. (2010). Semiparametric efficiency bound in time-series models for conditional quantiles. *Econometric Theory*, 26(2):383–405.

Kou, S. G. (2002). A jump-diffusion model for option pricing. *Management Science*, 48(8):1086–1101.

Kumar, D. and Maheswaran, S. (2015). Long memory in Indian exchange rates: an application of power-law scaling analysis. *Macroeconomics and Finance in Emerging Market Economies*, 8(1-2):90–107.

Lwin, K. T., Qu, R., and MacCarthy, B. L. (2017). Mean-VaR portfolio optimization: A nonparametric approach. *European Journal of Operational Research*, 260(2):751–766.

Manganelli, S. and Engle, R. F. (2004). Value at risk models in finance. In Szegö, G., editor, *Risk Measures for the 21st Century*, pages 123–143. Wiley, Chichester, UK.

McNeil, A. J. and Frey, R. (2000). Estimation of tail-related risk measures for heteroscedastic financial time series: an extreme value approach. *Journal of Empirical Finance*, 7(3):271–300.

Merton, R. C. (1974). On the pricing of corporate debt: The risk structure of interest rates. *The Journal of Finance*, 29(2):449–470.

Mörters, P. and Peres, Y. (2010). *Brownian Motion*. Cambridge University Press.

Nolde, N. and Ziegel, J. F. (2017). Elicitability and backtesting: Perspectives for banking regulation. *Annals of Applied Statistics*, forthcoming.

Parkinson, M. (1980). The extreme value method for estimating the variance of the rate of return. *Journal of Business*, 53(1):61–65.

Patton, A. J., Ziegel, J. F., and Chen, R. (2017). Dynamic semiparametric models for expected shortfall (and value-at-risk). *arXiv preprint arXiv:1707.05108*.

Rockafellar, R. T., Royset, J. O., and Miranda, S. I. (2014). Superquantile regression with applications to buffered reliability, uncertainty quantification, and conditional value-at-risk. *European Journal of Operational Research*, 234(1):140–154.

Sévi, B. (2014). Forecasting the volatility of crude oil futures using intraday data. *European Journal of Operational Research*, 235(3):643–659.

Taylor, J. W. (2018). Forecasting value at risk and expected shortfall using a semiparametric approach based on the asymmetric laplace distribution. *Journal of Business & Economic Statistics*, forthcoming.

Wang, P., Zhang, B., and Zhou, Y. (2015). Asymmetries in stock markets. *European Journal of Operational Research*, 241(3):749–762.